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# Towards a model of problem networks for studying problem solving in mathematics education

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*In this paper, we present an ongoing research project developing a modelling tool for conceiving and analysing problem-based teaching and learning trajectories, in form of networks of problems. We explain the origins and the emergence of the model, the underlying theoretical choices. We emphasise how the model allows taking into account argumentation and proving aspects in the context of problem-solving activities.*

*Keywords: Problem networks, inquiry based mathematics education, ckç model.*

## Introduction

This paper discusses a part of an ongoing project named *Problem-based learning trajectories in discrete mathematics education* (ProDiME), realized as a collaboration between the two authors and V. Durand-Guerrier. This project seeks to conceive and analyze teaching trajectories and their different implementations in French and Hungarian classrooms, with the help of a modelling tool based on “networks of problems”. The first step of the project is to develop this modelling tool by crossing our earlier works. In this paper, we explain how our respective preliminary research motivates the emergence of a model in terms of “networks of problems”, focusing particularly on aspects concerning argumentation and proof in the context of problem solving activities.

Our project is closely connected to questions of Inquiry Based Mathematics Education (IBME). Although different interpretations and approaches exist to IBME, with slightly divergent characteristics (Artigue & Blomhoj 2013), they share a number of main principles, one of them being the central role accorded to problems in the learning of mathematics, and the conviction that new mathematical knowledge can only emerge by active participation of the learners in the construction process of mathematics. From this conviction follows the challenge of concerting problem solving activities with curricular learning. Different strategies exist, in different approaches to IBME, to conceive complex problem-based teaching and learning trajectories (Artigue et al 2020), but a general observation is that the conception, analysis and implementation of these trajectories as well as the dissemination of the conception strategies behind them remain difficult.

In several approaches, and more particularly in our respective research, the idea of representing the structure of these trajectories in form of graphs emerged as a powerful tool both for research purposes and for communication with teachers. Our aim is to develop these attempts of representation into a theoretically well based modelling tool, relying on Vergnaud’s *Theory of Conceptual Fields* (2009) and Balacheff’s *ckç* model (2013). We thus discuss the question:

- How to establish a theoretical model allowing the description of set of problems, their interrelations, and their relations with mathematical content, for didactical purposes?

In this paper, we emphasize particularly a specific aspect of this work:

- What is the role of argumentation and proof in problem solving activities and problem-based mathematics education, and how can we take into account this role in the frame of our model?

Reflecting argumentation and proof processes appears to be a crucial aspect in this work, for several reasons. First, the process of problem solving implies a series of decisions based on (correct or wrong) convictions and conjectures. A (rational) problem solver has implicit or explicit arguments to choose steps of the problem solving procedure, and solving a problem supposes reasons to be convinced that the solution is found and correct. We suggest interpreting problem solving steps as transformations of the original problem, and Balacheff's *ckç* model allows us to take into account the above-mentioned argumentative aspects as *controls* applied during the choice and execution of these steps.

A second aspect is the connection of problem-solving activities with the construction of mathematical knowledge. Argumentation and proving play a crucial role in assuming the emergence of new elements of mathematical content.

In the following, we first present the emergence and the role of problem networks in our respective preliminary research, then we discuss how a combination of these works leads us to the definition of a modelling tool. We will emphasise questions related to argumentation and proof in this presentation.

### **“Series of problems” in history of mathematics and in mathematics education**

Gosztonyi's research was partly inspired by an interdisciplinary history of sciences project introducing the term “series of problems” as a methodological tool for the analysis of texts from various historical contexts (Bernard, 2015). She developed analytical tools and methods by reinvesting them from one context to another.

