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# Constructing a deductive argument when solving word problems

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*In earlier work the authors proposed to modify the process of solving word problems so that in addition to searching for an answer one needs to prove the validity of the symbolic representation of the situation. This presentation evaluates the results of an experiment in which university students performed the same modified word problem by discussing the tasks among themselves in small groups. A general inductive approach is used to analyse their discursive activity.*

*Keywords: Word problems, magnitudes, carrying a deduction, small group discussion.*

## Introduction

This is a follow-up development of a tool to teach and learn mathematical reasoning in school mathematics by solving word problems. Kilienė and Norvaiša (2022a, 2022b) suggested proving the validity of a symbolic representation of a situation described by a word problem. The whole task is planned to be done in two steps. First, a problem solver searches for, conjectures and justifies a symbolic representation which may be some equation. Second, he is requested to prove the validity of the justification by constructing and validating a deductive argument. So far this was analysed from a mathematical perspective.

In this paper we extend our theoretical research on the construction of a valid deduction for empirical consideration. For this aim we used the small group discussion methods. At the same time we continued to use the analytic framework describing the meaning of reasoning-and-proving suggested by Stylianides (2008). Recall that it looks at the activity of proving from three different perspectives, which are called components: the mathematical, the learner's (psychological), and the pedagogical (teaching) perspectives. The mathematical perspective of the analytic framework was already studied in Kilienė and Norvaiša (2022a, 2022b). Here we study the solution of a word problem with reasoned judgement from the learner's and the pedagogue's perspectives.

A distinctive feature of the tool for developing mathematical reasoning skills tested in this paper is the use of contextual knowledge. The logic that organizes mathematical discourse is treated in two aspects: syntactic and semantic. The semantic aspect in our case consists of the use of knowledge about continuous magnitudes in proving the validity of a symbolic representation of the word problem. We share the view that teaching formal predicate logic in school, without providing students with a perceptible context for mathematical reasoning, is not an appropriate approach (Durand-Guerrier et al., 2012). Research on the possibilities of teaching and learning mathematical proof in the context of university mathematics was carried out by Selden and Selden (2015). Related work have been done by Miyazaki et al. (2017), Koichu et al. (2021) and Marco et al. (2022).

## Modified algebraic word problems

We consider traditional word problems as a tool for learning the abstract concepts of school mathematics rather than as a tool for relating mathematics to the real world. One of them is the concept of a magnitude. Magnitudes usually appear in a *contextual representation* of a word problem.

They are used to express relationships among entities within a context of fictional situations. A verbal description of a problem in which one or more questions are raised, the answers to which can be obtained by establishing and solving an equation with respect to a numerical value of unknown magnitude, we call an *algebraic word problem* (AWP). An equation with respect to a numerical value of unknown magnitude is an example of a *symbolic representation*. By solving a word problem with a reasoned judgement we mean a proof that the symbolic representation obtained from the contextual representation is valid.

The collection of magnitudes of the same kind is a set of objects with an addition operation and a scalar multiplication such that the axioms on a certain list hold (Michell & Ernst, 1996). We recall an axiom of addition of line segments. Let  $AB$  and  $BC$  be two end-to-end concatenated segments on a line. Suppose that  $|AB|$  and  $|BC|$  are the numerical values measured with respect a unit length, say, a kilometre. Then the numerical value  $|AC|$  of the segment  $AC$  is given by the equality

$$|AC| = |AB| + |BC|. \quad (1)$$

Length and time used in this paper are examples of fundamental magnitudes. It is possible to generate such magnitudes by some sort of concatenation of a sequence of equal units because they do not involve the measurement of any other magnitude. Other magnitudes exist that cannot be determined except through their relationships to other, fundamentally measurable, magnitudes. These are called derived magnitudes, and speed is a typical example.

The primary goal of our suggestion is to teach and learn mathematical reasoning in school mathematics. With this aim, in addition to the usual question in an AWP, we have two tasks. The first of these is to justify a symbolic representation that answers the question. A justification means a set of arguments used to give reasons why a symbolic representation is justified. The second task is to prove the correctness of this representation by constructing a valid deductive argument. We use the symbol AWP+2T to denote an AWP together with these added tasks. This study examines the results of an experiment in which the following AWP+2T was performed.

**Word problem.** Two buses left the city at the same time to summer camp, located 72 km away. The average speed of the first bus is 4 km/h higher than the average speed of the second. The first bus arrived at the summer camp a quarter of an hour before the second.

