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Mathematics and debate: a dialectical approach

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In this paper we address the possibility of applying the methodology of debate to the educational setting of a mathematics class. Starting from Aristotelian dialectics, we examine its suitability for scientific, but not-demonstrative uses in light of current views in philosophy of mathematics then propose a taxonomy of debatable motions relevant for mathematics education. We conclude by listing the educational benefits of this novel method and what steps we consider necessary to test it in the classroom setting.

Keywords: Debate, metacognition, argumentation, educational philosophy.

Introduction

The use of debate as a teaching tool has greatly increased in recent years, both in upper secondary and tertiary education. Although debate is considered useful to improve soft skills, like argumentative skills and critical thinking skills, disciplinary debate—that is debate that is purposely aimed at increasing knowledge, skill, and competence in a specific academic subject—is all but absent, from classes and journals alike. We will here address the possibility of disciplinary debate in mathematics, starting from what constitutes the basis for current competitive debate techniques and theories (i.e., ancient dialectics) inquiring its suitability for investigating topics that have a disciplinary relevance, then proposing a classification of the possible approaches to use debate in mathematics classroom practice¹ that should empower students to enhance their problem solving and mathematical reasoning skills.

Debate in education is competitive debate

In everyday speech, the term *debate* is typically used as a synonym for *argumentative dialogue*. Yet, there are differences between debate and other types of argumentative dialogue (Walton, 1989) the essential difference being that in the former the outcome is decided by a third party that is normally advised to render their decisions disregarding the subjective or objective merits of the proposition that is the object of debate and considering only the merits of the support as presented in the debate itself (Freeley & Steinberg, 2009). Also, in debate, the stance, pro or con, of the debaters relative to the question to be debated is not chosen by the debaters themselves according to their interests and beliefs but it is assigned to them by the said third party. According to the results of the first² worldwide empirical-based review of the results of debate activities in education (Akerman & Neale, 2011) the

¹Unless otherwise stated, we will refer to secondary education when talking about classroom practice and students.

²The authors of that review explicitly claim that: “while a wealth of anecdotal evidence exists on the value and impact of these activities, no previous work has reviewed and analysed the empirical evidence base” (Akerman & Neale, 2011, p. 6).

majority of classroom activities, in formal educational settings, that can be classified as debate is constituted by *competitive debate* and

to establish a basic competitive debate a series of essential elements is necessary, namely a proposal or resolution, an affirmative side, which defends that the resolution is true, and a negative side, which denies that the resolution is true. There must also be a format of turns and times. (Cirlin, 1999, as cited by Prieto et al., 2021, p. 7)

Thus debate as typically practised in schools and universities today is in the form of the so called competitive debate and it is characterised by being organised in a formal environment governed by a strict set of rules, commonly called protocol. These rules do not merely regulate a pre-existing practice but create the very possibility of actually engaging in a competitive debate, in other words the protocol can be classified as a set of constitutive, not regulative, rules, if we are to use the terminology championed by Searle (1969). Its status as a sort of regulated mental sport, coupled with the fact that the majority of the propositions object of debate—or motions, as they are typically called—are usually not directly related to typical school subjects or academic disciplines may lead to the conclusion that, while being very effective in developing soft skills (Akerman & Neale, 2011), debate is not contributing to the acquisition of hard skills, namely the ones that are developed through the study of a particular subject or discipline. But it was not always the case in the past, nor it has necessarily to be the case today.

Dialectics as scientific inquiry

The roots of contemporary argumentation theory can be traced back to Greek philosophy and, in particular, to Aristotle's theory of dialectics (Walton, 2021). Aristotle recognized two species of argument: deduction³ (συλλογισμός) and induction (ἐπαγωγή). In addition, he classified deduction according to the following scheme: (a) demonstration (ἀπόδειξις), (b) dialectical deduction (διαλεκτικὸς συλλογισμός); (c) contentious deduction (ἐριστικὸς συλλογισμός); (d) paralogism (παραλογισμός) (Aristotle, ca. 350 B.C.E./1960, Top. I 1, 100a25–101a24). Demonstration is the kind of deduction that grants scientific knowledge (cf. Aristotle, ca. 350 B.C.E./1960, APo. I 2) and it is characterised by having premisses that are true and first or that are derived by first premisses (Aristotle, ca. 350 B.C.E./1960, Top. I 1 100a27–29), in modern parlance we would say that a demonstration, or proof, is a deduction, or a tree of deductions, that moves from axioms or propositions derived from axioms. According to this definition of demonstration—a definition more than two thousand year old but, in its essence, compatible with current mathematical practice—there should be no room for using a mathematical statement as a debate motion, since a mandatory prerequisite to have a debate is discussing a contentious statement, not something that can be proved. According to Freeley and Steinberg (2009):