#### **Series and networks of problems in the Hungarian context of mathematics education**

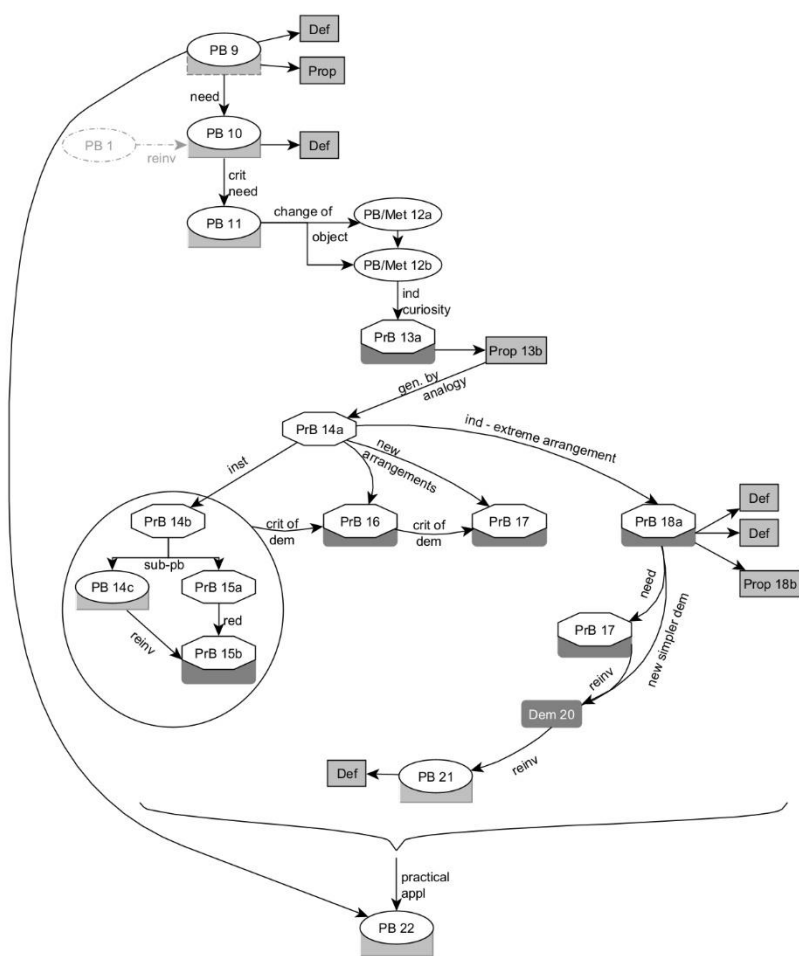
Gosztonyi's work set out from the analysis of Hungarian traditions of teaching by “Guided Discovery”. She showed that one of the specificities of Guided Discovery is the conception of long and complex teaching trajectories based on “series of problems”, meaning a conscious ordering of problems for reaching one or several specific teaching purposes (Gosztonyi, 2015). She showed that, although this idea appears quite coherently in different resources related to the approach, the identification of series problems, the reconstruction of the underlying principles of structuration, the related teaching purposes and aspects of implementation is quite a challenging task, both for researchers and for teachers who are novices to the approach. Based on the analysis of some specific texts where the design choices were explicated, she suggested a representation of the underlying structure of series of problems in form of problem networks, making the dense links between problems visible. In a collaborative project involving expert teachers of Guided Discovery, the problem network representation was reinvested as a mediator in discussions between teachers and researchers helping to explicate teachers' design choices (Gosztonyi & Varga, 2023).

#### **An analysis of Clairaut's Elements of Geometry**

The idea of problem network representation was also reinvested in the analysis of a text from another historical context, namely Clairaut's *Elements of geometry* (1741) from 18<sup>th</sup> century France, which shows interesting analogies with texts of Hungarian Guided Discovery (Gosztonyi, 2022). Although

we don't exactly know its original destination, the preface talks about introducing “beginners” into geometry, and the book was interpreted by many as a didactical text, a kind of teacher guide. Clairaut proposes to write an “Elements of geometry” not in a deductive order but structured by problems. The intention is to allow beginners to “acquire more easily the spirit of discovery” (p. x). We present here some elements of the analysis of this book, which emphasise the connections between problem solving, proving and the construction of mathematical knowledge.

The text is organised into four books, and structured mainly by numbered “paragraphs”; but different types of content (as definitions, theorems, demonstrations, problems, methods) are not explicitly distinguished. Gosztonyi, in collaboration with Bernard carried out a discursive analysis of the text to identify these types of content and interrelations between them, explicating the hidden structuration strategies of the text. One notable characteristic is the rich fabric of metadiscourse explaining interrelations between elements of the content and arguing transitions from one problem to another. The analysis of the metadiscourse allowed us to represent the structure of the text in form of a graph (Figure 1), and to interpret different types of links (represented on the figure as annotations of arrows).



**Figure 1: Scheme for the analysis of Clairaut's *Elements of Geometry*, Book III, paragraphs 9-22 (Gosztonyi, 2022)**

the discursive structures of these two types of problems and their solutions are quite similar.

