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Some thoughts on an enculturation function of mathematical proof

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We highlight an enculturation function of proof. We argue that this function cannot be omitted in the use of proof in the teaching and learning of mathematics, despite or precisely because of the alleged latency of this function. Thus, it deserves more focused educational attention. An enculturation function of proofs should be conceptualized and grounded in an appropriate and didactically justified framework as a prerequisite for its deliberate use for educational purposes. As a starting point, we propose a framework for further reflection and inquiry that connects to complementary approaches in mathematics education discourse on mathematical proof and proving that are relevant regarding enculturation and to the general education discourse about science propaedeutic competence.

Keywords: Enculturation, mathematical proof, proof functions, science propaedeutics.

Introduction

As is well known, a whole range of proof functions in mathematical practices has been proposed in the literature. Nevertheless, we see another, rather hidden and latent function of mathematical proof: the enculturation function. While the aspect of enculturation is passively addressed in several strands of mathematics education, e.g., in the work regarding authentic mathematical practice (e.g., Melhuish et al., 2022), we seek a clearer discussion leading to a coherent theoretical framework to investigate this topic further and in different contexts of teaching and learning mathematics at school and at university. We consider enculturation as the (learning) process of an individual about the characteristic features and dynamics of a social group to which he or she belongs (Grusec & Hastings, 2007, p. 547) and about the specific cultural practices lived by that group. In particular, the characteristic features and dynamics that an individual must recognize and acquire when becoming a practicing member of the social group include a wide variety of rules, values and norms, symbol use, language issues, discourse rules, and standards for validity claims. We consider mathematical practices as such cultural practices, emerging from and developing through the reproduction of activities by social groups of practitioners. An existing mathematical practice confronts the (learning) individual as a ready-made construct of ideas, methods, norms, and values (e.g., Larvor, 2016, p. 3). In this sense, an enculturation function of proof is to be seen within the context of mathematics education, but also as an aspect of the scientific discipline itself.

Based on a short and exemplary literature review highlighting various complementary aspects of enculturation and proof that have been discussed over the last decades, we describe aspects of proof and proving that may span a first dimension of a reflection frame on the enculturation function. We combine these aspects with a second dimension that connects to the general education discourse on science propaedeutic competence. Finally, we suggest some preliminary pedagogical implications.

A brief overview of theoretical lenses on enculturation via proof and proving

Proof and cultural artifacts

One possible view of mathematical proofs is to consider them as texts, i.e., as configurations of sentences constructed and used according to certain rules. How might such objects serve an enculturation function? One possible answer employs the concept of cultural artifact. We can consider a (not necessarily formalized) proving system of (syntactical) schemes and manipulation rules as a cultural artifact in an analogous way to Mariotti's use of the term in her work on Cabri (Mariotti, 2000), namely as "an object constructed with the aim of achieving a result [...], and for this reason it was designed to incorporate a certain knowledge [...]" (ibid., p. 36). In the case of a proving system, the result is an appropriate derivation of a mathematical proposition, and the system incorporates knowledge about how to achieve this result in the form of "utilisation schemes" that are available within it. In constructing a proof as a textually composed output within a proving system, the system can serve as a semiotic mediator with regard to meaning construction. Semiotic mediation is an integral part of an enculturation function, especially in the context of learning and teaching mathematics: "On the one hand, meanings emerge from the activity – the learner uses the artefact in actions aimed at accomplishing a certain task; on the other hand, the teacher uses the artefact to direct the development of meanings that are mathematically consistent." (ibid., p. 37). The analogy can even be drawn more closely for the case of computer-based proving environments (such environments have recently drawn attention in mathematics education, see Hanna et al., 2019), where we can quite straightforwardly recover Mariotti's notion of phenomenological experience created by "actions of the user and feedback from the environment, of which the artefact is a component" (Mariotti, 2000, p. 37). We complement the view of enculturation via proof under Mariotti's perspective of cultural artifact with the view of proof as genre, which connects to another notion of cultural artifact.

