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Establishing a top-down approach to recognizing professional teacher knowledge regarding probability: A didactically oriented reconstruction of the laws of large numbers

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This contribution focuses on recognizing professional teacher knowledge regarding probability, specifically the empirical, weak, and strong law of large numbers. The author utilizes a subject matter analysis, called “didactically oriented reconstruction” (DOR), to identify the necessary knowledge elements for successful teaching. The findings reveal the importance of stabilisation as a unifying idea, and knowledge elements are presented using a knowledge network. The author notes the lack of systematic knowledge requirements for teaching probability, specifically the law of large numbers. Overall, this paper aims to establish a clear and comprehensive top-down approach to the necessary knowledge for effective math teaching.

Keywords: Probability, teacher education, knowledge base for teaching, secondary school teachers, curriculum development.

Introduction

Felix Klein (1908), a German mathematician, already recognized the need for elementary mathematics from a higher standpoint in teacher training over a century ago. He argued that school mathematics and academic mathematics seemed detached, leading to a double discontinuity in teacher education. Unfortunately, this double discontinuity is still present in teacher training for secondary schools today, and it is often not clear what knowledge teachers need to acquire and to what depth in order to be able to teach mathematics successfully.

One of the important concepts that teachers need to know for teaching probability and statistics is the empirical law of large numbers (EmpLoLN). The empirical law of large numbers was introduced by Bernoulli as he stated that even the “stupidest man knows by some instinct of nature per se and by no previous instruction” (G. Gigerenzer et al., 1989, p. 29) that the greater the sample, the surer the conjecture. As the empirical law of large numbers is an important concept for understanding statistics and probability theory, teachers should know this “by heart”, so they will be able to teach it effectively. However, the relevance of the weak and the strong law of large numbers for teachers is again hardly proven. The weak law of large numbers (WeakLoLN) is defined by Athreya and Lahiri (2006, p. 239) as followed:

Let $\{Z_n\}_{n \geq 1}$ be a sequence of random variables on a probability space (Ω, \mathcal{F}, P) . Recall that the sequence $\{Z_n\}_{n \geq 1}$ is said to *converge in probability* to a random variable Z if for each $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|Z_n - Z| \geq \varepsilon) = 0$$

The strong law of large numbers (StrongLoLN) is defined by Athreya et al. (2006, p. 240) as followed:

A sequence $\{X_n\}_{n \geq 1}$ of random variable on a probability space (Ω, \mathcal{F}, P) is said to obey the strong law of large numbers [...] with normalizing sequences of real numbers $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ if

$$\frac{S_n - a_n}{b_n} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ w.p. } 1,$$

where $S_n = \sum_{i=1}^n X_i$ for $n \rightarrow \infty$.

The differences between mathematics at the school level and at the academic level are illustrated through these three different laws of large numbers. The empirical law of large numbers has an informal nature and cannot be proven. It is a natural law. The weak and strong laws of large numbers follow an abstract axiomatized logic that corresponds to mathematics at the academic level. To help bridge this gap in teacher knowledge, this paper aims to recognize teacher knowledge using a subject matter analysis called “didactic oriented reconstruction” (DOR) based on Kirsch’s (2000) aspects of simplification. This teacher knowledge is being proposed as an elementarised academic knowledge for teachers, which is a specialised knowledge that teachers must have acquired in their subject-specific training in order to demonstrate the necessary subject-specific depth for teaching mathematics and acquiring new knowledge. The paper begins by examining teacher knowledge, refining this definition for teacher knowledge based on the theoretical framework and gives a brief overview on the methodology and the research interest of this paper. Subsequently, the paper presents selected results of the didactically oriented reconstruction on the basis of the unifying idea of stabilisation. For this purpose, knowledge elements are presented as a knowledge network. The network demonstrates how those different law of large numbers in probability and statistics are connected to each other, and how these concepts can be simplified to help teachers better understand them.

Theoretical Framework

There is a considerable amount of literature on the conceptualisation and operationalization on teachers’ professional knowledge (e.g. Heather C. Hill, 2007; Blömeke et al., 2010; Krauss et al., 2008). Lee S. Shulman (1986a; 1986b; 1987) stated several categories of teacher knowledge. As Dreher et al. (2018) pointed out it “is not clear how to conceptualize and operationalize the construct of teachers’ professional [content knowledge]” (p. 321). For example, Krauss et al. (2008, p. 876) declared content knowledge as “a teacher’s understanding of structure” and pointed out possible notions of “content knowledge”, while using (3):

- (1) The everyday knowledge that all students should have.
 - (2) The school-level mathematical knowledge that good students have.
 - (3) Mathematical knowledge as a deep understanding of the contents of secondary school mathematics curriculum.
 - (4) The university-level mathematical knowledge that does not overlap with the content of school curriculum (e.g. Galois theory or functional analysis)
- (p.876)

In Germany, teachers for upper secondary level are supposed to gain mathematical content knowledge at mathematical courses aimed at not only teachers but also at students studying mathematics. As described above there is a significant difference between school mathematics (level (1) to partially level (3)) and academic mathematics, which prospective teachers learn at university. However drawing the line whether knowledge is located on level 3 or level 4 is not that clear. There hasn’t

been much research on how much content knowledge a mathematics teacher needs for successfully teaching mathematics.