From a point of view of argumentation and proof, it is interesting to observe Clairaut's ambiguous use of the term “problem”. In the few cases the term explicitly appears in the main text, it is used in the classical sense of construction or measurement problems. In the preface however, the term is used in a more general sense, covering also research questions leading to the discovery of theorems, which is closer to modern uses in the context of teaching by problem solving, as Pólya's (1945). In our analysis, we distinguished ‘problems’ and ‘proposition-problems’ (PrB on the figure): problems are construction or measurement problems the solution of which is a procedure, while ‘proposition-problems’ are conjectures or research questions leading to a theorem, and their solution is a demonstration. The

Clairaut suggests that two types of motivations can lead to the introduction of new problems: need or curiosity. The first book is mainly structured around problems motivated by practical needs of land surveying but we can already observe a combination of the description of procedures and argumentative, demonstrative passages in the description of the solutions. Our analysis pointed out that new ‘propositions’ emerge repeatedly as constitutive elements of a problem’s solution. In later books, proposition-problems and their demonstrations take a more central role, with a shift to motivation by ‘curiosity’. Figure 1 illustrates the structure of an extract from book III, underlying rich connections between problems and proposition-problems. The structuring strategies appear to be quite coherent throughout the different books of the text, nevertheless, new types of links (as inductions or generalisation by analogy) emerged together with the proposition-problems.

### **Analysing problem solving paths in teaching mathematics and computer science**

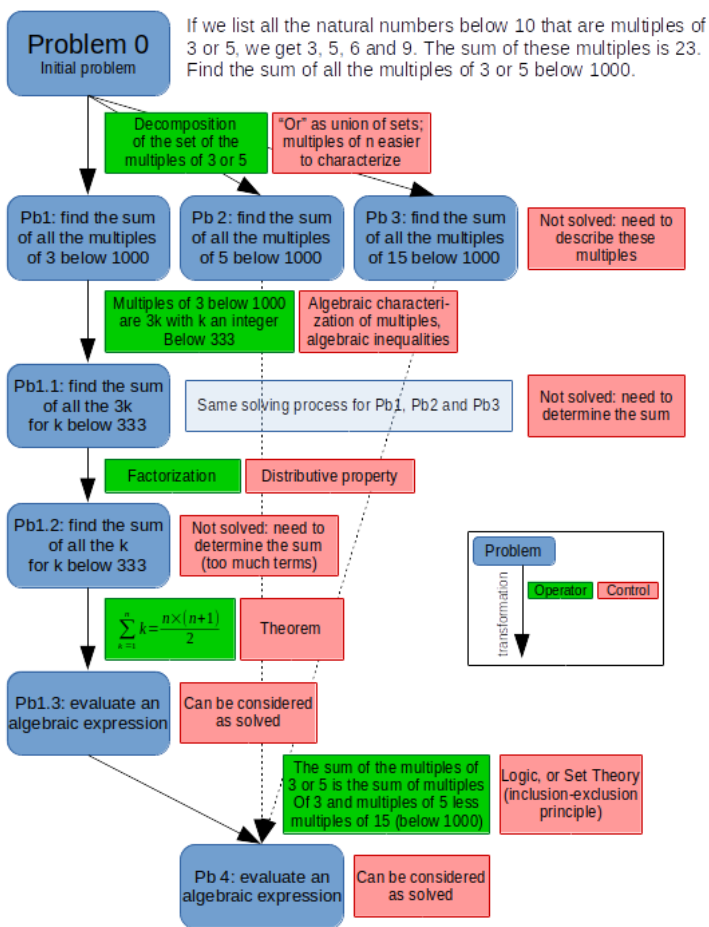
Modeste, the second author of this paper, arrived at the idea of problem networks from another direction. He was interested in the analysis of problem-solving strategies and processes and of proof and validation, at the interface of mathematics and computer science (Modeste & al., 2019).

Modeste (2020, Modeste & al, 2019) as well as Giroud (2011), inspired by the notion of problem used in the theory of algorithmic complexity (e.g., see Garey & Johnson, 1979), described problems as follows. A problem  $P$  is considered as a couple  $P=(I,Q)$  where  $I$  is a family of mathematical objects (instances), and  $Q$  is a single question that can be asked about any element of  $I$ . This definition allows considering various levels of generality of a problem, and various domains of application.