Proof as a genre and genre as social action

Selden and Selden (2014) brought up the idea of considering mathematical proofs as a genre, or rather the types of proofs in mathematical practice as a set of different proof genres. However, if we examine the notion of genre more closely, we return to the notion of cultural artifact as used in the field of Rhetoric. Here, 'genre' is conceptualized "as a conventional category of discourse based in large scale typification of rhetorical action; as action, it acquires meaning from situation and from the social context in which that situation arose" (Miller, 1984, p. 37). Moreover, genre is seen as a cultural artifact, a "bearer of culture", "as a product that has particular functions, that fits into a system of functions and other artifacts" (Miller, 1994, p. 69). Finally, genre rules of proof "form a normative whole that we can consider as a cultural artifact, that is, a representation of reasoning and purposes characteristic of the culture (Miller, 1984, p. 38). Miller's approach in Rhetoric Discourse to the concepts of genre and cultural artifact can be adapted to support the importance of social action for an enculturation function of mathematical proofs. Participation is a crucial aspect of enculturation: "For the student, genres serve as keys to understanding how to participate in the actions of a community." (ibid., p. 38f.). Miller's approach also indicates that enculturation through proofs occurs at the 'genre level', i.e., including social norms and rules such as the various written and unwritten rules for producing a proof that the mathematical community will accept or even view as persuasive, explanatory etc. Therefore, enculturation through mathematical proof cannot occur exclusively at the

level of formal reasoning and not merely through learning syntactic rules and the formal distinction between premises and conclusions. As Miller notes, “the failure to understand genre [hence proof; the authors] as social action” has implications for learning processes, “it turns what should be a practical art of achieving social ends into a productive art of making texts that it fulfills certain formal requirements” (Miller, 1994, p. 67). Essentially, learning proof as a genre provides the learner with an understanding of what appropriate goals and purposes we have with proofs and why we have them.

Approaching proof in a community of practice

Meaning construction and participation in the actions of a community of practice as aspects of enculturation are also central to Hemmi’s (2006) understanding of learning as “social participation in practices of social communities” (ibid., p. 33). Hemmi examines how students encounter mathematical proofs as members of a mathematics department viewed as a community of practice. Drawing on sociocultural theories of learning, she describes and analyzes how students negotiate the meaning of mathematical proofs while participating in this community. In line with the considerations presented above, Hemmi considers proofs as artifacts: “Access to artefacts both through their use and through understanding their significance is crucial to facilitate students’ access to the practice of mathematics” (ibid., p. 39). Accordingly, students become part of a community of practice by working with the artifact of proof while negotiating its meaning.

Socio-mathematical norms and norms in the learning and teaching of proof

The negotiation of meanings and norms that enable and regulate participation in the proof-related actions of a community of practice has already been highlighted by Yackel and Cobb (1996). They describe how mathematical norms are established in the classroom. From their constructivist perspective, proof learning can be viewed as enculturation, which in turn is a core part of enculturation. The ‘regular’ use of mathematical proofs is seen not only as a goal of this enculturation process but also as the primary tool for developing the process. A central component of this view is that the appropriate regulative socio-mathematical norms are constructed ‘in action’ through a process of negotiation between the students and the teacher. Applying the concept of socio-mathematical norms to the case of proof learning thus fits quite well with the idea of considering mathematical proof as a cultural artifact. Studies such as Blanton & Stylianou (2002) point to how a ‘genre level’ of proof and proving manifests itself in the classroom. Here, norms are established, e.g., for what argument can count as proof, what representations are allowed for constructing the proof, and what steps of the proof must be presented or can be omitted. From a constructivist and sociocultural perspective, the ‘genre level’ of proof is constructed within and as a result of the process. As for the genre level established in teaching, proofs have other constituents besides premises, conclusions, intermediate statements, and rules of inference. They can be viewed as texts presented through certain means of representation and interpreted by members of the actual classroom community. Both presentation and interpretation are meaning-constitutive constituents of any proof or proving action.

Meta-discursive rules

The view of mathematical proof and proving as a discourse extends and complements the view of proof as a cultural artifact. According to the discourse view, enculturation through proof is reflected in successful participation in the discursive process of proving. Such a view culminates in Sfard’s

theory of mathematical thinking as communicating. Sfard's theoretical framework can be used to develop the idea of recognizing discursive proving actions (Müller-Hill, 2013) as the true mediators of enculturation. A key concept in Sfard's approach is the notion of a meta-discursive rule. Meta-discursive rules can be understood as *subconscious regulators* of discursive choices and are responsible for both the manner and the possibility of successful communication (see, e.g., Sfard, 2001). Sfard's commognition perspective highlights meta-discursive rules as *tacit* meaning-constitutive parts of proof and proving actions. It explains why the implicit communicative rules governing proving activities in a mathematical community of practice „form a normative whole that we can consider a cultural artefact, that is, a representation of reasoning and purposes characteristic of the culture” (Miller, 1984, p. 38). It suggests that in the case of proof, we have a situation similar to that of genre: „ what we learn when we learn a genre is not just a pattern of forms or even a method of achieving our own ends. We learn, more importantly, what ends we may have” (ibid., p. 38).

