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# Incremental stabilization of cascade nonlinear systems and harmonic regulation: a forwarding-based design

Mattia Giaccagli, Daniele Astolfi *Member IEEE*, Vincent Andrieu and Lorenzo Marconi, *Fellow, IEEE*

**Abstract**—In this work, we address the problem of designing a control law for a system in feedforward form to be globally incrementally exponentially stable. To do that, we develop an incremental version of the so-called forwarding  $\text{mod}\{L_g V\}$  approach. Then, we apply such a control design to the problem of compensating matched disturbances assumed to be given by the superimposition of a finite number of harmonics with unknown amplitude. For this, we propose a dynamic controller made of  $L$  linear oscillators processing the regulation error and a stabilizer making the closed-loop system incrementally globally exponentially stable, uniformly with respect to the external signals. This guarantees that the closed-loop system asymptotically converges to a periodic trajectory having the first  $L$ -Fourier coefficients of the error to be zero. Then, we specialize our design for the class of linear systems with a scalar nonlinearity and of minimum-phase systems possessing contractive zero dynamics.

**Index Terms**—Harmonic regulation, contraction theory, incremental stability, forwarding, minimum phase systems.

## I. INTRODUCTION

Contractive systems are receiving a lot of attention from the control community. Roughly speaking, a system is said to be contractive (or incrementally exponentially stable) if the distance between any two trajectories starting from different initial conditions is decreasing exponentially in time, uniformly in the difference of such initial conditions. The study of contractive systems has a long history (see, e.g., [1]–[5] and references therein) but their interest is still active thanks to the several useful properties that contractive systems share, such as trajectories well-defined in positive times, periodicity when excited by external periodic signals and robustness with respect to (small) model uncertainties. These features motivate why contraction theory is used to deal with engineering problems such as output regulation ([6], [7]), multiagent synchronization ([8], [9]) and observers design ([10]).

From this viewpoint, particular attention was given to answering the question “how to design a control law for a system to define a contraction”. The existing control strategies can be divided in 3 groups. i) Designs that focus on specific techniques applied to particular classes of systems, such as backstepping design [11] for systems in strict feedback form or LMI-based design for Lur’e systems [12], [13]. These approaches provide tractable conditions, which however apply only to the considered class of systems. ii) Designs that make use of the similar notion of *convergent* system, e.g. [7], [14] and [15]. Note that, however, the notions

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of convergent, contraction and incremental stability are similar but not completely equivalent, e.g. [16]. iii) Optimization tools, e.g. the *Control Contraction Metrics* [17] or [18]. These designs are applicable to more general classes of systems, with the drawback that they usually require the solution to an (online, pointwise) optimization problem.

In this work, we focus on the class of continuous-time nonlinear systems in the so-called feedforward form (also denoted “in cascade form”). For this class of systems, existing results belonging to the first and second groups of the aforementioned techniques cannot be directly used. One could apply the tools belonging to the third group, but the relative drawbacks would come alongside (i.e. solution of an online optimization problem pointwise, loss of analytical solutions, ...). As a consequence, our first main objective is to derive design tools for such a class of systems. To this end, we focus on the so-called “forwarding approach” that has been developed for stabilization purposes (see [19]–[22] and references therein) and we propose an extension to the incremental framework. In particular, as our first result, we develop an incremental version of the so-called *forwarding mod}\{L\_g V\}* for multi-input multi-output (MIMO) systems that has been proposed for the single-input single-output (SISO) stabilization case in [22] (and inspired by [19]), extending *de-facto* our preliminary result [6]. The advantage of such an approach, especially for high-dimensional systems, is that it provides a design with several degrees of freedom, that potentially allows splitting the problem of control law construction for large dimensional cascade systems into multiple (possibly recursive) easier steps. Note that with respect to the previous classification of methodologies, our result takes place in the first group.

The proposed incremental forwarding tool is then applied to the context of harmonic regulation of nonlinear systems, e.g., [23], [24]. In this case, the objective is the design of a control action such that the trajectories of the closed-loop system are bounded in forward time and such that a regulation error with respect to a periodic reference is periodic (with the same period) and does not have a frequency content at certain frequencies. This is generally achieved by first extending the plant with a dynamical system possessing an *internal model* property and a feedback design for the extended closed-loop system. We highlight that in the more general context of output regulation and internal-model based regulators [25], existing designs can be divided into two groups: the first group considers systems that admit a globally defined normal form possessing stable zero-dynamics [26, Section IV]. In such a case, output regulation is achieved (semi)globally in the initial conditions and in the size of the references, by means of a feedback control law composed by a term depending on the internal model and an high-gain feedback (see for instance [24], [27] and references therein). The main drawback of this approach is that it requires the existence of a normal form, which is not always well-defined globally and that might be practically difficult to be found, especially for MIMO systems. The second group of results works in the “original” coordinates and

