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# Pre-service teachers' conceptions and beliefs of reasoning and proof in school mathematics

Jakub Michal

Charles University, Faculty of Education, Prague, Czech Republic; [Jakub.Michal@pedf.cuni.cz](mailto:Jakub.Michal@pedf.cuni.cz)

*This paper describes pre-service teachers' conceptions and beliefs regarding reasoning and proof in teaching at the primary level before an intervention targeting their ability to design teaching mathematics aimed at reasoning and proof. Group interviews were conducted and analysed. It transpired that future lower and upper-secondary mathematics teachers see reasoning and proof as an important part of mathematics education. Yet, they have different understandings of the notion of reasoning. The pre-service teachers also shared some worries connected to teaching reasoning and proof. These concerned mainly time limitations, the need to adjust teaching because of high-stake examinations or difficulty in engaging most pupils in the classroom with what they consider demanding tasks. Results serve to fine-tune the intervention and to observe what changes the intervention caused.*

*Keywords: Reasoning, proof, conceptions, beliefs, pre-service teachers' beliefs*

## Introduction

According to the Czech National Curriculum for primary schools (age 6-14), one of the aims of teaching mathematics is to increase pupils' argumentation competencies and critical reasoning skills. Grammar school National Curriculum (age 15-18) mentions, among other things, that mathematics education should result in the development of argumentation involving hypothesis formation, testing or refuting and the ability to defend one's procedures using valid justifications. As there is no thematic unit dedicated to reasoning and proof (R&P) at primary or grammar school, these skills are to be developed throughout an entire school curriculum and it is up to every individual teacher in what ways and depth R&P is present in his or her classes. It also depends on what exactly teachers understand under R&P, what teaching conceptions they have, their understanding of proof, and its different roles in mathematics and mathematics education (Knuth, 2002; Lesseig, 2016).

There have been attempts to shape these factors through university education. Cihan and Akkoç (2019) emphasize a need for intervention studies aiming to develop pedagogical content knowledge (Shulman, 1987) for teaching proof. One such intervention to enhance pre-service math teachers (PSTs) ability to plan lessons with a focus on R&P skills at primary school was designed by members of an international project MaTeK ([www.projectmatek.eu](http://www.projectmatek.eu)). During four sessions, based on reading relevant literature and aimed at gradually producing more elaborate lesson plans, PSTs learn about the importance of R&P and how to include R&P in their mathematics lessons at a primary school.

Before the intervention, group interviews took place to find out what conceptions and beliefs the PSTs had. The focus was on what they considered to belong under a notion of R&P, what roles of R&P they could see, what approaches to R&P in a classroom they would take and what concerns they had related to teaching R&P at primary school. The goal of this paper is to answer the research question: *What role and importance do R&P in teaching mathematics pupils of age 11-14 have for PSTs in their last year of university studies before the intervention?*

## Theoretical background

There are several dimensions of a teacher's professional knowledge. Shulman (1987) mentions a need for teachers not to only possess subject and pedagogical knowledge, but also what he calls *pedagogical content knowledge*, involving different ways of explanations, visualizations or activities to make new concepts more comprehensible for pupils. He explains this domain as follows: „Amalgam of content and pedagogy that is uniquely the province of teachers”. (Shulman, 1987, p. 8). This framework was adjusted by Lesseig (2016) for teaching proof. She mentions four dimensions a teacher should possess knowledge in: *Common Content Knowledge*, *Specialized Content Knowledge*, *Knowledge of Content and Students* and *Knowledge of Content and Teaching*.

In this paper, Lesseig's (2016) framework is used not only for proof but for R&P. Common Content Knowledge is assumed as the PSTs in our sample already completed the majority of their math classes. The focus is on the other three dimensions unique for teachers, mainly what roles proof has for PSTs in their teaching, what they understand under R&P, what their classroom argumentation would rely on and what they would accept as a valid argument. These understandings of R&P by a teacher or a community is what Harel and Sowder (2007) call *proof schemes*, which among other things involve social aspects of proving, where one needs to convince himself or herself and others. A similar proof conception is mentioned by Mason et al. (2010) or Stylianides (2007), who consider accepted socio-mathematical norms in a given classroom as a substitution for the axiomatic system used by mathematicians.

