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Andrea Hofmann, Trond Stølen Gustavsen. Opportunities for proving the rule for fraction multiplication – episodes from a task-based interview. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04410923

**HAL Id: hal-04410923**

**<https://hal.science/hal-04410923>**

Submitted on 22 Jan 2024

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# Opportunities for proving the rule for fraction multiplication – episodes from a task-based interview

Andrea Hofmann<sup>1</sup> and Trond Stølen Gustavsen<sup>1</sup>

<sup>1</sup>University of South-Eastern Norway, Norway; [andrea.hofmann@usn.no](mailto:andrea.hofmann@usn.no)

*The data presented here is part of a study in which we conducted six task-based interviews with students in grade 8. We explore opportunities for a grade 8 student to develop a proof of the rule for multiplying two proper fractions in a task-based interview involving paper folding while being provided with ample guidance. Our analysis focused on to what extent the arguments produced by the student can be classified as proofs, and in what way guidance by the interviewer can lead to the development of a proof. Based on our findings we suggest that it is important to focus on verbal explanations and to discuss generic examples in the development of a proof. We also propose that insight gained from this task-based interview can inform future interventions, including whole-class interventions with proof-based teaching.*

*Keywords: Fractions, proving, task-based interview, proof-based teaching.*

## Introduction

Proofs have central roles in mathematics, verification and explanation of results being two of them (De Villiers, 1990). While in the new Norwegian curriculum, that has been implemented from the school year 2020/21, “Reasoning and argumentation” is one of six core elements (Ministry of Education and Research, 2019), the process of proving is not fully integrated into the primary and lower secondary school curriculum. Research indicates that students encounter difficulties in understanding proof and proving and providing valid arguments to establish the truth of a mathematical claim (Aricha-Metzer & Zaslavsky, 2019). Additionally, research suggests that multiplication of fractions is one of the most challenging concepts for students in elementary and middle school (Rule & Hallagan, 2006), which is one of the reasons we selected it as the mathematical content in our study. To gain deeper insight into students’ reasoning processes, we conducted task-based interviews with students in grade 8, using tasks previously developed for a proof-based teaching intervention. Proof-based teaching is an approach to teaching mathematics in which students learn new content while conjecturing and proving statements (Reid & Vallejo Vargas, 2019). Before conducting the task-based interviews, we implemented an intervention where students collaborated in groups to prove the rule for multiplying two proper fractions. We analyzed video recordings of the students’ group work, with a focus on their interactions with representations (Gustavsen & Hofmann, 2023). In this paper we aim to examine the opportunities and challenges that emerge when a student is provided with ample guidance in developing a proof during a subsequent task-based interview. Our analysis shifts from the use of representations in the classroom intervention to the use of a framing theory in evaluating the student’s individual proving processes.

Our research question is: *Which opportunities and challenges can be identified in a task-based interview where an 8<sup>th</sup> grade student is guided in developing a proof of the rule for fraction multiplication?*

## Theory

### Fractions and paper folding

A common way to introduce fraction multiplication is by finding a fraction of a fraction (Tsankova & Pjanic, 2009). This relates to the operator subconstruct of the fraction construct (Behr et al., 1983). Paper folding can be used as a model for this subconstruct (Wyberg et al., 2011), and according to Adom and Adu (2020) the use of paper folding has a positive effect on students' performances in the topic of fractions. However, the translation from physical manipulatives to symbolic representations can be challenging for students (Behr et al., 1983). Furthermore, when solving tasks of multiplying two proper fractions, students can also use an area model (Tsankova & Pjanic, 2009), which can be linked to the measurement subconstruct of the fraction construct (Behr et al., 1983).

### Proof-based teaching

We adopt the definition of “proof” from Stylianides (2007): A *proof* is a

connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

Furthermore, we use the concept of “generic example” as in Balacheff (1988). In proof-based teaching, proving is integrated into the learning of mathematical content and becomes the means by which students develop conceptual understanding in mathematics. A *framing theory* specifies a “‘toolbox’ of statements accepted by the classroom community” (Reid & Vallejo Vargas, 2019, p. 809) and “the sequence of content to be learned so that it is possible for the learners to prove statements” (Reid & Vallejo Vargas, 2019, p. 809). It contains, among others, target theorems and target proofs.

### Proving in task-based interviews

In a task-based interview, the student interacts with an interviewer and engages in solving a mathematical task or a sequence of tasks that is carefully designed for the research focus one wishes to address (Goldin, 2000). Aricha-Metzer and Zaslavsky (2019) found out from their studies of task-based interviews that the interviewer and the nature of the examples provided played a crucial role in the students' developments of proofs. They also suggest “that more attention should be given to facilitating students' ability and inclination to build on the potential strengths of using examples for proving” (Aricha-Metzer & Zaslavsky, 2019, p. 321).

