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# Aspects of representing in proof-based teaching of fraction multiplication

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*This small-scale case study investigates the possibility of proof-based teaching of core curriculum content. Through a series of tasks designed to facilitate students' formulation and proof of the rule for fraction multiplication, we examined how a group of grade 8 students engaged with different representations. Our analysis focused on how the students represented objects and operations, statements of properties and relations, and inference. Based on our findings, we suggest that proof-based teaching of fractions should emphasize full sentence verbal representations of the operator aspect of fractions, and of mathematical statements and inference.*

*Keywords: Fractions, representations, proof-based teaching.*

## Introduction

A mathematical proof establishes the truth of a statement through a sequence of deductions that are acceptable within the mathematical community. Proof is an important aspect of how mathematics is developed and practiced, and it is a key feature of the discipline that sets it apart from other fields of study. In primary school mathematics the role of proof is more limited, and it is not fully integrated into the curriculum. Proof-based teaching aims at extending the role of proof by facilitating a classroom situation where “students learn mathematics through explanatory proving that builds on a shared body of knowledge” (Reid & Vallejo Vargas, 2017, p. 235). In facilitating the learning of the curriculum, the aim is that “the learners themselves conjecture and prove important theorems in mathematics” (Reid & Vallejo Vargas, 2019, p. 809). In this small-scale intervention, we investigated how two classes of grade 8 students interacted with different representations related to fractions when working on a sequence of tasks designed in view of proof-based teaching.

Research indicates that a large portion of the students struggle with fraction tasks, which is unfortunate as students' understanding of fraction is important for their further learning of mathematics (Siegler & Lortie-Forgues, 2017). One reason for students' challenges with fractions is the complex nature of the fraction construct as composed by several subconstructs each with many possibilities for representations (Charalambous & Pitta-Pantazi, 2007).

Proving is central to the discipline of mathematics, and the learning and the teaching of proving has attracted considerable research attention over the past decades. Although the role of multiple representations of fractions (Behr et al., 1992), and the role of representations in proving has been addressed in the research (Stylianides, 2007), the role of representations in proof-based teaching of the rules in the fraction arithmetic has not been explicitly addressed. To understand more about how students are facilitated and challenged by a variation of representations in proof-based teaching, we designed a sequence of tasks where students are invited to (re-)discover, formulate, and prove the rule for multiplication of fractions. The purpose was for students to learn core curriculum content and at the same time engage in problem solving and proving which are both central in mathematics education. Our research question is as follows: *How do a group of Norwegian 8th grade students*

*engage with representations of objects and operations, statements of properties and relations, and inference when working on a sequence of tasks designed to facilitate the formulation and proof of the rule for fraction multiplication? How are the students facilitated and challenged by the representations?*

## **Theory**

### **Proof-based teaching**

According to Reid (2011), proof and proving should not be viewed as a standalone subject to be taught, but rather as a method of instruction. This approach, referred to by Reid (2011) as proof-based teaching, involves among others, four key elements: *the toolbox*, *conjecturing*, *proving*, and *a framing theory*. The toolbox consists of statements and assumptions that can be used without comment (Reid, 2011; Reid & Vallejo Vargas, 2019), and several researchers in mathematics education have emphasized the connection between working with proof and *conjecturing* (Lin et al., 2012). Reid (2011) notes that proving and conjecturing can interact in two ways: proving can use insights gained from enquiry to form a conjecture, and conjectures can be the statements that students prove. In proof-based teaching, statements are established through *proving*. “Premises may be accepted on the basis of prior extra-mathematical experiences, or simply as conventions agreed upon in a community, without evidence. Fundamental principles, definitions, and notations fall into this category” (Reid & Vallejo Vargas, 2019, p. 809). In certain classroom situations, it may be acceptable to consider “theorems (from the mathematicians’ point of view) to be taken as axioms” (Stylianides, 2007, p. 295). However, inductive and heuristic argumentation, which is a part of the conjecturing activity, does not count as evidence in proof-based teaching, where “statements are based on mathematical evidence, on an argument that meets the criteria for a proof” (Reid & Vallejo Vargas, 2019, p. 809). In proof-based teaching, a proof is a “connected sequence of assertions for or against a mathematical claim” (Stylianides, 2007, p. 291) where “the modes of representation used in arguments must be accessible to students” (Reid & Vallejo Vargas, 2019, p. 810; Stylianides, 2007). *A framing theory* in proof-based teaching specifies the toolbox and “the sequence of content to be learned” (Reid & Vallejo Vargas, 2019, p. 809), including notions, axioms, target definitions, target theorems, and target proofs.