Another approach to conceptualise subject knowledge is school-related content knowledge (Dreher et al., 2018). School-related content knowledge (SRCK) stands in between academic mathematics and school mathematics and is defined as “1) knowledge about the curricular structure and its legitimation in the sense of (meta-) mathematical reasons as well as knowledge about the interrelations between school mathematics and academic mathematics, and its 2) top-down- and 3) bottom-up directions” (Dreher et al., 2018, p. 330). The question of what expertise is needed by mathematics teachers by combining academic expertise with the SRCK is answered. But here, too, the question of how much (academic) expertise is needed for successful practice remains unanswered. Furthermore, the expert knowledge or the SRCK with regard to stochastics is not discussed in detail.

The lack of cautiously conceptualizing teacher knowledge can also be found in the content area in probability. Often researchers approach their conceptualization from bottom-up as in “beginning with [the] practice [of mathematics teaching] (Loewenberg Ball et al., 2008, p. 395). Whilst we do find results of what teacher know in probability (such as Batanero et al., 2014; Muñoz-Rodríguez et al., 2020; Policastro et al., 2019), it does not tell us in a systematic way, how much knowledge teachers should have. As Stohl (2005) pointed out, “teachers may misinterpret the law of large numbers and misguide students” (p.348), it does not seem clear, what teachers need to know about the empirical law of large numbers.

As mentioned above, this paper uses the term “elementarised academic knowledge for teachers” for content knowledge. This elementarisation mentioned above is defined by Griesel (1974) as follows:

It is a matter here of transforming mathematical content and theories down to a lower lever with the aim of adapting them to the learners’ lever of intellectual development. [...] Elementarisation is partly associated with reductions, partly with expansions (p. 117).

Elementarisation therefore does not only mean the omission of content or structures, but also the addition of (didactic) elements so that a topic content becomes comprehensible for the target group. Therefor, this elementarised academic knowledge can also be assessed in terms of its knowledge dimensions, which are defined below:

Table 1: Differentiation of different knowledge dimensions

Name	Definition	Characterisation
School content knowledge (SK)	SK is content knowledge that is taught in school under consideration of learning objectived.	The contents of learning mathematics are not just simplifications of mathematics as it is taught in universities. The school subjects have a life of their own” with their own logic; that is, the meaning of of the contents cannot be explained simply from the logic of the respective scientific disciplines. [...] Rather, goals about school (e.g. concepts of general education) are integrated into the meanings of the subject-specific contents (Bromme, 1994, p. 74).

School-related content knowledge (SRCK)	SRCK is the interrelation of mathematics as a school subject and as a scientific discipline (see above).	SRCK as a special kind of mathematical content knowledge about interrelations between academic and school mathematics (Dreher et al., 2018, p. 329).
Academic content knowledge (AK)	AK is content knowledge required to pursue an academic career in mathematics.	AK is characterized by pure academic mathematics detached from the curriculum.

Elementarised academic knowledge is therefore determined as SK and SRCK (see table 1). In this paper the didactically orientied reconstruction as a theoretical top-bottom-approach is used to generate knowledge elements either assigned to SK, SRCK or AK. This approach is described in the next section.

Approach of the analysis

Subject-related approaches played an important role from the beginning of the development in mathematics education in Germany (Hefendehl-Hebeker et al., 2019). One of those approaches for developing essential concepts for teaching and learning mathematics is a subject matter analysis. As many mathematics education researcher in Germany have done them in the last centuries, less is known *how* to approach a subject matter analysis in a systematic way. The aim of this paper is the normative generation of knowledge elements on the basis of a didactic analysis of the material. For this purpose, the didactically oriented reconstruction was developed and briefly explained in the following (Huget, 2022).

