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Conceptualising the regular pentagon in collaborative argumentation

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This case study examines a group of Hungarian mathematics pre-service teachers solving an open problem collaboratively, where they must determine whether a folded shape is a regular pentagon. The study looks at how different factors, such as using perceptual evidence, ostensive arguments, and mathematical arguments, impact proof development. The study draws from theories such as Cognitive unity and the van Hiele model to analyse the students' argumentation. The video recording of the session was used to collect data. The results showed that students' argumentation had a strong dynamic between perceptual and mathematical character; moreover, the conceptualisation of the regular pentagon evolved as they advanced in the construction of their proof.

Keywords: Cognitive unity, van Hiele levels, problem-solving, argumentation

Introduction

We report on a case study focused on a group of students engaged in solving an open problem based on a paper folding procedure; openness means that the start is clearly defined, but the target is not specified, and the target itself must be revealed (Pehkonen, 1997). This research is part of a broader project on collaborative problem-solving. In the reported group work, students conjecture that a folded shape is a regular pentagon. The demand for proof arises immediately, yet what needs to be proven only develops gradually during collaborative problem-solving. The study looks at how different factors impact this development. These factors include using perceptual evidence, ostensive arguments, and mathematical arguments. In the frame of the theory of Cognitive unity (Boero, 2016; Garuti et al., 1998), the authors speculate on the effect of these factors and emphasise their significance in shaping the learning environment for students. The case study focuses on identifying those factors that can influence students' development of mathematical arguments and mathematical concepts.

Outline of the theoretical framework

In this paper, we are interested in analysing students' argumentation process to describe the complex relationship between a statement *intuitively* achieved and the production of a mathematical proof. The theoretical background of our study draws from different theoretical sources to provide an adequate frame for supporting the complex analysis we aim at. First, we will use an elaboration of the construct of Cognitive unity to describe and analyse the process of searching for proof and leading to the confutation of a primary (original) conjecture statement.

Cognitive unity

The practice of open problems has shown its effectiveness in triggering spontaneous argumentation produced to accompany and support the solving process (Silver, 1995).

Seminal studies report on the continuity between the argumentation produced to support the solution of a problem and the construction of a proof validating such a solution, called Cognitive unity (Garuti et al., 1998). Cognitive unity concerns the relationship between argumentation supporting the production of a conjecture and the mathematical proof produced to validate the statement expressing the conjecture, that is, a proof built within a theory using axioms, definitions, and available theorems. A person's solution to a problem and the supporting arguments are based on a personal system of conceptions (Balacheff, 2013); personal conceptions are mental constructs and may or may not have a counterpart in a piece of theory.

During the problem-solving process, any argument relies on the solver's system of conceptions, and it may happen that a solution comes to the solver's mind as a self-evident fact without any accompanying explanation. As Fischbein (1987) writes,

All these statements are accepted as being immediate and self-evident without feeling the need for a proof either formal or empirical. Self-evidence is, then, a general characteristic of intuitive knowledge" (p. 13).

Sometimes, a justification for such an intuition is produced later, prompted by the request to explain the solution provided. This phenomenon is referred to as *structural argumentation* (Pedemonte, 2007, p. 28). The original construct of Cognitive unity has been further elaborated, as discussed in detail by Mariotti and Pedemonte (2019), for studying the cognitive complexity of the relationship between a structural argumentation and the production of a mathematical proof. While intuitive arguments can sometimes be easily converted into a proof (Zazkis et al., 2016), intuition can also hinder the development of proof (Mason et al., 2013); in particular, intuitive arguments, including either *perceptual evidence*, that is, properties that appear as self-evident for the solver, or *ostensive arguments*, that is arguments based on directly verifying the property at stake, can present the serious difficulty to be turned into mathematical arguments.

Van Hiele levels

In problem-solving, the necessary conceptions must be mobilised and recalled from memory. In the process of argumentation, the mobilised conception does not necessarily remain unchanged but often evolves and enriches. We use the van Hiele levels as criteria for describing the conceptual development by analogy with how van Hiele- theory models the development of an individual's geometric thinking (Pegg, 2014).