For example, we can consider the problem  $P_1=(I,Q)$  where  $I$  is constituted of all the pairs of positive integers  $(m, n)$  (that is, the set  $\mathbf{N}^2$ ), and  $Q$  is the question “how many different words can be written with  $m$  letters ‘a’ and  $n$  letters ‘b’?”. This description allows to distinguish this general problem from more restricted problems – like the problem  $P_2$ , asking the same question  $Q$  but for a reduced set of instances where  $m$  is fixed at 2, or  $P_3$  asking the same question for the subset of instances where  $m=n$ . It also allows to consider instantiated problems, where you only take one element of the family of instances. A problem can also be modified by adding constraints to the question  $Q$  or by changing it.

Another important choice coming from Modeste’s earlier work is the use of Balacheff’s  $ck\phi$  model. Based on Brousseau’s *Theory of Didactical Situations* (Brousseau & Warfield, 2020) and Vergnaud’s *Theory of Conceptual Fields* (2009), Balacheff (2013) describes a concept as composed of 4 elements: a set  $\mathbf{P}$  of problems that give meaning to the concept, a representation system  $\mathbf{L}$  (the signifier), and invariants that are separated into a set  $\mathbf{R}$  of operators which permits to act on problems (that is, transform a problem into another), and a set  $\mathbf{\Sigma}$  of controls that permit to decide whether and how an operator  $r$  applies to a given problem, and to evaluate the effects an operator has had on a problem (in particular to determine whether or not a problem is solved).

This last aspect, the distinction between operators and controls is one of the notable developments of Balacheff’s model, and this is the aspect on which we can rely for considering the important role of argumentation and proof in problem solving activities. The separation of operators, controls, and representation system is relevant to deal with problem solving activity, to describe the actions one can have on problems, but also to explicate controls as elements justifying the action, thus emphasising proof and proving aspects. The representation system component allows taking into



**Figure 2. A mathematical solving of Problem 0**  
(Modeste & al., 2019)

account the semiotic registers in which the problem is represented, and to describe operators and controls available in this register (including those linked with changing register).

Combined with the above-mentioned definition of problems, Modeste et al. (2019) implemented the  $ck\phi$  model for a priori analyses, describing possible problem solving paths, the way new problems are generated in the process and their relations to the previous problems. They illustrated how such a model can be used to make an a priori analysis of a given problem, modelling and analysing solving strategies, and explicating the possible operators and controls. This can be represented with a flowchart as in Figure 2, presenting a mathematical solution of Problem 0 (Figure 2).

Arrows between problems (blue) represent the transformations by an operator (green) selected according to a control (red) on operators and on the status of problems.

## Towards a modelling tool based on networks of problems

In the ProDiME project, we combine the analysis of “series of problems”, that is, longer teaching trajectories conceived by a conscious ordering of problems, based on complex systems of problems, with the analysis of problem solving paths, that is, a set of problems generated by successive transformations of a given problem during problem-solving process. Although these are different types of analyses, there are numerous possible connections between them. For example, a teacher can conceive a series of problems with the intention of facilitating the solving of more complex problems, by posing some reduced versions of the target problem earlier in the series. In this case, the a priori analysis of possible problem solving paths can contribute to the conception of series of problems or to the interpretation of an existing one.

We suggest that modelling in terms of networks of problems can be efficient in both cases, facilitating the connection of these two aspects. We consider several possible uses of networks of problems:

- a priori analysis of problems and their possible solutions;
- a priori analysis of series of problems conceived by teachers;
- a posteriori analysis of students’ problem solving processes and learning trajectories;
- design of teaching trajectories (by researchers as well as by teachers).

For developing our model, we rely on Balacheff's *ckø* model and, more indirectly, on Brousseau's and Vergnaud's related theories. We consider mathematical activity from a cognitive and epistemological point of view as confrontation with problems and as successive transformation of problems into other ones up to reaching a problem which is solved (i.e. not being a problem anymore). A problem is considered, for an individual, as a question or a task of which the solution method is not directly accessible; or, as a disequilibrium between the individual and the "milieu" (TDS).

Vergnaud introduced the term "conceptual field" as "a set of situations and a set of concepts tied together". As he explains, "a concept's meaning does not come from one situation only but from a variety of situations and that, reciprocally, a situation cannot be analysed with one concept alone, but rather with several concepts, forming systems" (Vergnaud, 2009, p. 86). By modelling in terms of networks of problems, we intend to describe the structure of such conceptual fields, and potential or actual trajectories constructing and exploring them. In the case of series of problems conceived by teachers or researchers, we may identify the "conceptual field" covered by the problems of the series, in case of problem-solving paths, we may consider the conceptual field generated by the initial problem (which might be different according to different paths traced by different individuals).