In our analysis of a task-based interview with proving activities, we use the concepts of *base argument*, *ensuing argument* and *proof threshold* as in Stylianides (2007). The base argument “represents the prevailing student argument at the initial stages of the proving activity” (Stylianides, 2007, p. 299), and it depends on the prior knowledge of the student. The ensuing argument is “the

principal argument that seems to result from, or mark the endpoint of, the instructional intervention” (Stylianides, 2007, p. 314). The notion of a proof threshold refers to the point at which the argument meets the definition of a proof, i.e. the point at which the three components mentioned above (“set of accepted statements”, “modes of argumentation”, “modes of argument representation”) “meet the disciplinary standards for these components in the development of proofs” (Stylianides, 2007, p. 314).

## Methods

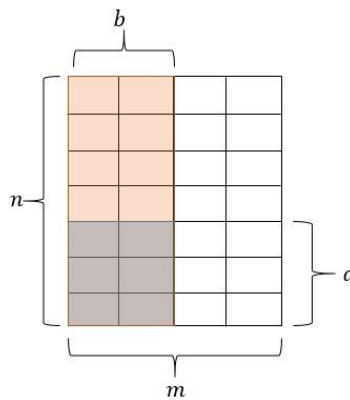
We conducted individual task-based interviews with six students in grade 8 (age 12-13), varying in length from 25 to 40 minutes. All six interviews were video-recorded. Our objective was to investigate what the student could achieve with the help of the interviewer. The interviewer’s role was two-fold: to facilitate the student’s reasoning processes, and to evaluate whether an argument given by the student qualifies as proof, as given by the definition above. The present study focuses on one student from the group examined in Gustavsen and Hofmann (2023). The second author was guiding this task-interview. We selected this particular student because he was central in the group’s process of making a poster with a proof of the rule for multiplication of two proper fractions. Based on the video recording of the classroom intervention, we hypothesized that this student had an understanding that he did not communicate, neither written nor orally, and that his argumentations had a potential for further development. The task-based interview approach, in which the student revisits the same tasks, offers a means of gaining a deeper understanding of the student’s reasoning processes (Goldin, 2000). For this reason we decided to use the same tasks, with the same target theorems in mind, as in the classroom intervention reported on in Gustavsen and Hofmann (2023), as a starting point, adapting the tasks to each student (Goldin, 2000), in the course of the interview. We were mindful that the student might remember procedures or results from the intervention.

### Target theorems

We use the operator subconstruct, meaning that  $\frac{a}{n} \cdot \frac{b}{m}$  is the same as  $\frac{a}{n}$  of  $\frac{b}{m}$  ( $a \leq n$  and  $b \leq m$  and  $n \neq 0$ ,  $m \neq 0$ ). The target theorems are a key component of the framing theory we developed for the proof-based teaching intervention. Specifically, we aimed for some students to be able to prove the following two theorems, probably by using a generic example.

Target theorem 1: Folding a (rectangular) sheet of paper into  $m$  equal parts in one direction and into  $n$  equal parts in the other direction, divides the sheet of paper into  $n \cdot m$  equal rectangles.

Target theorem 2: Consider a grid with  $m$  equal columns and  $n$  equal rows. Assume that  $b$  of the columns are colored. In the restricted grid consisting of the  $b$  columns, suppose that  $a$  rows are colored again. The part of the grid that is colored twice, is a representation of  $\frac{a}{n} \cdot \frac{b}{m}$  and  $\frac{ab}{nm}$ , and thus these two expressions are the same.



**Figure 1: Visualization of target theorem 2**

We designed “successive tasks of increasing complexity, culminating in one or more questions that challenge the most able of the subjects in the study population” (Goldin, 2000, p. 323). In the beginning the student was asked to fold fractions, for example  $\frac{1}{2}$  and  $\frac{1}{3}$ . Then the interviewer introduced fraction multiplication tasks, such as  $\frac{1}{4} \cdot \frac{1}{3}$  and  $\frac{5}{8} \cdot \frac{1}{3}$ , that the student was supposed to represent with paper folding, before going over to drawings as representations and, finally, a proof of the rule for fraction multiplication. We used paper folding as a representation of a grid, which for instance means that we represented the restricted grid with  $b$  columns, as in Target theorem 2 above, by folding a paper into  $m$  equal parts in one direction, representing columns, and then folding the paper such that  $b$  columns were visible. These  $b$  columns were then colored. The  $n$  rows were represented by folding the paper into  $n$  equal parts in the other direction, and finally  $a$  of these  $n$  rows were colored again.