*Proof schemes* can be divided into three categories: *external*, *empirical* and *deductive* (Harel & Sowder, 2007). *External proof schemes* rely on external authority (textbook, parent), a form of argument or manipulation with symbols without understanding the meaning behind it (ibid.). *Empirical proof schemes* are such arguments and justifications which rely on specific examples and perceptions (ibid.). These examples might come from what Stacey and Vincent (2009) call *experimental demonstration* which might use physical experimentations with manipulatives or purely numerical examples. Importance of such methods is also stressed by de Villiers (2010). *Deductive proof schemes* rely on deductive logic and comprise what a mathematician would consider proof (ibid.). According to Harel and Sowder (2007), many teachers consider “empirical proofs” to be sufficient explanations of general statements.

As for the role of the proof from a pedagogical perspective, Hanna (1990) distinguishes between *proofs that prove* and *proofs that explain*. Knuth (2002) adds some other roles, most notably *to verify*, *explain why*, *communicate mathematics* or *discover or create new knowledge* and *systematize statements*. In his research, teachers considered proofs mainly as means of justification but not as tools to develop an understanding (Knuth, 2002).

An important factor shaping teachers' approach to teaching is beliefs about the teaching, learning and curriculum: “Beliefs filter teachers' perceptions, interpretations, and decisions about how to respond to particular classroom events” (Whitcomb, 2003, p. 543). The beliefs are formed by many factors such as teachers' previous experience as pupils. This factor is difficult to change later as teachers tend to teach in the way they were taught (Whitcomb, 2003). This might be crucial concerning R&P, as many PSTs encounter proof at the university for the first time where it is taught mainly in a rigorous manner, which might influence their beliefs and way of teaching R&P.

## Methodology

The sample consisted of 19 PSTs (denoted as S1-S19) in their final year at the university, studying to be lower and upper-secondary mathematics teachers. They had already passed most of the university mathematics courses and two courses in didactics of mathematics. During their third mathematics education course, an intervention was planned to enhance their capacity to design teaching mathematics supporting pupils' R&P.

For the sake of the intervention, the PSTs made groups of 2 to 4 (see Table 1) and stayed in the same group for the whole intervention. S1 to S8 are full-time students, the rest are part-time students (their teaching time is about 1/3 in comparison with the full-time study).

**Table 1: The number of PSTs in groups**

Group	G1				G2				G3				G4				G5			G6	
Student	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15	S16	S17	S18	S19		

The leader of the mathematics education courses conducted interviews with the groups online. The length of the interviews varied from 33 to 51 minutes depending on the number of participants and the number of thoughts shared by the PSTs. The interviews were semi-structured and their main focus was on two questions: *What role do you believe reasoning and proof have in primary school mathematics (age 11 - 14)? What importance would you give it in your teaching?* The PSTs were also asked about lesson planning and the use of resources. These questions do not concern R&P and thus are not within the scope of this paper.

All the interviews were video recorded. The recordings were transcribed and analyzed using techniques of qualitative analysis (Saldaña, 2013) in MAXQDA. The data were coded in an inductive-deductive way. The initial coding framework was based on the framework developed by MaTeK for the analysis of interviews with in-service teachers. It was adjusted and extended for this study and enhanced with new codes emerging from the data. The part of the coding framework particular to this study consists of, for example, Beliefs about R&P, Purposes of R&P, Classroom environment during R&P activities or Active participants in R&P (see Figure 1). For example, the statement: "For the majority of pupils to understand, I think it [R&P] is quite time-consuming." was coded as Beliefs about R&P and: "[Purpose of R&P is] to be able to confirm their [pupils] solution is correct." was coded as Purposes of R&P.