follows a passivity-like approach. Within this approach, we recall for instance [14], where the problem is posed for incrementally passive systems coupled with a linear output, the work [7] where the problem is addressed under the notion of *convergent* systems with the control that is designed by explicitly solving the so-called *regulator equations* and, more recently, [28] for systems in Lur'e form having convergent properties.

In this work, we follow a different approach, similar to the one proposed in [29]. In particular, the idea is to first extend the plant with a dynamical system composed of a bunch of linear oscillators at the external reference's frequency and its multiplies processing the regulation error and to look for a stabilizer that makes the closed-loop system incrementally globally exponentially stable, uniformly with respect to the external signals. After adding the oscillators, the (extended) system is in feedforward form and the incremental forwarding mod $\{L_g V\}$  previously developed can be applied as the feedback design. The proposed result overcomes the main limitation of some existing techniques. With respect to designs such as [29], where harmonic regulation was obtained semi-globally in the initial conditions and only locally in the references/disturbances (that is, only with the amplitude of the external signals being sufficiently small), our design allows to handle external signals spanning the whole set of finite  $\mathcal{L}_\infty$ -norm. Moreover, the proposed approach does not require the existence of a globally defined normal form [24], [27], or (incrementally) passivity-based conditions [14]. Because of the incremental properties enforced by our design, we show sufficient conditions to achieve harmonic regulation globally in the size of the external references and globally in the domain of attraction for systems that do not explicitly require the existence of a globally defined normal form. Then, as a case study, we specialize our design for linear systems with a scalar nonlinearity and for the case of minimum-phase systems possessing incrementally stable zero dynamics. In the first case, we translate our conditions into an (easy-to-check) test design. In the second case, we demonstrate that the proposed conditions recover existing techniques, showing that our design is not restrictive but rather comprehensive of existing tools.

The paper is structured as follows: in Section II we present the considered framework and we provide the structure of the incremental forwarding mod $\{L_g V\}$  control law for the incremental stabilization of nonlinear systems in feedforward form. Then, in Section III-A we apply our result to the case of global harmonic regulation. We then specialize our result for the class of linear systems with a scalar nonlinearity in Section III-B and for the case of minimum-phase systems possessing contractive zero-dynamics in Section III-C. Conclusions are in Section IV.

*Notation:* We indicate with  $|\cdot|$  the vector norm. Given a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a  $C^1$  mapping  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we denote the Lie derivative of  $h$  along  $f$  at  $x$  as  $L_f h(x) = \frac{\partial h}{\partial x}(x)f(x)$ . Given a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a 2-tensor  $P : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  both  $C^1$ , we indicate with  $L_f P(x)$  the Lie derivative of the tensor  $P$  along  $f$  defined as  $L_f P(x) = \lim_{h \rightarrow 0} \frac{(I+h \frac{\partial f}{\partial x}(x))^T P(x+h f(x,t))(I+h \frac{\partial f}{\partial x}(x)) - P(x)}{h}$ , with coordinates  $(L_f P(x))_{i,j} = \sum_k \left[ 2P_{ik} \frac{\partial f_k}{\partial x_j}(x) + \frac{\partial P_{ij}}{\partial x_k}(x) f_k(x) \right]$ .

Given a square matrix  $A$  we indicate  $\text{He}\{A\} = A + A^\top$  and with  $\det(A)$  its determinant. Given  $L$  square matrices  $A_1, \dots, A_L$  we indicate with  $\text{blkdiag}(A_1, \dots, A_L)$  the block-diagonal matrix having  $A_1, \dots, A_L$  on the main diagonal and zero everywhere else. We indicate with  $I$  the identity matrix and with  $0$  the column vector where each element is the number zero (the dimension is clear from

the context). We indicate with  $\otimes$  the Kronecker product. Given a  $n \times m$  matrix  $B$ , we indicate with  $\text{vec}(B)$  the  $nm$  column vector where the elements are the ordered elements of matrix  $B$ .