Synopsis concerning the enculturation function of mathematical proof

Our synopsis leads us to three principal “proof as a”-aspects: as text, as activity, and as discursive process. The goal of this consideration is not to classify concrete proof occurrences under one of these aspects as unambiguously and disjunctively as possible. Instead, it is a matter of explicitly thinking about and addressing all these three intertwined aspects in every concrete proof or proof-related event that is questioned concerning its accompanying enculturation function. Certain features of a concrete proof or proving action, e.g., the choice and use of symbolic notation, can be reasonably considered under more than one of these three aspects. *Proof as a text* (orally or written) encompasses linguistic and pragmatic features of proof as a product, such as notation, representations, rhetoric, vocabulary, grammar, and structuring features related to a formal or informal framework, including rules of inference, axioms, and definitions. It also touches on procedural aspects such as reading and understanding a given proof text. *Proof as discursive process* highlights the process of developing and communicating a proof, the argumentative and negotiable nature of proof and proving and critical issues of participation and normativity. Meaning is thereby socially constructed and further developed through subjective acquaintance and intersubjective establishment of norms and rules for discursive proving processes within a mathematical community. Finally, *proof as an activity* intimately connects both a product perspective with a process perspective, and social aspects of proof and proving with a view on individual proving action. It is characterized by appropriate actions and utilization schemes for mathematical proof and proving ‘in action’, complemented by proper motives and goals pursued. In this sense, the meaning of proof arises from the performance and reflection of concrete, goal-directed proving actions driven by genuine motives for proof and proving.

A reflection frame for the enculturation function of mathematical proof

We propose a two-dimensional framework of reflection. One dimension is a developmental dimension of enculturation through proof and proving. We choose science propaedeutic competence as a central element to bring a developmental dimension into play. Mathematical proof is a core element of mathematics as a scientific discipline. Therefore, a general model for the development of scientific propaedeutic competence should be a reasonable candidate for a developmental dimension of enculturation through proof and proving. According to Huber (1997, p. 348), general science propaedeutic competence comprises three levels (these levels are not to be understood as distinct

stages of development, though, and we will not use them in this sense). Level (1) is “meta-scientific knowledge in the sense of learning basic scientific concepts, structures, and methods”: For the case of proof and proving, this level is about accumulating knowledge about basic concepts, structural and functional components, quality criteria, and knowledge about different functions proofs might fulfill. Level (2) is “the formation of an attitude towards the scientific expert culture”: An attitude of the learning subject toward mathematical proofs and proving is formed in the course of the development of appropriate attitudes, values and beliefs, as well as personal affects toward the mathematical endeavor and the associated habits of reasoning, justifying, and questioning. It includes motivational, social, and ethical aspects. Level (3) is “critical reflection of scientific knowledge and methods in larger contexts”: A growing ability to reflect on mathematical proofs in larger contexts can be described as analyzing and reflecting on historical and current conditions and constraints of proving, including inter- and transdisciplinary issues. This is concerned, e.g., with the relativity of the concept of mathematical truth and the corresponding role of definitions and axioms, etc. We combine Huber’s levels with the “proof as a”-aspects as a second dimension for our reflection frame (Table 1).

Table 1: Reflection fields on enculturation via proof and proving

	meta-scientific knowledge	formation of attitude	critical reflection
proof as text			
proof as activity			
proof as discursive process			

The three “proofs as a-” aspects allow for a differentiation of the given levels: *Meta-scientific knowledge* about proof as a text specifies knowledge about the distinction between assertions and premises, axioms, theorems, and definitions, about admissible rules of inference, and appropriate representations. Meta-scientific knowledge about proof as an activity means knowledge about acceptable ways and means of proving, such as heuristics, algorithmic techniques, and proof principles in their application. It also means knowing about the motives, goals, and purposes of such modes of action, both ‘local’ goals (verification, persuasion, explanation, communication, aesthetics, discovery, conjecture) and ‘global’ motives (systematization, theory building, and explanation in the objective sense of unification). Knowledge about the motives of proving activity also includes getting acquainted with so-called fundamental ideas of mathematics. Finally, meta-scientific knowledge about proof as discursive process complements the development of meta-scientific knowledge with knowledge about basic language patterns and registers and about (meta-)discursive rules and roles in (different stages of) formal and informal discursive proving processes. *Forming an attitude* toward proof as a text emphasizes the conscious evaluation of notation, clarity, rhetorical devices, choice of representations and definitions, and the sense-making structure of a proof. The formation of an attitude towards proof as an activity involves the growing appreciation and adoption of its values, motives, and goals. The formation of an attitude toward proof as discursive process can develop through experiencing and recognizing the value and merits of the discursive nature of proof, as, e.g., in collaborative work processes on mathematical proof problems, and the contribution of this discursive nature to the nature of mathematics as a human endeavor. Regarding the *critical reflection* of mathematical proofs as a text, this may mean thinking about the context-dependence of proof formats and proof standards concerning the degree of formalization and completeness, about the

influence of the choice of axioms and definitions concerning the expansion and stability of the mathematical body of knowledge, and the cultural imprint and social impact of this textual genre. Critical reflection on proving as an activity focuses on the motives and goals of proving itself and on reflecting on and questioning the effectiveness and appropriateness of the chosen courses of action in the context of proving. Critical reflection on proving as discursive process complements the other two perspectives by reflecting on discursive facets of the construction of mathematical knowledge, including participation issues, disciplinary features, error recovery and peer review culture.