A crucial question is to describe the nature of the relations between two (or more) problems, the solution processes and the related mathematical knowledge, in order to explicate the ways one can go across various problems making meaning (that is, learning). In the following, we discuss this question by referring back to the examples presented above.

Our respective preliminary works contain elements to interpret these links, but the different approaches have to be conciliated with each other and with some other works, as Giroud (2011). Most notably, we consider that Pólya's (1945) classical list of questions, intended to support problem solving processes, can also be read in this sense: indeed, many of Pólya's questions suggest transforming problems into other problems or linking new problems with earlier solved ones. In this sense, Pólya's suggested problem solving strategies do not deal with individual, isolated problems, they rather situate new problems in a progressively constructed network of earlier solved ones.

The types of problem-solving steps implied by Pólya's questions are quite coherent with the types of links we could identify in series of problems of different authors and teachers related to the Hungarian Guided Discovery approach (Gosztonyi, 2015): as finding a simpler or a more general problem, an analogous problem, reinvesting a method of solution... Many of the types of links identified in Clairaut's book (see Figure 1) seem to be also similar to those described in the Hungarian context. Nevertheless, new types of problems emerged in our successive analyses. We are thus not proposing, at this stage of the research, a full classification of possible links, rather an open inventory of links.

At the same time, the definition of problems as  $P=(I, Q)$  explained above allows us to give a coherent description of many possible relationships between problems.

A first example comes from the idea of generality or specificity of problems. A problem  $P'$  is a specialization of a problem  $P$  if  $P'$  can be obtained by a restriction of the set of a problem's instances or by fixing the values of some variables describing the set of instances. A particular (extreme) case is the instantiation of a problem, when all the parameters of the set of instances are fixed, that is when

we look at one particular case. Another way of specializing a problem  $P$  into a problem  $P'$  is when adding constraints to the question (see for ex. Figure 2).

Another type of relation is sufficiency. When you try to solve a problem  $P$ , you can transform it into a problem  $P'$  (or many problems  $P'_1, \dots, P'_n$ ) such that solving  $P'$  (or  $P'_1, \dots, P'_n$ ) is sufficient to solve  $P$ . We can also say that  $P$  is necessary to  $P'$ . Of course, identifying this relation relates to the controls of the operators used to transform  $P$  into  $P'$ , and can be or not identified by the individuals solving the problem. This type of relation was clearly identified and detailed by Giroud (2011).

Clairaut's 'need-based' transitions (i.e. when a problem is reduced to another one the solution of which is necessary to solve the original problem) can be interpreted in this sense. And indeed, Clairaut often arguments these reductions at length, and these are typically the points where new 'propositions' emerge in his text (Gosztonyi, 2022).

The relationships above (generality/specificity, necessity/sufficiency) are mostly of a logical or epistemological nature. We can explore relationships of different kind. For example, if we consider representations as semiotic registers, we can describe certain links between problems as conversions from one register to another. The new problem is not always equivalent to the first one.

Another type of link between two problems can be related to operators. If an operator  $r$  has been useful to solve the problem  $P$ , and if there are similarities between  $P$  and  $P'$  (which can be of various nature) it might be relevant to use  $r$ , or an adaptation of  $r$ , to operate on  $P'$ . This could be qualified as operator analogy. We can imagine that there are also analogies that concern controls and could be called control analogy. In fact, when two problems belong to a bigger family of problems, there can be general principles applied on them which can be considered as controls. In this case, the model allows clarification of the relationship between the two problems, by identifying a kind of "ancestor", a problem of which they both are specialisations. In this case, we can consider that using similar operators or control are something deeper than an analogy.

The search for generalization (of problems, operators, or controls) and for *ancestor problems* can play crucial role in establishing new mathematical knowledge and in increasing abstraction of concepts. In the context of Guided Discovery for example, mathematical knowledge is often built through progressive generalisation, by exploration of analogies between problems (Gosztonyi, 2015).

The interpretation and classification of possible links will be developed through further analysis of examples. At the conference, we will present the analysis of specific problems and series of problems from the domain of discrete mathematics. We will emphasise the emergence and role of controls (i.e., proving) in these series of problems. We will also explain how we plan to use the problem network model in collaboration with teachers.

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