1. What was the average speed of each bus?
2. Justify a symbolic representation of the problem situation.
3. Construct a valid deduction proving that the symbolic representation is correct.

## **Methods and theoretical framework**

The conceptual structure of the present study is formed by two themes, namely, word problem solving as a discursive activity, and constructing a valid deduction as a learning activity. For the second theme we continue to use the *analytic framework* described in Stylianides (2008). As for the first theme we use a *general inductive approach* based on coding and categorization of raw data (Thomas, 2006).

This study aims to understand how students construct a valid deduction of a symbolic representation and how it is possible to develop the mathematical proving ability. We attended to the specific content

of the arguments, what knowledge of mathematics and logic is used in the proposed task, what topics receive the most attention in the discussions and what mistakes are made by students. Our study pursues three **research questions**:

1. What resources of mathematics and logic do students use, and how, in small group discussions when constructing and validating a line of deductive reasoning supporting a symbolic representation of an AWP?
2. What forms of communication and topics of debate do students invoke, in the small group discussions, when solving AWP+2T tasks?
3. What possible discrepancies are there between the nature of a valid deduction implying a symbolic representation and its understanding by the students, and what are the teachers' possible pedagogical responses?

The third research question is formulated in order to answer the learning and pedagogical components of the analytic framework (Stylianides, 2008).

Solving a word problem with a reasoned judgement requires a theoretical knowledge. Usually, the symbolic representation of an AWP is based on relationships between magnitudes describing the problem situation. For example, comparing several ways of expressing the length of a segment.



**Figure 1: Graphic representation**

Suppose that the segment  $AC$  is measured in two ways: parts and whole. According to the axiom of addition (1), we have the implication: if  $|AB| = s_1$ ,  $|BC| = s_2$ , and  $|AC| = s$ , then  $s = s_1 + s_2$  (Figure 1). This is a general statement that together with numerical values of magnitudes from the problem situation, hereinafter called facts, can be used to infer the AWP symbolic representation. Another general statement used to infer the AWP symbolic representation is the average speed  $V$  formula: if the total distance travelled is  $S$  and the total time taken is  $T$ , then  $V = S/T$ . For short, we refer to this theoretical knowledge as arithmetic of magnitudes.

### **Participants, data collection and data analysis**

A class of 26 first year mathematics students in university took part in the study. Both authors of this presentation were the instructors of the class. The class studied the bridge course between school and university. It included the elements of predicate logic. The Lithuanian school mathematics curriculum provides superficial knowledge of the arithmetic of magnitudes and predicate logic. They are not enough to solve the modified word problem. The students who participated in the experiment acquired the necessary knowledge in less than two months at the beginning of their university studies. Also, they performed two worked examples of the AWP+2T before the experiment.

The students were working on the AWP+2T in seven small groups. Groups consisted of three to five students. The small-group discussions were audio-taped and fully transcribed (overall, about 16,935 words). The students' written work and the answers to a questionnaire served as complementary data sources.

Categorization was guided by the research questions, which provided a focus for conducting the data analysis. We used two kinds of category. The second kind of category analyses collaborative argumentation using conceptual tools developed in prior research discussed by Koichu et al. (2021). The first kind of category is intended to describe the resources of knowledge obtained as inferences from the students' utterances which can explain what the students might have held as true while constructing their arguments. We divided these categories according to the type of knowledge, into resources of logic and resources of magnitudes.