At the visual level (VH1), students recognise a geometric figure in its global appearance, often by comparing it to a known prototype (perceptual evidence). At the analytical level (VH2), students recognise and investigate the components and properties of a figure. Students can relate figures to their properties at the informal deductive level (VH3). Argumentation is based on intuition rather than on mathematical foundations (ostensive argument). Finally, at the deductive level (VH4),

students can argue deductively about geometric figures based on their defined properties. They can construct geometric proofs consciously and be aware of what counts as a mathematical proof.

The empirical setting

In this case study, we aimed to evaluate the collaborative problem-solving achievements of five Hungarian pre-service teachers (S1-5 in what follows) pursuing a degree in mathematics. They were in the third year of their training. The participants had previously completed introductory mathematics courses and courses in problem-solving.

The investigation reported in this paper concerned an open problem based on a paper-folding procedure producing a folded shape. The participants were asked to discover some properties of the folded shape in group work. The instructor was asked to provide no mathematical support to the group. The session conducted in the autumn of 2022 was video recorded.

The geometrical and didactical analysis of the task

The problem is drawn from the book of Lam and Pope (2016). The starting point is the folding procedure that produces an approximate regular pentagon from an A4 sheet. The procedure consists of three main steps, see Figure 1.



Figure 1: Main steps of the folding process

The resulting figure, $ABCDE$, is a symmetrical pentagon, and $AB = BC = DE$, $AE = CD$ (but AB is not congruent to AE). Concerning the angles, $\angle A = \angle E = \angle D = \angle C$, but not equal to $\angle B$; see Figure 2.

The first part of the task consists of the succession of foldings that leads to obtaining a pentagon, which seems to be "very regular." A feature of the process is that it is challenging to identify geometric relations during folding. The formal follow-up of instructions and the pursuit of accuracy presumably suppresses the immediate awareness of geometric relations corresponding to each folding act, i.e., the association of the fold with axial symmetry. On the other hand, it is not easy and immediate to identify the geometric elements and their consequent geometric relationships on the unfolded sheet; folds may lose meaning when the sheet of paper is unfolded, and it may become difficult to imagine the corresponding straight lines and their extensions beyond the margins of the sheet of paper. For example, due to the limited size of the sheet makes it difficult to recognise the geometric role of FF' as an axis of symmetry (Figure 2).

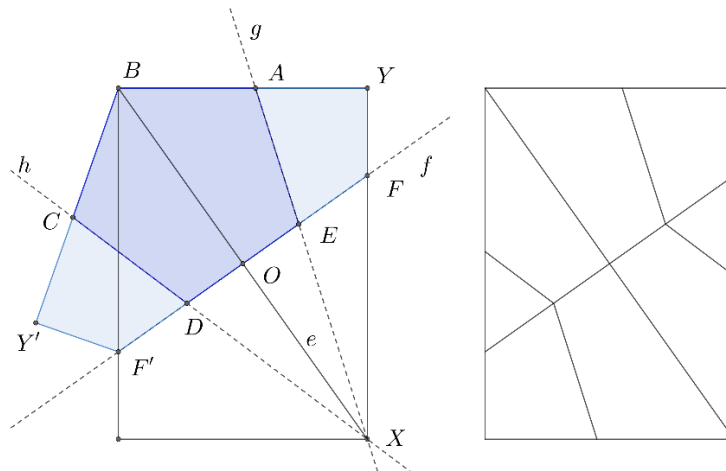


Figure 2: Geometric considerations of the folding with symmetry lines (left) and what is visible on the unfolded sheet (right)

In summary, formalising the folding act as a geometric transformation and geometrically interpreting the related properties of the folded shape is a highly demanding cognitive task. Moreover, the power of the problem comes from the tension between the immediacy of the conjecture about the regularity of the obtained shape and the fact that such a conjecture is not true.

Analysis of the data

The process of the problem-solving activity can be divided into five main episodes. Figure 3 shows the timeline with the turning points of the group discussion.