The task-based interview was structured in the sense that the interviewer followed a sequence of tasks that we had designed according to the framing theory with the two target theorems. The students had the opportunity to engage freely with the tasks, “in order to allow observation of their spontaneous behaviors and their reasons given for spontaneous choices before prompts or suggestions are offered” (Goldin, 2000, p. 323).

### **Analysis**

We transcribed the task-based interview, and we identified the student’s “base arguments” (Stylianides, 2007). We analyzed each argument in light of the three components of an argument, “set of accepted statements”, “modes of argumentation”, “modes of argument representation” (Stylianides, 2007), and compared each argument with the framing theory to decide whether it counts as a proof. We then analyzed how the interviewer interacted with the student when he did not accept the student’s base argument as proof. For instance, what did the interviewer do when the student’s argument did not qualify as a proof because the *mode of argumentation* was not valid, or what kind of *modes of argument representation* did he offer? In addition to the transcription, the written work of the student and the interviewer was also part of the analysis.

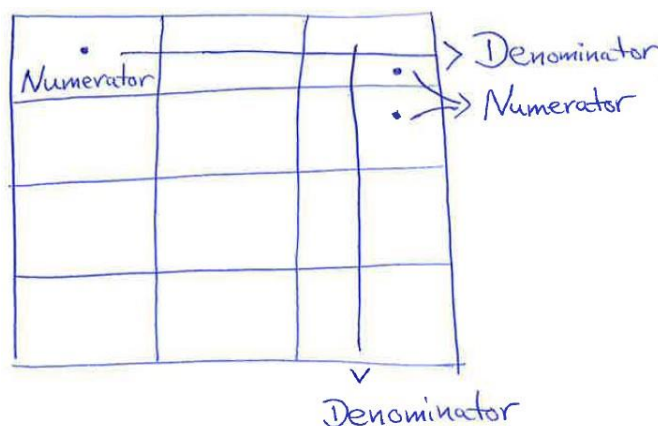
### **Findings and discussion**

We now present our findings. The main focus is on target theorem 2.

The ultimate goal of this sequence of tasks was for the student to provide a proof of the rule for multiplying two proper fractions. When asked to explain the result of a fraction multiplication with paper folding in examples with numbers, the student was able to do so. In one case, a little guidance from the interviewer was needed, but the student seemed to understand the operator subconstruct in relation to fraction multiplication in these examples (Behr et al., 1983). For instance, when asked what part  $\frac{5}{8}$  of  $\frac{2}{3}$  constitutes of a whole sheet of paper, the student was able to explain with the help of paper folding why it is  $\frac{10}{24}$ .

Once the student had folded and explained some examples with numbers, the interviewer asked for the general rule for fraction multiplication. The student was expected to give a proof of this rule in a way that another student in the same grade would understand *why* this rule is valid, emphasizing the role of proof in providing an explanation, as suggested by De Villiers (1990). In the following arguments, the student had to make a transition from representations with paper folding to drawing grids.

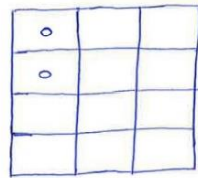
The student made a drawing (see Figure 2) and was able to explain where in the drawing “numerator times numerator” and “denominator times denominator” can be found.



**Figure 2: Translation of the student’s drawing (base argument no. 1)**

The interviewer asked whether the student believed that another student in the same grade would be able to comprehend the explanation. The student responded negatively, but asserted that he himself understood it. The interviewer then requested the student to improve the drawing so that other students could understand why the rule for fraction multiplication holds true. Based on the student’s engagement with the different tasks, it appeared that he had a good understanding of why the rule is true. However, being able to explain it in a clear and understandable way to another student requires a higher level of proficiency. We conclude that the provided argument is below the proof threshold (Stylianides, 2007), since it does not “meet the definition of proof on all three components” – “set of accepted statements”, “modes of argumentation”, “modes of argument representation” (Stylianides, 2007, p. 314). From the student’s second argument we can see that he had an understanding of what the numerator and the denominator in his drawing represent, as he said and wrote “numerator is how much you have” and “denominator is how many there are” (see Figure 3).