### **Representations and fraction multiplication**

The learning and teaching of rational numbers has been studied by several authors, including members of the Rational Number Project (Behr et al., 1992). They proposed a model with five subconstructs: part-whole, quotient, ratio number, operator, and measure. This model has been widely used in subsequent research (Charalambous & Pitta-Pantazi, 2007).

Fraction multiplication is often introduced through the operator subconstruct. Multiplying a whole number by a fraction can be understood as repeated addition, while multiplying a proper fraction by a proper fraction can be understood as a part of a part. Paper-folding has been proposed as a useful representation for this concept (Wyberg et al., 2012). Note however that fraction multiplication can be conceptualized through an area model (Webel & DeLeeuw, 2016) which may relate to the measure subconstruct. In proof-based teaching, different semiotic registers can be used, including symbolic

mathematical writing, visual or physical representations, and verbal explanations. Each system of representation has built-in relations and operations (Duval, 2006, p. 109).

### **Representations in proving**

Mathematical proofs often utilize a combination of verbal, symbolic, and visual representations to communicate a connected sequence of assertions to the reader. However, the process of creating or understanding a proof may involve multiple conversions between different types of representations (Duval, 2006). Visual representations, for instance, can be useful in proving, but the use of multiple representations can also pose challenges in terms of converting between them (Duval, 2006) and synthesizing them into a cohesive argument. Even when some proofs rely on minimal verbal and symbolic representation and require the reader to construct the connected sequence of assertions themselves, discursive representations are typically necessary for effectively communicating a proof. Three important discursive operations in mathematics are highlighted by Duval (2006): denotation of objects, statements of relations or properties, and inference. These operations are essential in both conjecturing and proving, with inference being particularly vital in the proving process. In this study, we investigate students' *representations of objects and operations, statements of properties and relations, and inference*. These three categories of representations align well with the three key concepts in proof-based teaching: *the toolbox, conjecturing, and proving*. Specifically, representations of objects and operations are essential for the use of a toolbox in proving, statements of properties and relations are conjectures until they are proven, and inference is crucial in proving them. While we do not claim that there is a strict correspondence between these triads, there is a meaningful connection between them, and in the present paper we will focus on the three categories of representations through their relevance to the three central aspects of proof-based teaching.

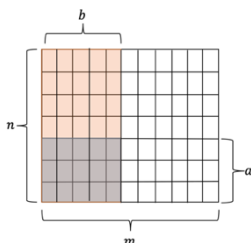
### **Methods**

To inform the design of a research project on proof-based teaching of fractions, we conducted a small-scale intervention consisting of seven tasks for students to complete during a 60–90-minute lesson on fraction multiplication. Ideally, the intervention would have taken place among a cohort of learners unfamiliar with fraction multiplication. We were however offered an opportunity by a grade 8 teacher who planned to review important fraction topics which would subsequently be extended. Hence, two classes of grade 8 students (approx. 13 years old) were invited to participate with an alternative provided in a separate classroom for those who declined. Prior to attending this school, the students had attended various elementary schools in the area. To our knowledge, the students did not have any prior experience with proof-based teaching. We conducted the intervention in two separate instances, with 90 and 60 minutes allocated respectively, to accommodate the different schedules of the two classes. The second author led both classes, which were divided into seven and six groups respectively. We were able to record the work of seven groups and collected written work from the students. This paper presents an analysis of the work of one group of three students in the second instance, while also incorporating partial analysis of the work of two other groups of three students.

### **Task design and framing theory**

Before designing the tasks, key literature on the teaching and learning of fractions were reviewed. To ground the intervention in the traditions of the Norwegian schooling system, several textbooks were

also reviewed. It was decided to root the tasks in the operator aspect of fraction using paper-folding in combination with an invitation for the students to use diagrams and symbolic representations. This was due the prevalence of the operator approach in Norwegian and international textbooks. The tasks were developed collaboratively by the authors and a group of three researchers in mathematics education. To guide the intervention and the analysis, a framing theory was formulated that took the representations of fractions by paper-folding and the representation of multiplication as a fraction of a fraction as an axiom and as a definition respectively. The tasks were also reviewed by teachers at the intervention school. Based on the review of textbooks we assumed that the students had some prior knowledge of some elements of the framing theory. However, the first three tasks, accompanied by whole-class discussions and instructions from the second author, aimed to establish the necessary axioms and definitions. In task 1 the students were asked to represent the fractions  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{3}$  and  $\frac{2}{3}$  using paper-folding. In task 2 the students were invited to discover the theorem that folding an A4 paper in  $n$ -ths in one direction and in  $m$ -ths in the other direction, divides the paper into  $n \cdot m$  equal parts. In tasks 3 the students learned to fold a fraction of a fraction and the multiplication by a fraction as a partition operator was written on the blackboard as the definition of fraction multiplication. The rest of the tasks were distributed one at the time for the groups to work on. In task 4 the students were invited to fold  $\frac{2}{3}$  of  $\frac{3}{4}$  and interpret it as  $\frac{2}{3} \cdot \frac{3}{4}$ . In task 5 the students were expected to find  $\frac{5}{8} \cdot \frac{1}{3}$  with the aid of paper-folding. In task 6 students were invited to use drawings inspired from paper-folding to find  $\frac{3}{7} \cdot \frac{5}{11}$  targeting a diagram as in Figure 1. Finally, task 7 asked the group to make a poster explaining to other grade 8 students not present, why the rule  $\frac{a}{n} \cdot \frac{b}{m} = \frac{a \cdot b}{n \cdot m}$  holds true.