**Table 1: A partial list of resource codes and categories with illustrations from transcripts**

| Code (category)                               | Quotations from transcripts that was coded into respective category   |
|---|---|
| L (resources of logic)                        | A student refers to rules of logic in order to explain their need   |
| LI (need of implication)                      | “It seems to me that it is necessary to write, if such a formula, then ... with our values”   |
| LIF (implication or fact)                     | “But I suggest to write it down as a fact”  |
| LNF (nature of facts)                         | “But this is from the situation description”  |
| LPS (proof search)                            | “You understand that we need an implication with the fact as the consequent”  |
| LSC (speed comparison)                        | “If the first bus travelled at speed $x$ , the second one travelled at $x+4$ ”  |
| LTC (time comparison)                         | “If the driving time of the first bus was $t_1$ and the driving time of the second bus was $t_2$ then $t_1$ minus $t_2$ is equal to a quarter of an hour“ |
| LIT (inferring time)                          | “Take $P_1, P_2, MP$ and get that $t$ is equal to $72$ divided by $v$ ”   |
| LAP (arrangement and formulation of premises) | “Wait a minute, we will have four premises?”  |
| L $\vdash$ (deduction argument)               | “Then we need to write these $P_1, P_2$ , [premises] this sign $\vdash$ “   |
| LMP (applying MP)                             | “Because MP consists of two statements”   |
| LV (validation)                               | “And then according to $Q_1$ and $Q_2$ [inferences] we will get the equation”   |
| M (resources of magnitudes)                   | A student refers to magnitudes in order to explain their need   |
| MSC (comparing speed)                         | “It is written that the speed of the first bus is $4$ km/h higher than the speed of the second one”   |
| MSR (constructing symbolic representation)    | “Combining the equations into one system, we have common unknowns, we get a system”   |
| ME (equation)                                 | “And this is already an equation“   |
| MSF (speed formula)                           | “Since $s = vt$ then $v = s$ divided by $t$ ”   |
| MAA (axiom of addition)                       | “It's like with that measurement, that adding those together, we need to write one more premise”  |

The discussions of all groups took place while performing each task of AWP+2T sequentially. Therefore we grouped the categories into themes according to the appearance of the discussion segments they denote, as follows:

1. Derivation of symbolic representation of the problem situation and finding an answer to the problem (tasks 1 and 2 of AWP+2T).
2. Finding premises of a deduction proving the correctness of the symbolic representation (the first part of task 3 of AWP+2T).
3. Logical inference of the conclusion of the deduction from premises (the second part of task 3 of AWP+2T).

## **Results and findings**

### **Mathematical and logical resources**

In this subsection we address research question 1. In order to successfully complete the proposed task students need specific knowledge from two areas. The first area should include the arithmetic of magnitudes. The second area of knowledge is predicate logic. An acquaintance with implication and a rule of logical inference Modus Ponens is required. Both areas of knowledge are needed to formulate general statements, together with the facts of the problem situation, used as premises to infer the symbolic representation of AWP+2T. In this problem, the symbolic representation can be an equation or a system of equations. Solving them provides an answer to the question of AWP+2T.

The general statement concerning the speed formula was included in the list of premises made by three groups. No general statements from the arithmetic of magnitudes were used as premises by all groups. However, the necessity of such a statement was discussed by the students of Group 5. Students had very little opportunity to become familiar with the arithmetic of magnitudes since this is not a part of the bridge course. However, the students of all groups, except one, demonstrated sufficient knowledge of implications and valid deductions.

A relationship between magnitudes used to prove the symbolic representation of AWP+2T we call the chosen mathematical structure. The same AWP+2T can have different symbolic representations. Six different symbolic representations were proposed by seven groups of students in the present experiment. Each of them is based on a chosen mathematical structure. Such choice requires some understanding of the problem situation and determines a way of solving the AWP. We hypothesize that knowing the dependence of a symbolic representation on the choice of mathematical structure can help in solving new word problems.

### **Dialogical moves and topics of debate**

In this subsection we answer research question 2. The most popular form of communication at the beginning of the experiment was individual work by each student. Apparently, it is a natural way to get an acquaintance with the word problem and to design a preliminary strategy to find an answer to the first question of the AWP+2T. Most students completed the first task successfully, which was to find the speed of each bus. It's nevertheless true that the third group of students had problems with finding the answer. About 15 minutes after the beginning they said: "How to solve it? I don't know,

I can't. I can't think of it." The second and third tasks were completed by discussing with each other within each group.

The second task asks for the justification of the AWP symbolic representation explaining the derivation. Students knowing that the third task will require applying the rules of logical deduction, began to use elements of logical reasoning when working on the second task already. For example, in the group 2 discussion, student A at the beginning of the group discussion said (No. 3): "I immediately saw that they should be done immediately with those implications." Developing the habit of argumentation is one of the main objectives when solving an AWP+2T.

Seven different groups of students chose six different symbolic representations of the AWP+2T connecting numerical values of three magnitudes: distance, time and speed. In most cases, the choice of a symbolic representation was made at the beginning of joint discussions with somewhat surprising solidarity, except for a few minor cases, where the possibility of an alternative choice was mentioned during latter discussions. We explain this by uncertainty among students that arises when choosing between possible symbolic representations whose validation difficulty is unknown. In the middle of the discussion, a student snapped "How long have we been solving this fifth grade task?"

