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Combinatorial reasoning in problem solving strategies: a pilot study with pre-service teachers

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This work is part of a wider investigation aiming to explore and describe the process of teaching combinatorics in Italian secondary schools both from the point of view of teachers' knowledge, practices and relations with the spontaneous strategies that students develop during combinatorial problem solving. In this work we will focus on a pilot study conducted with a group of pre-service teachers - as part of the whole research process - whose solving strategies and combinatorial reasoning were analysed under the lens of two different analytic tools. We present the results of this analysis together with a reflection on the two constructs used aiming to develop an instrument of analysis that would take into account both static and dynamical aspects of combinatorial problem solving.

Keywords: Combinatorial thinking, combinatorial strategies, pre-service teachers.

Introduction

Secondary school students and university students usually struggle with problem solving in combinatorics (Batanero et al, 1997; Godino et al., 2005). The difficulties are not due to the mathematical background needed to handle the computations itself, considering that combinatorics does not need specific prerequisites (Kapur, 1970), but are usually related to other weakness points like, for example, textual combinatorial variables such as the implicit combinatorial model (Batanero et al, 1997) or combinatorial operations (Fischbein & Gazit, 1988). In a previous study carried out with Italian students, focussing on the analysis of secondary school students' solving strategies in combinatorial problem solving, it was shown that a visible effect of teaching is the shift from strategies exploiting many representations of the set of the configurations and counting procedures toward attempts to directly use a single formula or develop arithmetic procedures (Lamanna et al., 2022). Whereas the global performance of the students who had attended a course was better (the number of correct answers was higher), the richness of the strategies decreased in a significant way after instruction and the behaviour turned from exploration of different representations to a mostly binary strategy leading to the identification and application of a formula or to missing answers. Indeed, while in secondary school teachers follow the tendency of classifying problems by categories associated with formulas (combinations, permutations, etc - this data derives from an ongoing research where several audio-recorded focus groups about the teaching of combinatorics have been conducted with a total of 25 secondary school teachers), as it happens in mathematical problem solving at large, many problems cannot be addressed by a direct use of a formula, or do not need a specific formula to be solved - for example in the case of combinatorial compound problems. While students could perform well in tasks "tailored" around the standard situation modelled by a formula

taught at school, this attitude could reveal to be a boomerang as far as the problem gets far from this standard. This phenomenon, well known in mathematics education, becomes even more relevant in a mathematical domain that secondary school teachers themselves identify with cases addressable through formulas. Since the use of different representations and the flexibility in the construction of examples and modelling of the context is crucial in the development of combinatorial problem solving skills as long as the problem is not completely addressable using a formula (Batanero et al., 1994), we decided to investigate the variable “teaching”, in order to figure out possible causes of the phenomenon of the shift towards the approach based on formulas induced by secondary school teaching of combinatorics itself.

In this paper we present a part of the whole research project, focusing on the way pre-service teachers (PTs) solve combinatorial problems in terms of strategies involved (Lamanna et al., 2022) and combinatorial reasoning (Lockwood, 2013). This choice was made in order to study solving strategies in combinatorial problem solving by university students that are not yet influenced by teaching practice and school institutional habits. The group we involved in the study was a set of university students attending a master’s degree in mathematics and training to become secondary school teachers. We based our work on previous studies aiming to assess combinatorial reasoning of PTs (Hubeňáková & Semanišínová, 2020) as a bridge toward the analysis of their specialized knowledge (MTSK - Carrillo-Yañez et al., 2018) in combinatorics. The authors showed, through the use of multiple solution tasks, that the knowledge to teach the topic of PTs in their sample is inadequate. They also offer an instrument to explicitly bridge combinatorial reasonings and mathematical knowledge, using Lockwood’s frame to model PTs’ reasonings and generate indicators for the MTSK. For this reason, we chose to rely on the same model, in order to understand solving processes and gain a lens of analysis that allows us to connect a teacher’s resolution to dimensions of knowledge. In this work we aim to investigate the relations between strategies and reasoning, in particular the different thinking processes that are developed and that lead to certain strategies. In doing so one of the aims is to investigate whether it would be possible to perform an analysis that contemporaneously embraces both these aspects of a combinatorial solution.