**Figure 1: A part of a proper fraction.**

## Analysis

To analyse the group's work, we employed a simplified version of the approach described by Powell et al. (2003). We divided the video recording into meaningful sections and assigned transcriptions, observations, codes and narratives to each section. Codes were both inductively and deductively developed, based on emerging themes in the data and theoretical concepts related to fractions, representations, and proof-based teaching. Examples of codes are *Deictic gesture*, *Verbal representation of a fraction as an operator*, and *Verbal representation of statement of properties and relations*. In addition to the analysis of the video, we analysed the poster, by classifying representations and hypothesizing about implicit inferences.

## Findings and discussion

We first briefly describe some of the group's work with the seven tasks in view of their interaction with representations and proof-based teaching, before we report on how *the students are facilitated*

and challenged by representations of objects and operations, statements, and inference. In working with the tasks, the students handled multiple representations of fractions: paper-folding, diagrams, verbal representations, and symbolic representations. Task 2 were discussed in the group as follows:

One, two, three, [coordinated with deictic gestures along the top edge of the sheet] times, one, two, three, four [coordinated with deictic gestures along the right edge of the sheet] Like that! It's very simple [...]. So, [...], three times four is 12.

The students understood that an  $n \times m$  array contains  $n \cdot m$  entries, and they explained this by using the concept of repeated addition. During the discussion, one student pointed out that there is no benefit to multiplying compared to counting all entries when one of the factors is 1. The group was able to convert fractions between verbal, symbolic written, paper-folding, and drawn representations, and used these different representations in group discussions and with the teacher. They used paper-folding as representation and were able to move freely between symbolic representation and paper-folding. In solving task 4 which involved finding the answer to  $\frac{2}{3} \cdot \frac{3}{4}$  the students were able to go from a symbolic representation to paper-folding aided by verbal representations and gestures.

Teacher: There you have three quarters.  
 V: First, and then two thirds of three quarters.  
 Teacher: How many parts do you have then?  
 V: Six 12ths.  
 M: One half, man! [...] 6, then it's one half.

The verbal representations seemed to be prompted by the intervention of the teacher. A translation of the group's written proof is given in Figure 2. Although they used the example  $\frac{2}{3} \cdot \frac{2}{3}$  to illustrate, there is an aspect of generality as they refer to a fraction as composed by a numerator and a denominator. They correctly state that the denominator of the first fraction times the denominator of the second fraction equals the numbers of squares in the whole unit, and that the numerator times the numerator constitutes the area "you are colouring". However, the colouring is not justified explicitly by the operator aspect (by taking  $\frac{2}{3}$  of  $\frac{2}{3}$ ) or by an area consideration (through the multiplication of the length  $\frac{2}{3}$  by the length  $\frac{2}{3}$  to obtain the area).

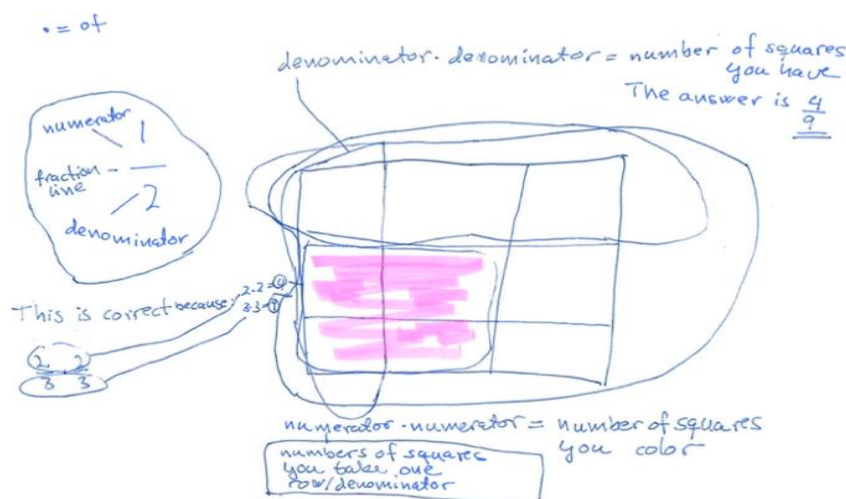


Figure 2: The proof provided by the group.

