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Using analogies to give meaning to the principle of mathematical induction

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We conducted a study on the teaching and learning of mathematical induction, with undergraduate students (N=117) enrolled in a Discrete Mathematics course at a four-year university in the United States. We focus here on the design and implementation of tasks using analogies with the potential to give meaning to the Principle of Mathematical Induction (PMI).

Keywords: Principle of mathematical induction, analogies.

Introduction

Mathematical induction is a critical topic for students enrolled in Discrete Mathematics courses. Research on the teaching and learning of mathematical induction revealed students' difficulties and their procedural approach to proofs, and proposed ways to improve the teaching of mathematical induction and increase students' understanding, their performance on mathematical induction tasks, and their convictions in the validity of proofs by induction (Ernest, 1984; Gonzalez, 2020; Kokushkin, 2020; Movshovitz-Hadar, 1993). At the heart of a proof by mathematical induction is PMI. In the textbook (Epp, 2019) used by the participants in this study, PMI is defined this way:

Let $P(n)$ be a statement that depends on n , where n is a natural number, and let n_0 be a natural number such that:

- i. $P(n_0)$ is true; and
- ii. for any natural number $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.

Then, the statement $P(n)$ is true for all natural numbers $n \geq n_0$.

Ernest (1984) analysed the way the topic of mathematical induction was presented in 17 textbooks and found that only two of them addressed the students' understanding of PMI and included an analogy of PMI. Ernest (1984) stressed the importance of using analogies to help students relate mathematical induction to their own experiences. The purpose of the study reported here is to gain insight into the use of analogies with the potential to give meaning to PMI.

Theoretical background

Analogy comes from the Greek mathematical word "analogia" meaning proportion (Pimm, 1981). Proportionality, interpreted as equality of ratios or agreement in the ratios of corresponding parts, can be seen as an analogy. Mathematicians and mathematics educators are familiar with Polya's (1954) book, *Mathematics and Plausible Reasoning: Induction and Analogy in Mathematics*, with examples of analogies that help us see the solution to a problem more clearly and can provide new ways of thinking about problems. Ruppert (2013) argues that we can clarify the role of analogy in mathematics

education only by using the cognitive psychology approach. From a cognitive psychology perspective, Gentner & Markman (1997) argue that *similarity*, seen as a relation between two sets in which each element from the first set is associated with an element from the second set, is like analogy since both psychological processes involve *structural alignment* and *mapping*. According to the structural alignment view, people understand analogies by mapping the relational structure of one situation (*the source*) onto another situation (*the target*). In this paper, we consider an analogy as a mapping from the source structure to the target structure. In the learning of mathematics, the use of analogies may lead to the creation of abstract schemas representing the underlying structure of the source and target, thus contributing to the transfer of learning from source to target (Novick & Holyoak, 1991).

The importance of visual representations and gestures used by the instructor during teaching was highlighted by Kokushkin (2020), and Richland et al. (2007). To exemplify the structural view of analogy, we used the falling dominoes analogy of PMI:

To visualize the idea of mathematical induction, imagine an infinite collection of dominoes positioned one behind the other in such a way that if any given domino falls backward, it makes the one behind it fall backward also. Then imagine that the first domino falls backward. What happens? ... They all fall down! (Epp, 2019, p. 277)



Figure 1: Falling dominoes analogy of PMI

As can be seen in Figure 1, the first domino in the line falls backward. Figure 1 replaces the figure from Epp’s textbook (2019, p. 277) where the dominoes are labelled 1, 2, 3, ..., k , $k + 1$, the dominoes labelled 1, 2, 3 are standing, and the domino labelled k falls backward. We found the replacement necessary to match the sentence “imagine that *the first* domino falls backward.” The following is a student description of the falling dominoes analogy:

If an infinite amount of dominoes are up in a line, if I push the first one over, it will knock down the next one, and so forth. This implies $P(1)$ is true (the first domino is pushed), and therefore if $P(k)$ is true (the domino k will fall), then $P(k + 1)$ is true (the next domino will also fall), thus indicating that all the dominoes will fall.