Pedagogical implications of the enculturation function of mathematical proof

One implication applies to the general awareness that every mathematical proof in a learning process deploys some enculturation effect. However, the enculturation function might be considered a latent function. By this latency, we mean that observable effects on the learners' enculturation process are delayed compared to actual exposure to a particular proof or proof action and that what is conveyed to the learning subject through proofs or proof actions about the norms and values of a mathematical community tends to be obscured. Thus, corresponding changes in the learning subject's (re)actions tend to be unconscious. Despite or perhaps because of this presumed latency, an enculturation function of mathematical proof is particularly relevant for the teaching and learning of mathematics. We highlight the need to make motives, ways and goals of actions, norms and rules for constructing and using proof visible and accessible to learners (Dawkins & Roh, 2016). Thus, implications include the importance of engaging students in proving tasks and explicitly reflecting and communicating about the use of proof. Teaching the nature of mathematical proofs should be done in the context of concrete proving discourses. We want to suggest two lines of further thinking and investigation.

Imitation and simulation

We experienced that in learning contexts proving happens to be taught in a so-called "as if" mode, as simulation: "simulation is as if real, contained within a purely artificial world with arbitrary and distant connections to the real" (de Castell et al., 2014, p. 340). If learners experience a discursive character of the teacher's proving action (at all), it might rather be perceived as an 'abstract discourse' which has the teacher as the only participant and which is predetermined by the necessities of algebraic transformation rules, or as an orally accompanied unveiling of an already finished proof product than a discursive process in which the proof is being developed. It might not be perceived a social discourse that has specific discursive goals like persuasion, which is rhetorically orchestrated by the teacher (Gabel & Dreyfus, 2017) and in which learners can actively participate to develop an understanding of the extent to which the relationship between certain actions of the teacher and their proof-constituting role is not merely a semantically arbitrary one. In contrast, with a view to enculturation it seems necessary to conceptualize learning environments as including concrete proving actions in a "just-like" mode, that is, as imitation (de Castell et al., 2014, pp. 337ff.): such actions should allow for a mimetic engagement of learners as a basis for developing an understanding of proof as discursive process across all three levels. Single learning environments or activities can hardly achieve this; an accompanying prerequisite is that mathematics teaching represents a reality of reasoning and proving in a mathematical community that is accessible to learners.

Individual learning, coherence, and participation issues

Shifting the focus to the learning individual, we view enculturation through proofs and proving as a long-term process that begins with exposure. During this process, the individual makes (possibly fragmentary and accidental) experiences with mathematical proof and proving on various occasions. A core factor of successful enculturation processes can then be understood as the coherence of the body of experience related to proof and proving that the learning subject can make (compare Stylianides and Stylianides, 2022). Thus, if we take the enculturation function seriously, learning activities related to proof and proving should be carefully designed and implemented regarding the learner's ongoing enculturation process and the anticipated extent of enculturation that the individual is expected to achieve. Coherence, as an aspect of meaning, is not self-generating in this process and must be actively constructed by the individual in interaction and negotiation with its sociocultural environment, which is particularly challenging in heterogenous learning groups. For example, the goals and expectations of university-teaching research mathematicians regarding student enculturation appear to be markedly different for students who want to become research mathematicians than for students who do not (e.g., Müller-Hill, 2013). The latter will not actively participate in the research community, yet they are supposed to be educated to be representatives of their discipline in different contexts. However, students who are not sufficiently enculturated will not have equal opportunities to participate in mathematical discourse and will not identify with mathematics as a disciplinary community of practice. For prospective mathematics teachers as multipliers of their subjective views, such issues are particularly sensitive, affecting not only self-efficacy beliefs, but also (epistemological) beliefs about the nature and cultural contingency of proof.

We advocate research on levels of maturity of enculturation through proof and proving, on the role of individual background experiences and belief systems in guiding the teaching and learning of proof, and on course designs that address specific issues related to the enculturation function of proof in teacher education (see, e.g., Bauer & Müller-Hill, 2023). Our reflection frame can serve as a basis for thinking about such issues and combine, complement, and reflect on existing approaches.

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