The most debated topic within the groups seems to have been the logical form of the premises: fact or implication? The use of implication was considered in the formulation of general statements in two cases. First, an implication was used to formulate a variant of the speed formula. For example, "if the bus travels a distance of 72 km in  $t$  hours, then its average speed  $v$  is equal to  $72/t$  km/h". Second, the implication was used to say that the numerical value of a magnitude is in arithmetic relation with another one of the same kind. For example, "if the speed of the second bus is  $x$ , then the speed of the first bus is  $x+4$ ". The necessity of both implications was often debated by the students. Those who proposed using the fact form instead, claimed that the problem situation refers to both speeds directly and therefore they should be treated as facts. These discussions ignore the fact that the problem situation specifies the numerical value of the difference between two magnitudes.

A similar discussion took place regarding the formulation of the duration of travel of each of the buses: fact vs implication. In other words, it was suggested to use the form of premise: "if the travel time of the second bus  $t_2 = t$ , then the travel time of the first bus  $t_1 = t - 0.25$ ", although the problem situation refers to the difference between driving times. For example, in Group 6 discussion, the implications were replaced by facts during the last stage of proof, when applying the inference rule Modus Ponens. This was preceded by a discussion about the difference between implication and fact.

In the experiment, there were groups of students in which the opinion of one or more people dominated and decisions were made by the majority. For example, as many as 12 monologues, whose written speech occupied more than three lines, took place in Group 2. There were also student groups where all members of the group spoke often but laconically. For example, the longest utterances in Group 6 occupied three lines.

### **Misunderstandings and response**

In this subsection, we address research question 3. Most students seem to have understood the meaning of valid deduction when they tried to infer logically the truth of a conclusion (a system of

equations) from the truth of the premises. The same cannot be said about the results of the 3<sup>rd</sup> group of students. Finding the answer by solving the equation was their work outcome in response to the third task of AWP+2T.

Facts about the difference in speeds and driving times of the two buses in the problem situation were restated in a new way without justification by some students. The sentence “the average speed of the first bus is 4 km/h higher than the average speed of the second” was replaced by a sentence saying that the average speed of the first bus is equal to the average speed of the second bus plus 4 km/h. A similar substitution regarding driving time was made by some students too. Given any two magnitudes  $a$  and  $b$ , if  $a < b$ , then there exists a magnitude  $c$  such that  $a + c = b$ . If those magnitudes are line segments with  $a = AB$ ,  $b = AC$  and  $c = BC$ , then relation (1) holds true for their numerical values. So, we interpret the substitution of one fact for another fact about the speed and driving time of the buses as the application of the stated axiom of magnitudes without explicit reference to premises used for a deduction. As mentioned earlier, some students made attempts to include the reference to the axiom of magnitudes into the list of premises during discussions in a group. These attempts were coded as AA meaning the category “axiom of addition”.

We recommend that instructors and teachers reflect on the obtained results with students and draw their attention to several points. First, the features of formulating general statements by implication should be discussed because this aspect of knowledge seem to be most difficult to students. Second, it is worthwhile explaining how the solution of the task depends on the choice of a mathematical structure used to infer logically the system of equations. We hypothesize that this understanding may be helpful when solving new word problems.

Many students spoke in very short phrases. Their meaning is hardly possible to understand without knowing the context and their own notes. General recommendation – it is important to develop the habit of formulating ideas about mathematics in such a way that the interlocutor could understand.

## **Discussion**

Reflecting on the data analysis we noticed that the process of validating a deduction has some characteristics of a search for a mathematical proof. For example, the search for a suitable rule of logical inference forced students to constantly return to the beginning of their reasoning, to alternatives to the decisions already made, to discuss methods of argumentation. This shows that the integration of predicate logic into the content of school mathematics provides an opportunity to better master logic and get a feel for the meaning of proofs.

After the experiment all participants were asked to fill out a questionnaire. It contained the question: Do you support the use of proof by deduction in school learning, with proper preparation beforehand? The most surprising thing was that 21 students out of 26 answered yes to this question. We speculate that the reason for this could be a sense of the meaning of predicate logic, that arose from doing the modified word problem.

Due to the lack of space, we have shortened the presentation of the results. We plan to prepare a framework for constructing proofs when solving word problems invoking a Grounded Theory



approach. For this purpose, new experiments with other students and teachers are needed, as well as modified word problems of various difficulty to be solved.

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