After coding all the data, the resulting codes with the PSTs' quotes were repeatedly read until broader categories related to our research question originated.

## Results

Figure 1 depicts the frequency of occurrences of the codes of the resulting coding framework in each group interview. Next, selected results particular to the identified categories will be presented.

Code System	G1	G2	G3	G4	G5	G6
> How suitable the used proving techniques is			3			
> PST's experience as a pupil	9	7	3	2	6	4
> Types of reasoning mentioned in the example	1		7	3	1	1
> Representation used in reasoning	9					1
> Purposes of R&P	9	8	2		6	
> Beliefs related to R&P as a classroom activity	23	35	13	19	10	14
> Roles of teacher/student in reasoning	1	1	3	1	2	1
> How reasoning is developed in the classroom	1		1			
> Source of R&P task	1				7	1
> Resources for R&P		2	1	1	1	3
> Frequency of R&P-activities		2	1		3	
> Motivational reasons of using/not using R&P in the class		7	1	6		7
> How pupils are motivated to start R&P						1
> Pre-service teacher's reflection on his teaching	5	9	8	7	2	3
> Pupils perception of R&P in teacher's opinion					1	1
> When R&P is included in learning trajectory	1	2		1		
> PST provides concrete example of task in the interview			1	1		1
> Topics in which R&P was mentioned	6	1	3	6	15	3

Figure 1: Code system with frequency of code occurrences in different groups

### What constitutes R&P and what purpose does it have in a classroom environment?

The PSTs mostly see R&P as a way to justify solutions: "...to be able to, like, confirm, their [pupils'] solution is correct." (S15) or as a way of justifying the validity of a mathematical statement. For some PSTs, *reasoning* also involves the process of derivation of a theorem or statement itself:

...it seems to me that it [R&P] has a big role – mainly the reasoning – in that sense that pupils can see, even if they might not understand it [the reasoning], that the proofs or the mathematical theorems do not simply fall from the sky. (S9)

In six cases *reasoning* was seen "only" as a description of steps a pupil takes to arrive at a result. The difference is that in this case pupil is not justifying why the step he or she took is valid, but only describes a sequence of steps performed:

I always want them [pupils] to describe everything too, how they got there [to a solution] in the first place, but it's more like, I want them to justify why they got this result and not another, rather than why it is valid that way. (S17)

All the PSTs agreed that R&P is important. They mentioned that R&P can serve teachers to assess pupil's thinking, to see if their pupils truly understand: "It's [R&P] definitely critical for me because to capture the thought processes and the way of thinking of my pupils, I definitely need to know how they got to a result." (S2)

Here again, the *reasoning* is probably seen as in the case of S17, but it serves as a means for thought verbalization thanks to which a teacher can discover misconceptions in pupils' understanding.

PSTs often recalled their school years and the lack of R&P present. Answers such as: "I guess I didn't enjoy math back then because no one ever did any reasoning with me." (S9) or "In high school, they just told us this is how it is, this is how it is done, this is how you count it. There was no understanding involved." (S11) suggest the PSTs consider the lack of R&P in primary or secondary education to harm pupils' relation with mathematics and their understanding.

In the last two quotes, *understanding* is achieved by encountering R&P in a topic. The role of the proof is explanatory (Knuth, 2002; Hanna, 1990). That is connected to the PSTs' belief that

employing R&P would help pupils obtain a deeper understanding and prevent knowledge without understanding: "...when you do a lot of work in the beginning, which might seem tedious, it can result in understanding the content more and then staying in the minds of pupils for longer." (S6)

S6 believes that even though using R&P in a classroom is time-consuming, it is worth the time as pupils will understand new notions better and more permanently. Other PSTs reacted that doing it this way will also save time later when a teacher can proceed more quickly as there will be no need to return to a not fully grasped concept over and over again. Focus on conceptual understanding can further be observed in this claim: "I think if the teacher guides pupils to understand it, then they can do the counting too." (S7) This, however, was tempered down by S8 in the same group who added: "...it seems to me that when the understanding is there, the counting ability is not always present; there is still a need to train it a little bit." Nonetheless even S8 later added that she would give more time to an explanation of concepts than processes, for which R&P is, according to her, suitable.