³We prefer to translate the Greek term *sylogismos* as “deduction”, instead of the more usual “syllogism” because that term is more in line with modern usage and also because our main concern here is not philology but investigating the possible relevance of Aristotle's works for our purposes.

it would be pointless to attempt to debate “Resolved: That two plus two equals four,” because there is simply no controversy about this statement. *Controversy is an essential prerequisite of debate.* (p. 43)

If someone shows a demonstration to someone else, there cannot be dialogue proper, let alone debate, and this was perfectly known to Aristotle, who, in classifying the various types of arguments used in discussions (cfr. Aristotle, ca. 350 B.C.E./1955, SE 2), named the ones that make use of demonstrations as didactic, or didascalical (*didaskalikoi*), adding that they must not consider the opinions of the pupil who, thus, cannot be considered a collocutor proper and has no role other than listening to the words of the teacher.

Yet, it is not necessary to debate a mathematical statement if one wants to use debate for scientific or educational purposes in mathematics: in fact according to Aristotle,⁴ demonstration is not the only method to acquire scientific knowledge. Writing about dialectical deduction, that is point (b) of the previous scheme, he explicitly stated that it is useful in regard to three things: exercise, discussions, and sciences (Aristotle, ca. 350 B.C.E./1960, Top. I 2, 101a25–28).

Dialectical deduction is the kind of deduction whose premisses are not first principles or propositions derived from first principles but are *endoxa*, that is premisses that are accepted by everyone, or by all experts or at least by the most respected ones (Aristotle, ca. 350 B.C.E./1960, Top. I 1, 100a30–b23). Thus, in a hypothetical dialectical discussion about mathematics, *endoxa* as premisses that are accepted by everyone, could be considered as any working hypothesis that both contenders are willing to concede, while *endoxa* as premisses accepted by experts, could be considered what Reuben Hersh calls established mathematics: “it includes proved statements ‘in the literature,’ and also some simpler statements that are so well accepted that no literature reference is expected” (cf. Hersh, 2013, as cited by Aberdein, 2013, p. 264).

Considering now the above mentioned three things about which dialectic is, according to Aristotle, useful, we argue that the first use is linked to the use of debate as a means to improve soft skills, the second use is linked to competitive debate, as typically practised in academic settings, and the third one is potentially linked to what we are investigating in this paper, that is disciplinary debate. In this regard, the Greek philosopher wrote:

For what concerns sciences conducted in a philosophical manner (*kata philosophian*), for the fact that we are able to examine the difficulties (*diaporesai*) from both sides, we will more easily see the true as well as the false in each case. Moreover, it is useful also in relation to the first principles (*prota*) of each science. For, to reason starting from the first principles of a given science, makes impossible to say anything about these, because first principles are the starting points (*archai*) of everything, so it is necessary to discuss them according to the educated guesses (*dia ton endoxon*) about each of them. This is the peculiarity that most characterises dialectic for, being fitted for

⁴Aristotle hints that there could be another method for acquiring scientific knowledge in the very same passage where he describes his theory of demonstrative science (Aristotle, ca. 350 B.C.E./1960, APo. I 2).

inquiry (*exetastike*), it possesses the gateway (*hodon*) to the principles of all disciplines. (Aristotle, ca. 350 B.C.E./1960, Top. I 2, 101a34–101b4; our translation)

Thus, according to Aristotle, there are two possible uses of dialectic for scientific inquiry: examining a problem's difficulties from every angle and discussing the first principles of a theory. We will call the first use diaporetic and the second one exetastic. These terms are already found in literature, see for example Bjelde (2021) or Walton (2021), but we use these terms in a slightly different meaning. Aristotle wrote that dialectic is useful for sciences conducted "in a philosophical manner" so, to better understand the two uses named above, it is convenient to have a brief look at the philosophy that is relevant for the science we are discussing, that is: philosophy of mathematics. Contemporary philosophy of mathematics can be identified with the three big schools of logicism, formalism, intuitionism (Shapiro, 2000) and concerns itself mainly with questions of ontology, metaphysics, and epistemology (Plebani, 2011). In recent years, alongside this prevailing view, another approach to the philosophy of mathematics that originated, among others, in the works of Pólya (1954) and Lakatos (1976), came to the fore. This approach, embodied by the so-called philosophy of mathematical practice, reacted against philosophy of mathematics conceived as foundation of mathematics and started to pose questions like:

How does mathematics grow? How are informal arguments related to formal arguments? How does the heuristics of mathematics work and is there a sharp boundary between method of discovery and method of justification? (Mancosu, 2008, p. 3)

We thus maintain that the exetastic use of dialectic is naturally focused on the themes of mainstream philosophy of mathematics, since it is mainly concerned with foundational issues, while the diaporetic use pertains to the themes characterizing the philosophy of mathematical practice, being this approach more concerned with mathematics 'in the making'.

Disciplinary debate and mathematics

Our goal is to devise a model of disciplinary debate in mathematics patterned on the scientific use of dialectic discourse, as outlined in the previous section. Being a debate, the model should sport all the above mentioned peculiarities (cf. Cirlin, 1999, as cited by Prieto et al., 2021, p. 7): (a) a resolution, (b) two opposing teams, (c) a protocol.

For what concerns point (a), the resolution should be contentious. This, as said, rules out the possibility of debating something that can be proved. Patterning disciplinary debate on dialectical inquiry, instead of demonstration, resolves this *impasse*, since dialectic is not tied to deduction from first principles but uses other forms of inference: deduction from plausible premisses, and non-deductive arguments. Are there contentious questions relevant for mathematics as an academic discipline? Yes, these are the very questions inquired by the exetastic and diaporetic dialectics. Mathematics is not simply mathematics presented in finished form as theorems proved from axioms (Cellucci, 2022). If someone wants to discuss the axioms they use, they cannot do it inside the axiomatic system itself, thus they have to resort to the exetastic use of dialectic. Likewise, if someone is resolving a problem or devising a proof, they might have to evaluate the relative merits of different approaches, hence resorting to the diaporetic use of dialectic.

Having clarified this theoretical point, how can we practically devise debatable questions in mathematics? Patterning our motions according to the two scientific uses of dialectic, we propose two kinds of motions: foundational⁵ motions and metamathematical⁶ motions. The first ones are related to the exetastic dialectic, and thence to the mainstream philosophy of mathematics, the others are related to the diaporetic dialectic, and thence to the philosophy of mathematical practice. Considering the main concerns of mainstream philosophy of mathematics, foundational motions may be further subdivided into ontological, metaphysical and epistemological motions. Broadly speaking, ontological motions are relative to the existence of mathematical objects, metaphysical motions are relative to the nature of mathematical objects, epistemological motions are relative to our capacity to know mathematical objects. The taxonomy of metamathematical motions may prove more difficult than that of foundational motions, mainly because the philosophy of mathematical practice has a broader and less established scope (Mancosu, 2008; Hamami & Morris, 2020) than mainstream philosophy of mathematics. Without claiming to be exhaustive, we propose a further subdivision of metamathematical motions into three subtypes which we call ‘worst error’, ‘comparative analysis’, ‘best explanation’, following Giangrande and Matteucci (2021):

Such approaches address some well-known issues of mathematics teaching, namely: overcoming the correct answer compromise; refining fundamental skills of mathematical literacy and competence; also, they enquire into the explanatory capacity of a proof. (Giangrande & Matteucci, 2021, p. 276)

It is worth noting that, although our proposal of a disciplinary form of debate is patterned on philosophical considerations, the resulting taxonomy is philosophically agnostic, and can be used without the need to take a particular philosophical stance. A recapitulation of the above taxonomy is presented in Table 1, along with some examples. Due to space constraints, we only gave examples of motions and not of actual debate transcripts (cf. Giangrande & Matteucci, 2021). For what concerns points (b) and (c), there is not much to add at this stage. There are many different protocols in use today for academic debate (Freeley & Steinberg, 2009) and their suitability to the kind of disciplinary debate we are proposing is more a matter of trial and error than a theoretical issue. We only remind you of three caveats (cf. Giangrande & Matteucci, 2021) when experimenting while devising an *ad hoc* protocol: flexibility (adapting in the absence of established practice), caution (calibrating activities based on the actual knowledge of the students), continuity (modifying existing debate rules, instead of creating new ones from scratch).