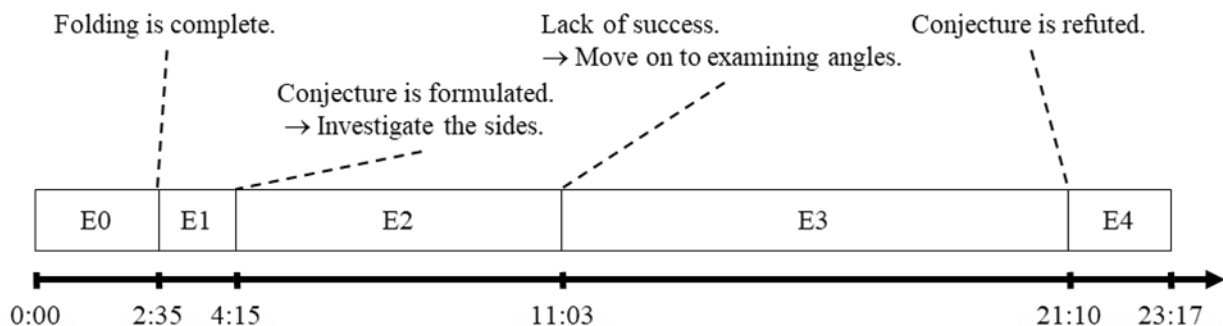


Figure 3: Timeline of the problem-solving process

Episode 0 (E0). Paper folding. The teacher reminds the group of the ratio of the sides of an A4 sheet of paper ($\sqrt{2}:1$).

Episode 1 (E1). Visualising the folded shape and directly comparing its sides. Formulation of the conjecture: The folded polygon is a regular pentagon.

Episode 2 (E2). Trying to calculate the measures of the sides of the pentagon without success.

Episode 3 (E3). Focusing on the measures of the angles of the pentagon. More precisely, calculating the measure of at least one angle of the pentagon using the trigonometric properties

relating two sides of a right triangle. Refutation of the conjecture: The pentagon is not regular.

Episode 4 (E4). Looking back and checking the calculation in different ways.

Although the definition of a regular pentagon was known to the students from their previous studies, episodes E1-4 show how during the group activity, the geometric characterisation of a regular pentagon gradually evolves: the definition is recalled, rebuilt, and deepened. We describe the key moments of this process using the van Hiele levels as developmental criteria.

E1 starts at the end of the folding procedure when the students finish folding and globally recognise the shape as the prototype of a regular pentagon (perceptual evidence); moreover, as shown in Figure 4, after folding they hold their pentagons and directly compare the sides to check if they are equal (ostensive argument). At the end of E1, the conjecture is formulated: The shape is a regular pentagon.



Figure 4: Recognising the shape (left) and empirical argumentation (right)

In this first episode, the conception of a regular pentagon is related to the first three van Hiele levels. Recognition from the global picture refers to thinking on VH1. On the other hand, comparing the sides of the pentagon is an activity corresponding to VH2. However, the group moves beyond these two phases very quickly. When they formulate the conjecture, the explicit reference to the congruence of the sides, used to support the regularity of the pentagon, shows the characteristic of the VH3 level, e.g., students connect the figure with its properties. Nevertheless, the arguments remain intuitive and based on perceptual and empirical rather than mathematical evidences, as seen in the following transcript (03:20-03:30).

S1: Apart from my folding inaccuracies, all sides are the same.
S5, S2: Regular pentagon?!

Immediately afterward, the need for proof also arises, and the second episode starts.

In E2, the group investigates the sides of the pentagon to calculate them somehow. Behind this activity, the actual conception of the group is clearly shown regarding the regular pentagon, i.e., a pentagon is regular if its sides are equal. The episode ends with a turning point after some failed attempts. A group member (S3) suggests that it might be possible to prove the conjecture by calculating angles instead of sides.

In this episode, some interventions can be related to VH3. It is the case of arguments associated with the act of folding and the general intuitive principle: "folding halves the figure into two congruent

parts." However, the following transcript (04:48-05:10) illustrates how, starting from level VH3, the group members collectively move and reach the next level, VH4:

- S3: Well, here it is ... [S3 shows the diagonal of the original rectangular A4 sheet.]
S2: That is the diagonal.
S5: This halves the diagonal, doesn't it? [S5 shows the midpoint.]
S2: Yes, yes! [S2 shows the first fold again.] Then we can work it out because we know this side and this side [BY and YX , see Figure 2], then ... We can calculate half of the diagonal. Moreover, that will be exactly the height of our pentagon [BO].