On the other hand, some PSTs admit that R&P is difficult and that only a few in the classroom might understand it. Nevertheless, they still see the importance of R&P presence in classrooms (see, e.g., the quote of S9 above), as pupils might develop a deeper understanding of the nature of mathematics and its system.

Another role of proof appeared in a quote from S15, who mentions convincing other classmates:

Well, or just to really convince my classmates that I was right, ... even though the majority agreed on something, that one individual was able to justify it and defend his solution, so for me that discussion... (S15)

### **Where does R&P belong in the teaching process?**

The quote of S6 above also talks about the *beginning*. It might mean any of the first lessons of a new topic or the beginning of a teaching trajectory. Consider the following quote:

...that initial derivation of, say, the formula for the square of a binomial or the difference... I mean the sum and the difference. It's like being offered right there at the beginning to actually start it with some of those squares, to draw them there. Break it down so the pupils can figure it out for themselves and try to formulate it afterwards. (S4)

S4 suggests starting with a R&P activity before the procedural knowledge is trained. Thus, R&P is not only used to convince pupils (Hanna, 1990) but also to derive the formula by the pupils.

S14 contrasts the approach above as she shares her struggle as a pupil to understand the proof of the *Pythagorean theorem*: "...there should have been more of that, that kind of realization maybe. Like numerical examples first, rather than straight to general. That was actually hard for us." In this case, the R&P activity did not serve to derive the formula or theorem but to prove it. S14 believes having some procedural knowledge first would be beneficial to understand the proving process better.

### **Use of visualizations for R&P**

PSTs also mentioned physical manipulatives, e.g., "...the proofs, when I do it, I do it so that they [pupils] can feel it, which seems to me to be the best way" (S10).

PSTs consider non-rigorous empirical justifications suitable to teach. They realize these are not rigorous and mentioned it is important to at least outline, using these, why a given formula/theorem

is valid. For PSTs, the language and symbolism used in R&P activity are also not important. They would support pupils to reason in their own words as the relevant language might be developed later.

On the other hand, three PSTs who try to incorporate the use of manipulatives into their teaching also mentioned examples where a visual manipulative proof could be employed but another proof performed by a teacher was presented: "...for instance, the sum of the interior angles in a triangle. I do that on the board myself and I show them how it works". This was mentioned by S10 in a reaction to S11 who admitted to struggling with making R&P of the same theorem accessible for pupils. S11 mentioned few pupils paid attention and understood the proof performed by her. The corresponding and alternate angles theorem was used, which pupils found difficult to follow. Opposed to this, S13 mentioned a proof using paper manipulatives. Similar phenomena were observed by S9 who complained about the difficulty to teach the divisibility rule for 3 or 9:

...like the divisibility by three it seems to me that it doesn't really fit in the curriculum because the proof to deduce why... actually the sum of the digits. It seems to me like it's something higher than what the pupils are able to comprehend. (S9)

S9 admits pupils did not understand his explanation which he wrote on the blackboard: "I know that it was observed by like 2, 3 pupils at most." In this case, it seems again only formal proof was presented even though it was possible to use visuals or manipulatives. Lack of knowledge of such methods by some teachers was mentioned by S12 who shared her experience of attending professional development courses where a visual way to explain *algebraic identities* was presented:

I was like, everybody knows this one, but I was actually the only one there who has seen it and knows it. And even the teachers at my school, who are usually older – I'm the youngest one there – they don't really know these modern, I would say, more modern techniques. (S12)

By *modern approaches*, S12 describes using manipulatives or visualizations for R&P. For her using these means was only natural, however, it was not for her older colleagues.

Not knowing ways how to make R&P accessible to a given age group might be one of the reasons why teachers do not use R&P in their teaching, according to PSTs. S7 believes the reason for in-service teachers not to know such methods to be the lack of exposure to didactics of mathematics.