As indicated in Figure 2 with the arrows in the background, two major processes takes place within the didactically oriented reconstruction. Starting from an objective under consideration of a target group and a possibly division of the larger mathematical content area into smaller areas, the passage through the five steps means a didacticisation of a mathematical content. In step 1 of the didacticisation, only the mathematical content is considered, detached from the geneticisation and the possible applications. Central concepts, fundamental structures, properties, definitions, theorems as well as proofs and proof ideas are to be given. In the second step, possible reference systems of the mathematical content are included, so that applications are possible. The didactic possibilities and also the historical-genetic developments are described. From step 3 of the didacticisation, further didactic considerations are made. The target group is considered and, based on it, the required pre-service knowledge is given. In step 4, different levels of representation are considered and necessary basic concepts are specified. In the case of the basic concepts, not only basic concepts that already existed in the literature can be specified. The process up to step 4 can also indicate new basic concepts, although basic concepts are not the focus of this work. In step 5, further basic didactic concepts are integrated. Here, an extension by specific didactic concepts used in the corresponding sub-area is possible. For the area of probability theory, probability concepts are an example of basic didactic concepts (Huget, 2022). After the didacticisation has been carried out for each smaller unit, the interconnections between the sub-areas within each step, but also with regard to the whole process,

are shown and reflected upon based on categories by Kattmann et al. (1997). Here, differences and commonalities may emerge that would not arise within a smaller sub-area. This is done by reconstructing with the support of the four different categories: Peculiarities, Commonalities, Diversities and Limitations. In the process, concepts or elements are discarded due to lack of relevance, commonalities and dissimilarities are identified, and then possible limits and possible applications are reflected upon. Following this, normative statements are generated by assigning knowledge elements to dimensions of knowledge (Huget, 2022).

This method was applied to the laws of large numbers mentioned in the introduction. The empirical, the weak and the strong law of large numbers were analysed with the aim of generating knowledge elements for teachers.

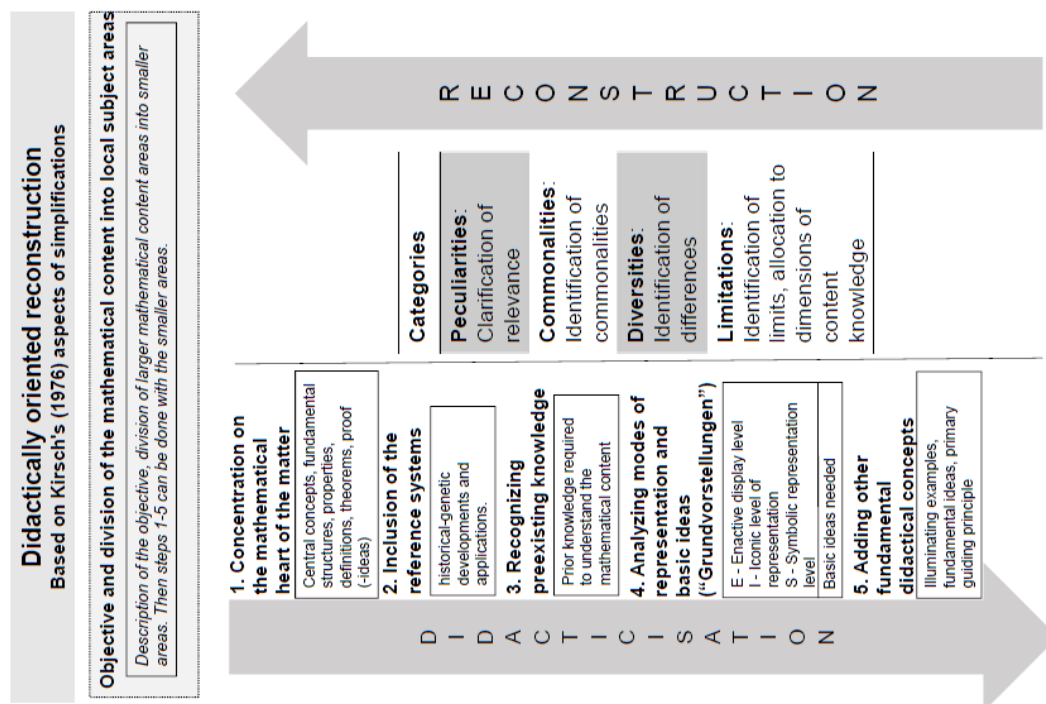


Figure 2: Knowledge network regarding the connected idea of stabilization (Huget, 2022)

Results of the analysis

After carrying out the didactically oriented reconstruction, various observations can be made. Specific didactic concepts in the field of statistics were taken into consideration. The implementation of didacticisation produces a structured list of subject-specific and subject didactic aspects of a mathematical subfield. Unifying ideas can be generated that may be relevant for understanding the topic area. In the reconstruction, knowledge elements are formulated without weighting, taking into account the objectives. The comparison of similarities and differences has shown that there are connecting ideas that allow knowledge elements to be presented in a networked manner.