Theoretical framework

Analysis of solving strategies

For the analysis of PTs’ solving strategies, we based on the classification summarised in Lamanna et al. (2022), where the following strategies were considered and described: a) *Enumeration*, found in previous research (Godino et al., 2005; English, 2005) and consisting in the explicit listing of all the possible combinations to be formed, according to the problem statement. b) *Tree- diagram*: a tree diagram is built as a help in producing all the configurations. c) *Formula*: a combinatorial operation is recognized as a solution to the problem. d) *Reference to another problem*: the solution of a problem is referred to the one of an equivalent one. e) *Sub-problem decomposition*: the original problem is divided into several combination problems of smaller dimension and the resolutions of which are combined to get the solution to the initial problem. f) *Sum, product or quotient rules*: elementary arithmetic rules are used to determine a solution without an explicit reference to a formula. g) *Other strategies*, generally giving a wrong answer with no justification. Such solving

strategies were developed by a group of university students with higher preparation in mathematics and whose strategies' distribution pattern was shown to be similar to the one observed in groups of secondary school students, who received instruction in combinatorics, studied in Navarro-Pelayo (1994) and Lamanna et al. (2022) in whose work the authors observe different use of the described strategies in particular, where multiple strategies appear, they observe a hierarchical organisation where a primary strategy is completed by a supporting strategy (for example, a reduction to a sub-problem that is locally solved through enumeration); in this paper we will refer to primary strategies with the term *strategy* and *procedures* all the supporting strategies. With this definition, the above-mentioned strategies could occur in solution either as primary strategies or procedures.

A model for combinatorial thinking

For the analysis of combinatorial reasoning, we based our analysis on the model proposed by Lockwood (2013) which schematizes a combinatorial reasoning through the interaction of three different dimensions (Lockwood, 2013): *a) counting processes*: refers to engaged processes (either mental or written) developed solving a counting problem. This dimension includes, for example, various reasoning, enumerative processes or implementation of different case breakdowns. *b) Formulas/expressions*: refers to mathematical expressions yielding to a numerical value (i.e. formula with combinatorial meaning or combination of numerical operations). *c) Set of outcomes*: refers to the collection of objects to be counted. This dimension does not only refer to the set of all the solving configurations but also to any set or subset that can be associated with a certain counting problem.

The two proposed tools of analysis provide two different insights on the same process of resolution; the classification of strategies offers a rather static image of a solution that, possibly, intertwines strategies and procedures in a more dynamic way. On the other side, Lockwood's model offers a more dynamical instrument to inspect a solution, also embracing some of the dynamical aspects of the combinatorial thinking consisting in a shift from one dimension to another during the resolution.

Research problem and research questions

In this paper we adopt the classification of Lamanna et al. (2022) to analyse the Italian PT's strategies, in order to compare them with the secondary school ones and look for similarities and differences in terms of frequency. Moreover, we analyse them from the point of view of reasoning (through Lockwood's model) and resources involved, in order to understand what dimensions are involved in the different strategies and what resources are more exploited. This will allow us to bridge teachers' combinatorial thinking and the strategies they aim to institutionalize.

We can, then, summarize the aim of this work through the following research questions:

RQ1 – Which are the dimensions of combinatorial thinking activated by a group of PTs during problem solving activities? Which solving strategies and procedures are activated within the same activity?

RQ2 – How can the two tools of analysis be integrated in a single tool to allow a deeper analysis of solutions provided to a combinatorial task?

In our opinion, exploring this all-round process from the point of view of the teacher would allow a better analysis and comprehension of the teaching practices developed within secondary school's combinatorics.

Methodology

To answer our research questions, we conducted an explorative case study with 10 Italian PTs, consisting in university students attending a master's degree in mathematics where they attended also course in statistics and probability but without especially focusing on combinatorics. The students were given an open-handed questionnaire of task analysis in combinatorics and were asked to produce their own answer in a written form. In this work we will focus on one of the items of the questionnaire where students were requested to solve in at least two different ways the following item (here proposed in an English translation):

The garage in Angel's building has five numbered places. As the building is very new, at the moment there are only three residents, Angel, Beatrice and Carmen to park their cars in the garage. This is a plan of the garage: | 1 | 2 | 3 | 4 | 5 |. For example, Angel could park his car in place number 1, Beatrice in place number 2 and Carmen in place number 4. In how many different ways could Angel, Beatrice and Carmen park their cars in the garage?

This item is a simple combinatorial problem (meaning that its solution could be referred to a single application of one of the combinatorial formulas) belonging to the distribution model (Dubois, 1984) in which students are requested to distribute objects (the cars) and whose solution is $D_{5,3} = \frac{5!}{(5-3)!}$ for a total of 60 configurations that allow us to classify this task as high dimensional. Once that students' answers were collected, a qualitative content analysis on all the solutions was performed coding answers with respect to the analytical tools presented in the theoretical framework. We will now provide an explicit example of this process since the core investigation for answering to RQ2 emerged from the process of analysis of the solutions. With greater detail, we started coding PTs' answers looking for both combinatorial thinking processes and solving strategies after which a relation between the two aspects emerged (both from the point of view of coding and elements observed). In Figure 1 we observe one of the students' answers with the relative English translation. Analysing the first statement, the student starts "*supposing that Angela parks in (the parking number) 1*" meaning that he is fixing one of the variables and, by adding the constraint, the task's resolution is reduced to a problem of smaller dimension (where the dimension of the problem is intended as the number of solving configurations, as stated in Lamanna et al., 2022) that is solved by enumerating explicitly all the possible configurations and counting them. At this point the student states that the initial constraint of fixing a variable could be done in 5 different ways ("*[...] has a total of 5 initial choices [...]*") and for each one of them "*there will be 12 possible cases*" as deduced from the resolution of the sub-problem. Finally, the student computes the total number of configurations through the application of the product rule extending the particular solution of the sub-problem to the solution of the starting task. From the point of view of the classification of strategies we can here observe an occurrence of the primary strategy of reducing the task to a sub-problem that relies on two procedures: enumeration and product rule - respectively occurring for the resolution of the sub-problem and the extension of the particular solution to the one of the starting task.