According to English (1993), from the student’s point of view, *effective* analogies are those in which the student has clarity with respect to the source structure, can make the mapping from source to target, and can select for mapping only the relations forming a coherent structure. The student *sees* the dominoes being pushed and falling as being similar to the statements $P(k)$ from PMI, there is a first element (first domino falling corresponding to $P(1)$ in PMI), there is a logical implication, although it is mapped from the target to the source (“if $P(k)$ is true then $P(k+1)$ is true” is mapped to “the domino k will fall, the next domino will also fall”), and a concluding statement (“thus ... all the dominoes will fall”) without any stated correspondence in PMI, the target structure. The student has clarity with respect to the source situation, and, possibly, with respect to the target situation, but

the relations are not always mapped from source to target. Those relations that are stated explicitly, should we consider all of them mapped from source to target, create a coherent structure.

We propose another way to account for *viable* analogies, analogies that are *good* representations of mathematical ideas, and can be effective analogies (English, 1993). This way offers a clear view of the mapping from the source to the target situation. Figure 2 represents a possible mapping of the relational structure from the source situation, the falling dominoes, to the relational structure from the target situation, PMI. The mapping from Figure 2 was created based on the way the first author approached the falling dominoes analogy with her students, during lesson.

Source (the falling dominoes situation)	Target (PMI)
The 1 st domino in line	$n_0 = 1$
The n^{th} domino in line	n
The k^{th} domino in line	$k \geq n_0$
The n^{th} domino falls backward.	$P(n)$ statement that depends on n
The 1 st domino falls backward.	$P(n_0)$
The k^{th} domino falls backward.	$P(k)$
The $(k + 1)^{\text{st}}$ domino falls backward.	$P(k + 1)$
If “the k^{th} domino falls backward” is true then “the $(k + 1)^{\text{st}}$ domino falls backward” is true.	If $P(k)$ is true then $P(k + 1)$ is true.
All the dominoes, starting with the 1 st one, fall backward.	$P(n)$ is true for all natural numbers $n \geq n_0$.

Figure 2: Example of structural alignment of the analogy with falling dominoes

In Figure 2, the objects from the source situation are concrete objects, like dominoes. These concrete objects, dominoes, are associated with mathematical objects from PMI, natural numbers. The similarity between concrete objects and mathematical objects is given by the mapping between the set of dominoes and the set of natural numbers. The mapping focuses on the relations between the concrete objects that are similar to the relations between the mathematical objects. There is a relation of order of the dominoes that can be mapped with the relation of order on the set of natural numbers $1, 2, 3, \dots, k, k + 1, \dots$, there is a first domino in line, and the idea of successor of the k^{th} domino being the $(k + 1)^{\text{st}}$ domino in line, and the alignment proposed maps the ranks of the dominoes in line with instances of the variables from PMI, even for the variable n that refers to *all* natural numbers. Also, by the nature of our physical limitations, the line of dominoes depicted in Figure 1 is finite, whereas the set of natural numbers is infinite. To show that the proposed structural alignment is viable, we check the psychological constraints that our alignment must satisfy (Gentner & Markman, 1997):

- I. *consistency* expressed as *parallel connectivity* (matching relations have matching arguments) and *one-to-one correspondence* (each object/relation from the source situation is matched with one and only one object/relation in the target situation);

- II. *relational focus* (the focus is on matching similar relations rather than similar objects);
- III. *systematicity* (matching connected systems of relations rather than disconnected relations).

The structural alignment proposed in Figure 2 has *consistency*. Indeed, the statement “falls backward” matches the statement P . The arguments of the statement “falls backward” are the ranks of the dominoes in the line, and are matched with arguments of the statement P , representing corresponding values of natural numbers. Hence, we have *parallel connectivity*. It can be easily seen from Figure 2 that we have matched each statement/argument from the source situation with one and only one statement/argument from the target situation. Therefore, the *one-to-one correspondence* is satisfied. The alignment has *relational focus*, since same types of statements are matched: the simple statement “falls backward” matches P ; the compound statement “If the k^{th} domino falls backward is true then the $(k+1)^{\text{st}}$ domino falls backward is true” matches the compound statement “If $P(k)$ is true then $P(k+1)$ is true”; and the statement “All the dominoes, starting with the 1st one, fall backward” matches the quantified statement “ $P(n)$ is true for all natural numbers $n \geq n_0$.” The relations (statements) and their types are important, and it does not matter that we have the representation of a line of dominoes instead of the set of natural numbers. Finally, the alignment has *systematicity*, and this can be observed from the types of statements that are matched and the way they are connected, the network of relations, from the basis statement referring to the first domino, to the implication statement referring to the k^{th} domino and its successor, to the conclusion statement referring to all the dominoes in the line, and their matching statements in the target situation.