S2 continues her previous reasoning by determining the length of the diagonal using the Pythagorean formula. S1, S3 and S5 understand what S2 explains. In this excerpt, the students use typical geometrical arguments showing awareness of what counts as legitimate proof in mathematics. Thus, we can recognise a shift to the VH4 level.

E3 is about determining a certain angle of the pentagon. The conception of a regular pentagon has been extended in this episode, focusing on the angles and their congruence. As a result of the group's collective decision, the students calculate that the measure of one of the angles is not that that was foreseen, namely 54 degrees. This leads the student to refute the conjecture.

The argumentative dynamics between students' intuitions and the elaboration of mathematical conditions requested for the proof can be interpreted using the construct of Cognitive unity: the arguments develop, attempting to maintain the link between the intuitive and formal plane. The empirical experience of the folding act resulting in the conception of "halving something" and of "overlapping something with something else" leads to the mathematical notion of congruence. The following two examples (18:33-18:36 and 19:09-19:33) illustrate the dynamics between intuitions and formalisation, consisting of a pair of back-and-forth between VH3 and VH4.

- S4: Wait a minute! This is. This is half of this sheet. [S4 shows triangle OGB and BG where G is the midpoint of the side of the sheet. See, Figure 5, left.] But then we know that this is $1/2$ [BG], and this is $\sqrt{3}/2$ [BO], and this is a right triangle.

Here, point G appeared on S1's sheet after folding the rectangle in half. S4 had seen this and asked for the sheet. S4 interpreted G as the midpoint of the side of the rectangle and immediately found the appropriate triangle. This is a manifestation of a geometric interpretation of the halving conception; moreover, at the group level, the joint action of two students led to the result.

- S1: [S1 points to the angle in question.] Because if we can say anything, it is half of the angle of the pentagon. So, it should be half of 108, shouldn't it?
S3: Which one?
S4: We will find out; I do not know.
S5: Yes, because ... [S5 folds and identifies angle OGB with the half of the "top" angle of the pentagon; see Figure 5.]
S1: Because this is the vertex of the ... [S1 also folds.] Yes.
S1, S4: So, the point is that this angle should be 54.

It is not easy to interpret the OGB angle on the folded-out sheet as half of one of the angles of the pentagon. As the photo shows, however, S5 folds back the sheet so that the overlapping of the angles and congruence of angles appear almost simultaneously.



Figure 5: Right triangle to calculate angle (left) and empirical confirmation of symmetry (right)

In the last E4 episode, the looking-back phase appeared spontaneously once the angle was determined. Maybe the strength of their original conjecture makes the students doubt the correctness of the calculation. Group members reconsidered how they had come to the result and checked the calculations they had made. In addition, one of them (S3) also confirmed the conclusion by folding in half the pentagon by a line different from the previous one. "The two top sides do not fit exactly. It did not come out ..." - S3 said. This moment illustrates the dynamics between intuition and proving well again and the extension of the regular pentagon conception to a new property: a regular pentagon is symmetric to any line joining the vertex and the midpoint of the opposite side.

Discussion and conclusion

The case study depicts how argumentation evolves during collaborative problem-solving and how students developed their conception of the regular pentagon while striving to convert perceptible evidence, empirical evidence, and ostensive arguments into mathematical proof. The analysis of students' argumentative dialogue reveal how their geometrical characterization evolves over time. The van Hiele levels used as developmental criteria accurately described crucial evolution moments. The group argued that a pentagon is regular if its sides are equal. Later, the group concentrated on angles. From level VH3, the group moves to VH4, which requires a rigorous mathematical description of a regular pentagon. The dialectics between students' conceptions and the mathematical viewpoint sustain the production of the proof. To retain perceptual and geometrical coherence, the conception of regular pentagon is expanded.

We claim that the character of the task, i.e., an open problem leading to an immediate conjecture that later reveals to be false, and the collaborative work setting can be considered the motor of the dynamics of the argumentation, fostering the development of students' formal arguments. This kind of task can also provide a powerful base for a discussion that can go beyond the solution of the specific geometrical problem, a meta-discussion focusing on the meaning of a mathematical definition, and this could be the direction for further research studies.

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