## II. INCREMENTAL STABILITY OF CASCADE SYSTEMS

In this work, we first consider systems in the following feedforward form

$$\dot{x} = f(x) + g(x)u, \quad (1a)$$

$$\dot{\eta} = \Phi\eta + v(x), \quad (1b)$$

where  $x \in \mathbb{R}^{n_x}$ ,  $\eta \in \mathbb{R}^{n_\eta}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control action,  $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_u}$  and  $v : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\eta}$  are sufficiently smooth function with  $f(0) = 0$ ,  $v(0) = 0$  and  $\Phi$  is a matrix of appropriate dimension. We assume that there exists  $\bar{g} > 0$  such that  $|g(x)| \leq \bar{g}$  for all  $x \in \mathbb{R}^{n_x}$ . We consider the problem of designing a state-feedback control law

$$u = \alpha(x, \eta) \quad (2)$$

such that the closed-loop system is globally (exponentially) incrementally stable in the sense that there exist  $k^*, \lambda^* > 0$  such that for any pair of initial conditions  $\chi_a = (x_a, \eta_a)$ ,  $\chi_b = (x_b, \eta_b) \in \mathbb{R}^{n_x + n_\eta}$  we have that

$$|\mathcal{X}(\chi_a, t) - \mathcal{X}(\chi_b, t)| \leq k^* \exp(-\lambda^* t) |\chi_a - \chi_b|, \quad \forall t \geq 0,$$

where  $\mathcal{X}(\chi_0, t)$  denotes the trajectory of the closed-loop system (1), (2) at time  $t$  and initial condition  $\chi_0$ . Such a property is verified, for instance, if there exists a Riemannian metric along which the closed-loop vector field generates trajectories for which the distance associated with such Riemannian metric is monotonically decreasing in forward time (i.e. for each  $t \geq 0$ ,  $\chi \mapsto \mathcal{X}(\chi, t)$  is a contraction, see [1], [4]). The open-loop system (1) is in the so-called *feedforward* (or *cascade*) form, for which forwarding-based control techniques have been developed for equilibrium stabilization purposes ([19]–[22], [30]). Practical examples of systems in this form are the disk-inertia pendulum in [22] or the TORA system [31]. The structure of the control law that we aim to develop is an incremental version of the forwarding mod $\{L_g V\}$  control design for systems of the form (1) first presented for stabilization purposes in [22] for the case in which  $\eta$  is scalar with  $\Phi = 0$ . This result extends *de-facto* the author's preliminary results in [6, Section III] in which the case of  $\Phi = 0$  and  $n_u \geq n_\eta$  has been studied. In this framework, we start by assuming the following.

**Assumption 1** (Pre-contractive feedback + Killing vector). *Consider system (1). There exist a  $C^1$  function  $\alpha_0 : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ , a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric and positive values  $P = P^\top > 0$  and three positive real numbers  $\underline{p}, \bar{p}, \bar{p}$  such that the function  $f_0(x) = f(x) + g(x)\alpha_0(x)$  satisfies*

$$L_{f_0} P(x) \leq -pI, \quad \underline{p}I \leq P(x) \leq \bar{p}I \quad (3)$$

$$L_g P(x) = 0 \quad (4)$$

for all  $x \in \mathbb{R}^{n_x}$ .

Assumption 1 asks for a pre-stabilizing feedback control action such that the  $x$ -dynamics generates a contraction with respect to a Riemannian metric induced by the matrix function  $P$ . Also, (4) implies that  $g$  is a ‘‘Killing Vector’’ field<sup>1</sup> for this metric (see [32]). In the linear case, this corresponds to a stabilizability assumption, where (4) is always satisfied as  $P$  is taken as a constant positive

<sup>1</sup>Given a  $C^1$  2-tensor  $P$  and a  $C^1$  vector field  $g$ , we say that  $g$  is a Killing Vector field for  $P$  if  $L_g P(x) = 0$  for all  $x$

definite matrix solution of a Lyapunov equation. The design of  $\alpha_0$  and  $P$  can be obtained following existing techniques ([7], [11], [12], [14]). Note moreover that, by [32, Theorem 2], the Killing vector property (4) guarantees that the  $x$ -dynamics is incrementally (exponentially) ISS (see [3, Definition 4.1]) with respect to any input  $u$ . From now on, we take  $f(x) = f_0(x)$  without loss of generality (i.e., Assumption 1 holds for  $\alpha_0(x) = 0$  for all  $x$ , for some  $P$ ).

**Assumption 2** (Non positiveness). *There exists a symmetric positive definite matrix  $H = H^\top > 0$  such that*

$$H\Phi + \Phi^\top H \leq 0 \quad (5)$$

To conclude, similarly to [22, Proposition 1], our last assumption is the following.