One unifying idea is the idea of stabilisation, which was generated when comparing the results from the didacticisation. Around this, various knowledge elements can be assigned, which are interlinked

by this idea. As shown in Figure 3, knowledge elements can be found for each law of large numbers, which are also interconnected with the support of the ideas. In this example, knowledge elements relevant to teachers can be identified. The element "Knowledge of stabilization of relative frequencies of an observed event with encreasing number of trials as empircal law of large number" is assigned to the school content knowledge (white background in Figure 3). The knowledge element "knowledge of the lack of mathematizability of EmpLoIN and designation as a law of nature" is connected to the other laws of large numbers through the knowledge element "knowledge of mathematizable statement for stabilization". In the case of the weak and strong law of large numbers, the limits of teachers' knowledge become apparent because they only need a little knowledge to formulate the laws of large numbers and the exact meaning of stochastic and almost certain convergence for this connecting idea of stabilisation. These are assigned to academic content knowledge (s. dark grey background in Figure 3).

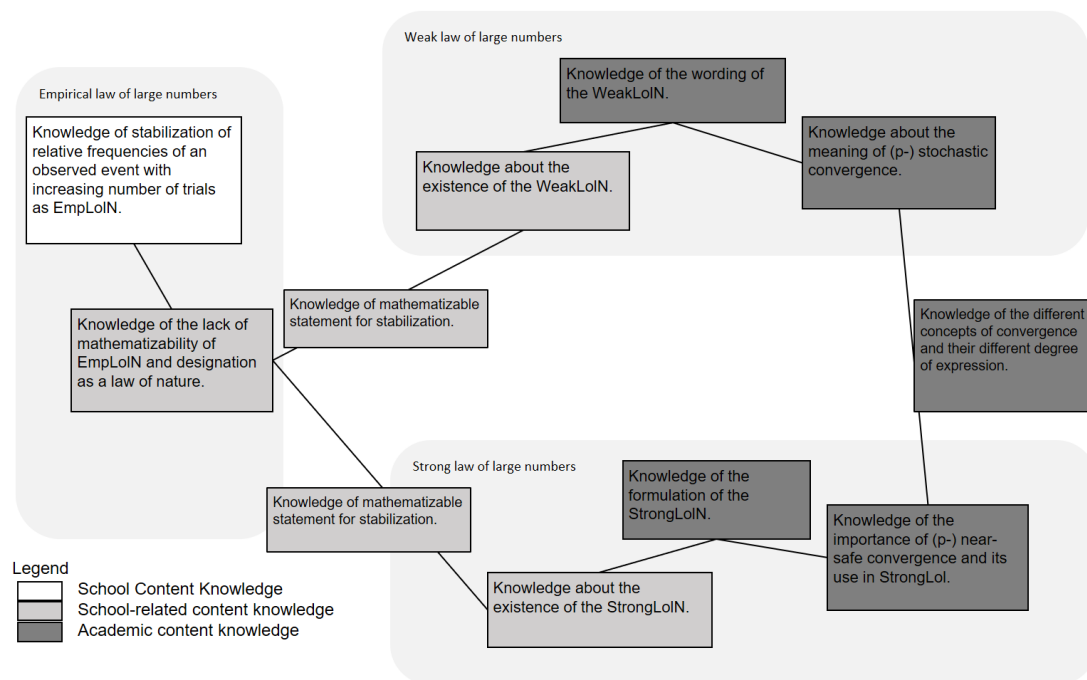


Figure 3: Knowledge network regarding the connected idea of stabilization

Conclusion

Conducting a didactically oriented reconstruction can lead to the identification of potential knowledge contents for teachers regarding the laws of large numbers. These knowledge contents can be organized by knowledge networks and connected ideas, providing a more comprehensive understanding of the subject matter.

It is important to note that there are specific elements of knowledge that are relevant for teachers to effectively teach the laws of large numbers. These were identified as elementarised academic knowledge. Furthermore, the relevance and depth of this knowledge may vary depending on the phenomena being focused on. This has implications for teacher training, particularly in Germany where stochastic training is still heavily focused on academic mathematics, with less emphasis on a specific content knowledge. As a result, changes in teacher professional education may be necessary

to ensure that teachers are equipped with the appropriate knowledge and skills to effectively teach stochastic concepts. However, the specific content-related changes required in teacher training are currently unclear and require further research.

It is important to acknowledge that the arrangement of knowledge elements and their assignment to knowledge dimensions in the didactically oriented reconstruction are normative and require empirical validation. Nevertheless, the transparency of the normative decisions made during the reconstruction process allows for others to follow and assess the validity of the resulting knowledge contents.

Overall, conducting a didactically oriented reconstruction can provide valuable insights into the knowledge contents necessary for effective teaching of stochastic concepts, and can guide the development of teacher training programs that better prepare educators for the classroom. However, further research is needed to validate and refine these findings.

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