Methods

The study reported here is part of a study that lasted three semesters. The first author taught the lessons. Participants were undergraduate students, STEM majors, enrolled in a face-to-face Discrete Mathematics course at a four-year university in the United States (21 participants in the first semester, 37 participants in the second semester, 59 participants in the third semester). The objective of the study was to design and implement tasks with analogies that have the potential to give meaning to PMI. During the first semester, we stated PMI and discussed in class the analogy with the falling dominoes. During the second semester, after teaching the lesson when we stated PMI and used the falling dominoes analogy and introducing the proofs by induction, we assessed our students’ knowledge in a quiz with the following task:

- a) State PMI;
- b) Explain whether the following is a good analogy for PMI: An immortal fairy princess enters a castle with an infinite set of rooms numbered 1, 2, 3, 4, She has in her hand the key to room 1, and each room has the key to the next numbered room. Therefore, she can enter every room in the castle.

None of the 37 students correctly stated PMI, and only 8 students considered the fairy princess and the castle analogy to be a good one. Instead of stating PMI, the students referred to its use and the steps of a proof by induction. The main reason for rejecting the analogy as being a good one was the infinity concept (e.g., “it is impossible to reach infinity” or “there is no significance of the princess being immortal”). Other rejection reasons referred to the analogy as a problem to be solved or a statement to be proved (e.g., “the number of rooms is missing” or “the statement needs to be proven”)

or showed a misunderstanding of the situation (“she can enter any room so that means she doesn’t need a key”).

In this paper, given the space constraints, we focus on analogy tasks from the third semester.

Previous research studies on using analogies in mathematics education indicate that mathematics teachers use analogies while teaching but they fail to assess their students’ understanding of those analogies, specifically the structural alignment between the source and the target (Richland et al., 2004). Informed by previous research, we designed the following homework task, Task3 (see Figure 3, below), using cognitive support (Richland et al., 2007) and allowing the students to create their own analogies (Vamvakoussi, 2019):

Given a statement, $P(n)$ where n is a natural number, PMI can be stated this way:

- i. If $P(1)$ is true; and
 - ii. for any natural number $k \geq 1$, if $P(k)$ is true, then $P(k + 1)$ is true;
- then, the statement $P(n)$ is true for all natural numbers $n \geq 1$.
- a) The following is an example of analogy for PMI: An immortal fairy princess enters a castle with an infinite set of rooms numbered 1, 2, 3, 4, She has in her hand the key to room 1, and each room has the key to the next numbered room. Therefore, she can enter every room in the castle.

Complete the empty cells in the following table:

PMI features	Corresponding features of the example
n natural number	
	“room 1”
	“rooms numbered 1, 2, 3, ...”
	“the next numbered room”
$P(n)$ statement that depends on n	
$P(1)$	
$P(k)$	
$P(k + 1)$	
If $P(k)$ is true then $P(k + 1)$ is true	

- b) Give another example of an analogy for PMI. Justify that your example is a good analogy.

Figure 3: Description of Task3

The students submitted their homework online via Blackboard/Discussion Forum, and they could read and comment on their colleagues’ answers. The scoring rubric for assessing the students’ answers took into consideration the psychological constraints described in the section referring to the theoretical background (Gentner & Markman, 1997): I. *consistency* (4 points for *parallel connectivity* and 2 points for *one-to-one correspondence*); II. *relational focus* (3 points); III. *systematicity* (1 point). For the category *parallel connectivity*, we created a 4-level scoring rubric (none, some, most, all) and assigned 0, 1, 2, 4 points respectively. For the category *one-to-one correspondence*, we created a 2-level rubric (no, yes) and assigned 0 or 2 points, respectively. For the category *relational focus*, we considered whether the focus was on matching similar relations (1 point), the relations were

appropriately described (1 point), and *all* relations were accounted for (1 point). For the category *systematicity*, we created a 2-level rubric (no, yes) and assigned 0 or 1 point, respectively. Students' answers were scored by two raters, and the inter-rater percentage of agreement was 95% (Cohen, 1960).

Analysis and results

Table 1 presents the mean scores for Task3 (parts *a* and *b*), described above in Figure 3.