### **Issues with teaching R&P**

The main obstacle the PSTs see is time limitations. The time concerns could already be observed in S6's quote. PSTs are worried to fit the obliged curriculum into the number of lessons they have available. S19 also complains that in the last year of primary school, he needs to focus mainly on the development of procedural knowledge as nationwide entrance examinations for secondary schools demand it and there is not enough time for R&P: "...if a teacher wants his or her pupils to be successful, if they're going to high school, then they've got to do without [R&P], so they're just got to adjust to the system." (S19)

The belief that R&P might not be understood by a majority of pupils is common concern among the PSTs (see also S9's quote above). It was also mentioned that R&P is mainly for the gifted: "I think the main idea of proving is that someone talented in math ... just wants to prove it, like it [a statement] will always be valid." (S15) The PSTs expressed similar thoughts about reasoning.

## Discussion and conclusions

As for the role and importance, the results show the PSTs see R&P to be an important part of primary school math education. Even though they had similar experience with R&P in their university mathematics courses, they expressed different views of the role of R&P. Most often, they see it as a way to justify or validate solutions or statements (Knuth, 2002). Contrary to the results of Knuth (2002), the PSTs in our sample also see R&P as a way to develop pupil understanding. It might be caused by the university courses on mathematics education where a strong emphasis on teaching mathematics in a way that puts understanding first is present. S15 talks about the role of proof to communicate mathematics (Knuth, 2002) and mentions the importance of discussion and classmates' conviction similar to Mason et al. (2010) or Stylianides (2007). Seeing R&P as a simple description of processes without the use of any conceptual knowledge was unexpected and will be focused on in the intervention.

When the PSTs recalled their school years, they thought negatively about the lack of R&P present in the classes. This view suggests not only that the PSTs consider the presence of R&P beneficial, but it might also positively influence their approach and involvement of R&P in classes as they might not be guided by what they experienced as pupils to teach similarly as Whitcomb (2003) mentions. It can also be inferred that PSTs consider conceptual knowledge development in classroom to be more important than procedural.

The PSTs realize there are different approaches to R&P in different grades and classes (Stylianides, 2007) and believe the inclusion of visual manipulative proofs to be a suitable way to teach R&P. Even though many proofs using manipulatives lack generality or prove only one implication of a given theorem that might be in a form of equivalence, PSTs did not mention these issues (similarly to Knuth (2002)). They did however mention that proofs do not need to be rigorous as long as there is at least some explanation involved. Despite the popularity of manipulative proofs, several PSTs shared their difficulties to teach proof where manipulatives could be employed. The fact that they used more abstract proof might suggest a lack of knowledge of such methods and a need to develop their *specialized content knowledge* and *knowledge of content and students* (Lesseig, 2016). This was however present only among PSTs who study part-time and have fewer lessons in didactics. PSTs did not mention a belief Harel and Sowder (2017) mention, that empirical justifications are sufficient, yet some of the proofs using manipulatives are empirical.

Some believe the inclusion of R&P tasks saves time later. The PSTs did not agree on when to involve R&P in the teaching trajectory. They also shared some concerns and limitations caused by the current school system - mainly the need to prepare their pupils for procedure-based entrance exams. They did not describe how their teaching process of R&P would look more in detail (what classroom layout would they use, who would be the initiating R&P activity or how R&P would be developed). These will be explored more during the following sessions of the intervention study.

The results of this study let us adjust the intervention accordingly. After the intervention, another phase of interviews will follow to observe what PSTs' conceptions and beliefs were influenced. Based on these, the intervention will be modified for the following rounds.

There are several limiting factors to this study. Firstly the sample size was small and results cannot be generalized. Secondly, there might also be differences in results between the group and individual



interviews. Nonetheless, results give us valuable feedback for the intervention as results suggest PSTs do realize the importance of R&P but have different, sometimes misleading conceptions about what R&P constitutes. Results also show a need to further supply PSTs with specific visual methods for R&P.

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