Numerator = How much you have  
 Denominator = How many there are



$$\text{Numerator} = x = 2$$

$$\text{Denominator} = \overline{12}$$

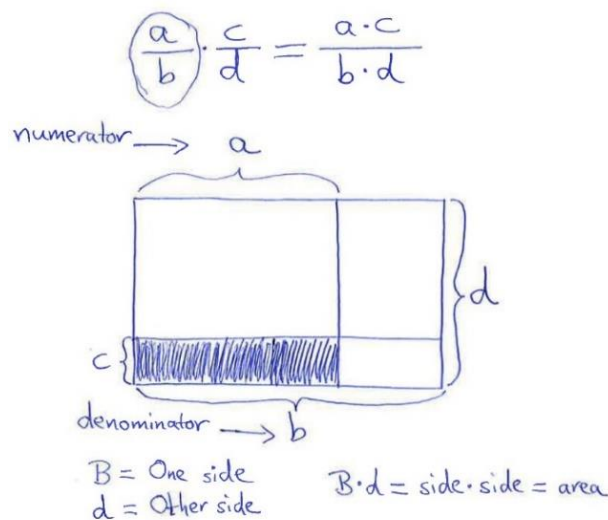
**Figure 3: Translation of the student's drawing (base argument no. 2)**

The interviewer then asked if the student could explain the drawing by using some examples.

- Student: Here is, we can say, denominator. Here it is twelve [points at the small rectangles in the drawing]. And, then it will be, numerator is  $x$  [writes].  
 Interviewer: Exactly. So how can I find  $x$ ?  
 Student: Then you take as many as you want to. And then, if I take one, two [draws], then it will be two. Then you have two out of twelve.

Since the student did not make explicit how this example could be considered as “a characteristic representative of its class” (Balacheff, 1988, p. 219), we do not characterize this as a generic example, and we do not consider the argument being above the proof threshold. We denote the two arguments of the student as base arguments (Stylianides, 2007). The interviewer remarked that the student had a very good explanation, but he did not accept this base argument as a proof and started to guide more explicitly towards a generalization.

The interviewer started making a new drawing, saying that he wished to represent  $\frac{a}{b}$  times  $\frac{c}{d}$  (see Figure 4). In a dialogue, the interviewer and the student agreed on what measures the variables  $a$ ,  $b$ ,  $c$  and  $d$  and the expressions  $b \cdot d$  and  $a \cdot c$  represented.



**Figure 4: Translation of the interviewer's and the student's drawing (ensuing argument)**

- Interviewer: And then the area of the part of that, which we want to find?  
 Student: This? [points on the drawing]  
 Interviewer: Yes.

Student: This is  $a$  times  $c$ .  
Interviewer: Exactly.  
Student: Because  $a$  is one side, and  $c$  is the other side.

The interviewer and the student agreed that fractions represent parts of a whole, and that in this context,  $b \cdot d$  is the area of the whole rectangle, and  $a \cdot c$  is the area that represents the part of the whole they wanted to find. In the following interaction, the interviewer and the student arrived at a common understanding that this explains the rule  $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ . An area model of fraction multiplication became explicit (Tsankova & Pjanic, 2009). The interviewer played a crucial role here in developing an ensuing argument that qualifies as a proof (Stylianides, 2007; Aricha-Metzer & Zaslavsky, 2019). One important feature was that the interviewer in his drawing introduced variables  $a$ ,  $b$ ,  $c$ ,  $d$  right from the start to ensure the generality of the argument that followed, and subsequently made sure that the argument is above the “proof threshold” (Stylianides, 2007).

## Conclusions and implications

From our analysis of this task-based interview we gained insight into how an instructional intervention by the interviewer can lead to the development of an argument that is above the “proof threshold” (Stylianides, 2007). Although the student demonstrated correct reasoning in examples involving numbers, and for instance showed understanding of representations of a numerator and a denominator in a grid, there seemed to be a threshold to express the full generality of the argument. One opportunity in this task-based interview was for the interviewer to introduce variables in the moment when the student had challenges to explain the full generality of his reasoning, and thus guide the student towards an argument that is above the proof threshold.

It seemed that the transition from paper folding as representation to drawing grids did not constitute a challenge for the student in itself. One implication for future interventions, also whole-class interventions, is on the other hand to spend more time on generic examples before moving to algebraic representations. Ideally, we would have preferred that the student had introduced variables without intervention by the interviewer. The student had some prior knowledge of variables, but he required guidance to apply them in expressing the argument in complete generality. Symbolic language, here in the form of variables, was a “mode of argument representation” (Stylianides, 2007) that the interviewer made available to the student.

Insights gained from the interviewer’s role in a task-based interview can inform our understanding of the teacher’s role in a whole-classroom lesson. Specifically, in a classroom setting, teachers play a key role in assessing whether a student’s argument meets the criteria for proof, and in guiding students in developing arguments that satisfy those criteria (Stylianides, 2007).

Since we could observe that interactions between the student and the interviewer contributed to formulating claims and developing arguments, as another implication we propose that it is important to explicitly provide ample guidance to students in formulating conjectures and arguments, both orally and written.



## Acknowledgment

We thank the teacher who arranged the task-based interviews with the students. We also thank David A. Reid, Sikunder Ali and Sigurd Hals for their support and contributions.

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