Table 1: Score means for Task3

Task3	Consistency		Relational focus (3 points)	Systematicity (1 point)	Total (10 points)
	Parallel connectivity (4 points)	One-to-one correspondence (2 points)			
Part <i>a</i>	2.4	0.9	1.9	1	6.2
Part <i>b</i>	2.5	1.7	1.4	0.9	6.7

As can be seen in Table 1, the mean scores for *parallel connectivity* for both parts *a* and *b* are very close, 2.4 and 2.5, respectively (on a scale from 0 to 4). For part *a*, most of the students' answers were scored with only 2 points, because they did not properly account for the implication statement "if $P(k)$ is true then $P(k + 1)$ is true". They mapped the implication statement with conclusion statements referring to the keys for the rooms (e.g., "there is a key for all the rooms") or the princess who opens the rooms (e.g., "she can continue forever"). For part *b*, although the situations proposed had the potential to be viable analogies, the basis statement corresponding to $P(1)$ and the conclusion statement " $P(n)$ is true for all natural numbers $n \geq 1$ " were absent, and the implication statement was expressed as a possibility, like in the following student-created analogy:

In Pokémon, in order to advance in the game, you need to fight the gym leaders. Each gym leader gives you a badge after winning and allows you to continue the story and challenge the other 8 gyms in the game.

The mean scores for *one-to-one correspondence* for Task3 part *a* and *b*, are 0.9 and 1.7 (on a scale from 0 to 2), respectively. For part *a*, the *one-to-one correspondence* was not observed in some of the students' answers where the statement $P(n)$ was mapped to "rooms numbered 1, 2, 3, ..." and at the same time the statement $P(k)$ was mapped to "keys to rooms". For part *b*, the *one-to-one correspondence* was not violated, but some students did not answer, or they did not provide enough detail. The mean scores for *relational focus* for Task3 part *a* and *b*, are 1.9 and 1.4, respectively (on a scale from 0 to 3). For part *a*, the implication statement was not properly described in some of the students' answers (e.g., "each room has the key to the next numbered room"). In addition, for part *b*, few students' answers focused on objects and their properties instead of the relational network, for example on the 'infinite' nature of the natural numbers:

In the world of high-performance cars, there are close to infinite amount of parts to combine with in order to find the best combination [...] Each year, people manufacture better and better parts. Therefore, you are infinitely trying to find the best combination of parts [...].

The mean scores for *systematicity* for Task3 part *a* and *b*, are 1 and 0.9, respectively (on a scale from 0 to 1). Students' answers contained connected systems of relations.

Discussion and conclusion

As seen in Table 1, the use of analogies of PMI has the potential to give meaning to PMI (total mean scores of 6.2 and 6.7, on a scale from 0 to 10). The total mean score of 6.7 for Task3 part *b* (student-created analogy) is greater than the total mean score 6.2 for part *a* (structural alignment for a given analogy). A possible explanation may be that the symbolic representations and their meanings posed difficulties for students in part *a*, whereas in part *b* the context of the situation was familiar, based on students' own experiences. Also, in part *a*, in the princess analogy, the basis and implication statement refer to having access to the key room, while the conclusion statement refers to being able to enter each room (given that she has the key to each room). The confusion between the implication and conclusion statement in Task3 part *a* and the absence of the conclusion statement from part *b*, may be related to the fact that, although we state PMI at the beginning of Task3, the conclusion statement does not appear in the structural alignment table from Task3 part *a* (see *PMI features* column, in the table from Figure 3). The analysis of the students' answers on Task3 revealed that the student-created analogies were approached holistically, and not by parts. The holistic approach may be a key factor for the *systematicity* observed.

With respect to student-created analogies, the difficulties of explicitly stating the basis statement, the implication statement, and the conclusion statement, mirrored the students' difficulties with proofs by mathematical induction (Gonzalez, 2020). It may be that our way of representing the mapping in Figure 2 did not lead to students' understanding of the way a mapping needs to be described, or that the underlining structure of the source or target situation in Task3 was not clear for our students. More research is needed with refined analogies tasks. In the future, we will continue using analogies tasks with cognitive support (Richland et al., 2007) and allowing the students to create their own analogies (Vamvakoussi, 2019). We will refine the design and implementation of Task3 by: i) making visible (pointing out, highlighting) the structure of the source; ii) focusing on the symbolization aspect of PMI; iii) clarifying the student understanding of the variable k from PMI (one can imagine a particular natural number k , $k \geq n_0$, or all the natural numbers between n_0 and k); iv) emphasizing the logical implication in the statement "if $P(k)$ is true then $P(k + 1)$ is true" (writing it as an implication $p \rightarrow q$); and v) adding the conclusion statement " $P(n)$ is true for all natural numbers $n \geq n_0$ " to the structural alignment (in *PMI features* column from Task3).

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