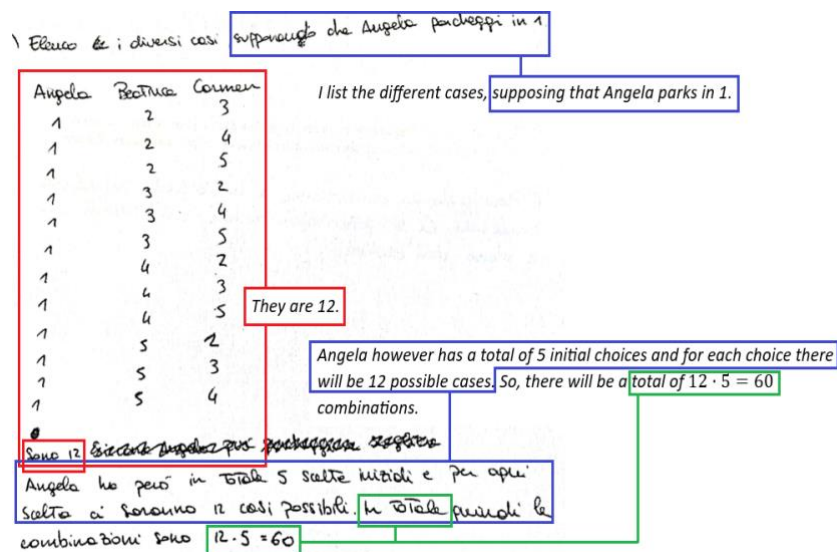


Figure 1: Analysis of one of the collected answers (with English translation)

The classification used provides a hierarchical picture of the occurring strategies and procedures but lacks in underlying the dynamical aspect of the process followed in the resolution. Under the lens of Lockwood’s model, we observe that we did not have the need of changing or introducing a different coding of the solution that, after the first analysis, was the one suggested in Figure 1. We can identify four different moments in the resolution: the operation of pivoting of variables (Godino et al., 2005), the resolution of a sub-problem, a reasoning about the choices made, and a final computation of the original number of configurations. We notice that these moments, previously coded with respect to strategies and procedures, can now be coded following the flux in the combinatorial thinking of the student: at the beginning there is a process of reasoning about the configurations that in Lockwood’s model is included in the dimension of counting (C) and that is followed by enumeration of all the elements belonging to a particular set of outcomes (O): all the configuration starting with the number 1 fixed in the first position, that is a proper subset of the set of all the outcomes. So, the solution moved from dimension C to dimension O of the model. At this point there is another reasoning made in order to extend the solution to the one of the original task, meaning that we are now, again, in dimension C before finishing with an explicit numerical expression, switching toward dimension of formulas and expression (F). We can, then, schematize the student’s resolution with the sequence $C \rightarrow O \rightarrow C \rightarrow F$, highlighting the dynamical thinking process followed in the resolution of the task. At this point one could argue that the second shift ($O \rightarrow C$) is not a continuous process, since in the solution there is not an explicit connection between O and C but, reasoning on the graph of Lockwood’s model, there is more like a ‘jump’ (in graph theory: a shift from two nodes of a graph not connected by an arch so a non-continuous shift). The transition, then, could be furtherly specified differentiating explicit connections from jumps so we opted for the following schematization of the resolution, in which continuous thinking processes are made more explicit: $C \rightarrow O \cdots C \rightarrow F$ with an expression that better embodies the process followed. Considering the final expression obtained we can observe that three processes could be identified: a general one, considering the expression as a whole and two separated ones given by the transitions $C \rightarrow O$ and $C \rightarrow F$. From this consideration naturally emerged a bridging similitude with the first classification made and, in particular, we

identified the general process with the solving strategy (reduction to a sub-problem) and the two processes with the procedures, in particular $C \rightarrow O$ is identified through the enumeration and $C \rightarrow F$ through the product rule, in the sense that each of the transitions corresponds to one of the procedures. In this sense we can state that it is possible to provide a coding that would embrace either the dynamical aspect of the combinatorial thinking and the statical classification of strategies involved – the problem only shifts toward the definition of a proper notation that would embrace all the elements involved. As an example, we propose – without claiming to propose a final and unambiguous notation – the following schematization for the solution we just analysed: $C \downarrow O \cdots C \uparrow F$, meaning that there is a reasoning leading to a reduction of dimension (the switch from C to O means that an explicit enumeration of all the configurations of a subset was proposed) followed by a reasoning and an extension from a the solution of a particular problem to a formula or numerical expression that solves the original problem. It is worth noticing that further work should be done in order to reach an unambiguous notation (for example, the last notation does not differ an extension toward a product rule or a formula).