Table 1: Summary of mathematical motions with examples

exetastic	mainstream	foundational	ontological	“Imaginary numbers do not exist”
			metaphysical	“A set is different from the collection of its

⁵We chose this name because these motions are broadly related to the foundations of mathematics.

⁶We chose this name because these motions are related to the how and the why we do mathematics, hence to metamathematics, understood in a broad sense.

dialectic	philosophy of mathematics	motions		elements”
			epistemological	“learning long division gives no insights into number theory”
diaporetic dialectic	philosophy of mathematical practice	metamathematical motions	worst error	“writing $\sqrt{\alpha^2} = \alpha$ is a worse mistake than writing $\sqrt{\alpha^2} = \pm\alpha$ ”
			comparative analysis	“the best method to solve a quadratic inequality is by graphing”
			best explanation	“the most explicative proof of Gauss’ formula for adding the first n positive integers is the one that uses triangular numbers”

Debate in mathematics class

The classic presentation of mathematics, according to the axiomatic method, in some respects deceives students by leading them

to believe that mathematics is created by geniuses who start with axioms and reason directly from the axioms to the theorems. The student feels humbled and baffled. [...] [Axiomatic demonstration] omits the real activity [...] conceals the flesh and blood (Kline 1970, as cited by Cellucci, 2022, p. 474)

Also to counter this unintended side effect (Lockhart, 2009) of standard classroom activities “many mathematics education researchers have called for teachers to engage students in the practice of doing mathematics as mathematicians” (Sriraman & Umland, 2020, p. 65). Actual classroom teaching of heuristic methods to students, however, did not always prove totally successful (Mousoulides & Sriraman, 2020). Yet, that the proofs presented in classrooms should “make students wiser by offering explanations and illustrating methods” (Harel & Weber, 2020, p. 188) is still to be considered a critical issue whose tentative solutions are only now slowly emerging, and this is true not only for proofs but also for any problem that is not a mere procedural exercise.

When teachers let the students confront with the inventive and explorative part of doing mathematics (e.g., devising the best way to solve a problem or construct a proof) the only alternative to appealing to a somehow mystical mathematical intuition is to resort to teaching strategies that involve the full spectrum of argumentations, including non-deductive inferences. Disciplinary debate as presented above, being based on dialectics⁷ and thus encompassing all kinds of reasoning, is an ideal candidate for: experimenting heuristics processes, favouring metacognition, examining mathematics in its making, breaking the didactical contract (as it destroys the convenient, albeit harmful, belief both for

⁷Significantly, while the only kind of argument allowed in demonstration is deduction, dialectics also allows non-deductive inferences, for example induction (Aristotle, ca. 350 B.C.E./1960, Top. I 13, 105a10–14).

the teacher and the pupil that mathematics is only for intuitive prodigies: the first is thus spared the effort of teaching the latter is spared the effort of thinking), confronting foundational questions, exercising critical thinking when learning and exploring mathematics.

There are also other methods, that were recently proposed, addressing the same issues above. For example, providing students with a list of heuristics to be checked and used when working out a proof or a problem (cf. Mousoulides & Sriraman, 2020) or the request to back every claim with an appropriate warrant in the context of an informal classroom debate (Luzniak, 2019). Compared to the first, our disciplinary debate has the advantage of conversational interaction that provides both cooperative and confrontational moments, compared to the second it avoids a problem that typically plagues classroom informal discussions, because in a structured debate nobody can take the floor exclusively, nobody can avoid participating. Also, as an additional benefit, nobody can take on the works of others, nor can anyone delegate their share of work to someone else.

Conclusions and perspectives

Poincaré (1947) once wrote:

If a new result has value, it is when, by connecting elements already known for a long time but until now scattered and appearing unrelated to each other, it suddenly introduces order where the appearance of disorder reigned. (p. 24–25; our translation)

We linked Aristotle's dialectical scientific inquiry with modern debate theory through the mediation of contemporary philosophy of mathematics and what could have appeared *prima facie* a contradiction in terms (i.e. debating mathematics) took shape in a reasonable form.

After laying the theoretical foundations showing the possibility and soundness of disciplinary debate in mathematics, we believe it is necessary to move on to a phase of classroom experimentation that should set the stage for the next step: building an effective database of debatable motions that follow the proposed taxonomy and prove themselves to be useful in improving the metacognitive abilities of the students in regard to the exploration of heuristic and inventive processes in mathematics. This step is essential because in debate, even competitive debate, it is difficult to guess the suitability of a given motion before having it actually debated in an appropriate environment.

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