## **Representations of objects and operations**

We identified several issues that arise when representing a fraction in various registers: *equivalent fractions*, *representing a fraction as an operator*, and *representing general fractions*. In the group studied in this paper, *equivalent fractions* presented limited difficulties, but the concept of common denominators offered confusion in other groups. In line with Behr et al. (1992) the intervention focused on fraction multiplication through the operator subconstruct. *Representing a fraction as an operator* was challenging for students. Moving from paper-folding to drawn diagrams, the operator interpretation of fraction multiplication seemed to be diffused with an area interpretation of fraction multiplication. The lack of a designated symbolic representation of a number as an operator (function) in elementary school mathematics, may have contributed to this. Multiplier (operator) and multiplicand are often distinguished in contexts, but the distinction does not convert to symbolic writing. In the intervention we used the verbal representation “of” as is common in taking a percentage of an amount. Furthermore, the students were not facilitated to *represent* multiplication of two *general fractions*, since the intervention only addressed a proper fraction times a proper fraction. This constitutes a breaking point of a representation, a limit which when exceeded makes the representation unsuitable (Vig et al., 2014). Importantly, the students did not realize or question this limitation. Furthermore, only a few fractions can be easily folded, which may also be considered a breaking point of paper-folding representations of fractions, and regarding symbolic representations, the students seemed to have limited experience with the use of variables and were thus further challenged in representing general fractions. In their proof the group used an example which we do not characterize as a generic example.

## **Representations of statements of properties and relations**

There are several examples of statements of properties and relations in the students’ work. For instance, “ $3 \cdot 4 = 12$ ”, “ $\frac{6}{12} = \frac{1}{2}$ ” and “denominator times denominator is equal to the number of squares you have [in the whole].” Statements of properties and relations were however not plentiful, and it is not completely clear if the root cause lies with the paper-folding or with the oral and written representations. However, we did note that the students had difficulties with explaining paper-folding orally and formulating their understanding in mathematical statements. The physical representations of paper-folding were not always followed by statements of properties and relations in a straightforward way, although interaction with the teacher seemed to bring the students closer to the formulation of statements. It seems that the students were able to formulate some statements during their work with the tasks, but struggled to formulate others, including a clear conjecture. The fact that the students were able to provide a written proof suggests that they had a certain level of understanding of the mathematics involved.

## **Representations of inference**

There are instances of representations of inference. For instance, in the written proof, they write: “This is correct because  $2 \cdot 2 = 4$  [and]  $3 \cdot 3 = 9$ ”. However, it is not completely clear what “This” refers to, and inference remains mostly implicit. Overall, there is not a clearly communicated connected sequence of assertions for or against a proposition.

Students have experience with inference in the form of computation, mostly in the symbolic register where computation can be done algorithmically. Describing inference in a multifunctional register is more challenging due to the complexity of a multifunctional register (Duval, 2006). In proving, the mono-functional registers of symbolic mathematical writing and discursive operations in multifunctional registers may be in tension with each other as the mono-functional registers often are given priority in school mathematics. In proofs there is often a need for merging different registers, and the students may not have enough experience with this.

We view human cognition as multimodal, and we value visual representations in proving. However, although a proof by diagram can be an illuminating way of indicating the validity of a mathematical statement, in our view a complete proof involves a written or verbal explanation that clearly outlines central parts of the inference involved in the proof.

### **Conclusion and implications**

Our study of a group of three students working on tasks designed to facilitate their conjecturing and proving of the rule for fraction multiplication, revealed insight into how students engage with representations in proof-based teaching. The students were able to use multiple representations of fractions, including paper-folding, area model drawings, verbal, and symbolic representations, and were able to convert between these different representations and use them in group discussions and with the teacher. While these representations appeared to facilitate their thinking and communication, the students encountered difficulties with representing fractions as operators and in structuring and formulating a conjecture and proof. Despite being able to express the operator aspect of fractions orally (guided by questions from the teacher), the students struggled to consistently apply this understanding in their subsequent work and in the written proof. Formulating statements as in a conjecture and inference in proving may be difficult for the students because of the complexity of multifunctional registers (Duval, 2006).

Implications for future interventions is to allow for more time for paper-folding activities, with a focus on fractions as operators in order to help students internalize this concept. Additionally, we recommend focusing more explicitly on helping student formulate statements and conjectures and inference using verbal representations and full sentences. Verbal representations are required at all stages of proof-based teaching, and we suggest that it is necessary to explicitly address this aspect in proof-based mathematics lessons.

### **Acknowledgment**

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