Results

Regarding the solving strategies observed, whose frequencies are summarized in Table 1, in spite of a reduced sample, we notice a trend that is similar to the one observed both in university students (Godino et al. 2005) and secondary school students (Navarro-Pelayo, 1994; Lamanna et al., 2022), where students tend to rely mostly on formulas and arithmetical rules after receiving an instruction. The observed similarity between the PTs’ strategies in our sample and the ones observed with secondary school students does not come as a surprise; in fact, in case study conducted with university students, Godino et al. (2005) showed that an higher preparation in university mathematics does not allow students to overcome the difficulties already observed at secondary school level and noting how, consequently, they develop similar pattern of strategies during problem solving. The students mostly rely on formulas and product rule – from other data gathered for our main line of research (audio recording of the discussion following the completion of the questionnaire), it emerged that some of the students stated not to remember the correct formula to be used before proceeding in different ways, explaining the occurrence in our data of different strategies and procedures.

Table 1: Frequency of use of different strategies in the group of PTs

(10 students, 21 solutions)	Strategy	Procedure
Enumeration	2	1
Formula	4	2
Reference to other problem	2	0
Sub-problems decomposition	3	0
Product rule	9	2
Quotient rule	1	1

However, this distribution is strongly influenced by the fact that each student was asked to provide at least two different solutions and so had to think about at least a second solution but, nevertheless, this data will be useful in further stages of the research in order to study the development and evolution of students' solving strategies with respect to the strategies developed by their teachers. It is worth noting that some of the students proposed an idea of solution without an explicit development of counting or enumerations; this last aspect highlighted one of the limits of the classification through Lockwood's model since without an explicit reference to methods used it is not possible to reconstruct or deduce the thinking process followed. With respect to the analysis done through Lockwood's model we observe that most of the resolutions end in the dimension of the formulas and expressions (18 out of 21) but the process leading to this conclusion is different. We observe different pattern of evolution in the combinatorial thinking of the PTs in particular, schematizing the different solutions, we can introduce a measure of combinatorial complexity in a solution: complexity 1 – the solution only develop within a single dimension of the model (for example there is a formula only or a list of all the configurations), complexity 2 – the solution develops between two dimensions of the model with just one transition (for example a case breakdown before the use of product rule, in a transition $C \rightarrow F$), complexity 3 – the solution develops between two or more dimensions of the model and/or there is more than one transition. With this classification, we observe that PTs mostly produce solutions of complexity 2 (10 out of 21 – mostly use of the product rule) and complexity 1 (7 out of 21 – complete listing of direct application of a formula).

Discussion and conclusion

In conclusion, in spite of the reduced sample, the analysis provided an insight on strategies developed and paths of combinatorial thinking followed by a group of PTs that will allow us, in later stages of the research, to better understand teaching practices in combinatorics and which dimensions of knowledge are involved (concerning dimension presented in the MTSK model - Carrillo-Yañez et al, 2018). From this point of view, answering to RQ1, PTs' strategies are aligned with ones observed in university and secondary school students with instruction, mostly relying on the use of the formula or arithmetical rules of combinatorics producing, from the point of view of Lockwood's model, solutions of reduced combinatorial complexity and mostly involving the dimension of counting and formulas/expression. In our work the two tools used allowed us to perform a deeper analysis of the gathered solution through a dynamical process embracing both strategies, procedures and combinatorial thinking. In this sense, the joint work of the two tools allowed us to take a first step toward a unified coding that would provide an all-round analysis of the solutions at once, considering that in the study of teaching and learning of combinatorics it is important to focus on developed strategies but also processes. The analysis of strategies was enriched by the encoded new information about the complexity of the combinatorial thinking standing behind the mere definition of "procedure used", highlighting the process that led to the development of the strategy and allowing a further refinement in the study of combinatorial problem solving; with better focus, providing an all-round deeper understanding of solution and also unveiling reasonings and aspects that would remain hidden performing an analysis from a single point of view. However, in spite of the fact that a joint analysis appears to be possible (so, partially answering to RQ2), to reach a networked framework of analysis

it is surely necessary to conduct further and deeper studies and developments, both from empirical and theoretical point of view.

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