**Assumption 3** (Incremental Forwarding  $\text{mod}\{L_g V\}$ ). *We know three  $C^1$  functions  $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$ ,  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$  and  $\varrho : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that, for all  $x \in \mathbb{R}^{n_x}$ :*

1) *the functions  $\bar{M}$  and  $\Delta$  are solution of*

$$L_f \bar{M}(x) = \Phi \bar{M}(x) + v(x) + \Delta(x); \quad (6)$$

2) *there exists a matrix  $\Lambda$  such that*

$$L_g \bar{M}(x) = \Lambda \quad (7)$$

*and such that the couple  $(\Phi, (H\Lambda)^\top)$  is detectable;*

3) *the function  $\varrho$  satisfies*

$$\Lambda \frac{\partial \varrho}{\partial x}(x) = -\frac{\partial \Delta}{\partial x}(x); \quad (8)$$

4) *the following inequality holds for some  $\lambda > 0$*

$$L_f P(x) + \text{He} \left\{ P(x)g(x) \frac{\partial \varrho}{\partial x}(x) \right\} \leq -\lambda I. \quad (9)$$

Assumption 3 corresponds to a MIMO version of the assumptions in [22, Proposition 1]. In our framework, item 1 corresponds to a more general version of [22, Eq. (6)]. A solution  $\bar{M} = M(x)$  of (6) is known to exist for  $\Delta = 0$ , i.e. a function  $M$  satisfying

$$L_f M(x) = \Phi M(x) + v(x) \quad (10)$$

always exists if  $x = 0$  is a globally asymptotically stable and locally exponentially stable equilibrium point for  $\dot{x} = f(x)$  and  $\Phi$  has no unstable eigenvalues (see [19, Lemma IV.2]). These conditions are satisfied under Assumption 1 and Assumption 2. However, such a solution can be practically difficult to compute (see [21, Section III.A]). The main idea of forwarding  $\text{mod}\{L_g V\}$  is to introduce the term  $\Delta$ , so that  $\bar{M}$  becomes an *easy-to-compute* solution, in which the mismatch between the exact solution  $M$  of (10) and the approximated one  $\bar{M}$  is represented by  $\Delta$ . About item 2, the term  $L_g \bar{M}(x)$  can be seen as a controllability assumption on the control  $u$  to act on the dynamics of  $\eta$  of (1) in any point of the state space  $x$ . Note that in case  $\Phi = 0$ , the term  $L_g \bar{M}$  can be seen as an approximation of the DC-gain between the input  $u$  and an output  $y = v(x)$  along the trajectories of the system (see [6] for a wider discussion). About item 3, as the dynamics of  $\eta$  in the most general case can have a higher dimension than the one of  $u$ , in order to mimic the result of [6, Proposition 4], we ask for the mismatch term  $\Delta$  to be mapped in a (possibly lower) space of the dimension of the input and to be integrable. Finally, item 4 asks for a robustness-like property for the autonomous system. Indeed, in order to rely on a free-to-choose PDE solution  $\bar{M}$  rather than the exact one  $M$ , the open-loop system must be *sufficiently contractive* to merge the mismatch represented by  $\Delta$ .

We're now ready to state the main result of this section.

**Theorem 1** (Incremental stability of feedforward systems). *Consider system (1) and let Assumption 1, 2 and 3 hold. Moreover assume that there exists  $L_M \geq 0$  such that  $\left| \frac{\partial \bar{M}}{\partial x}(x) \right| \leq L_M$  for all  $x$ . Then, for any gain  $\kappa > 0$ , the system (1) in closed-loop with the control law*

$$u = \alpha_0(x) + \kappa(H\Lambda)^\top(\eta - \bar{M}(x)) + \varrho(x) \quad (11)$$

*is globally incrementally exponentially stable.*

*Proof.* We consider the change of coordinates  $\eta \mapsto z := \eta - \bar{M}(x)$  with  $\bar{M}$  solving (6). By making use of (7), the closed-loop system can be then written in the form

$$\dot{\mathbf{x}} = F(\mathbf{x}), \quad F(\mathbf{x}) := \begin{bmatrix} f(x) + g(x) \left[ \kappa(H\Lambda)^\top z + \varrho(x) \right] \\ (\Phi - \kappa\Lambda(H\Lambda)^\top)z - \Lambda\varrho(x) - \Delta(x) \end{bmatrix} \quad (12)$$

with  $\mathbf{x} = (x^\top, z^\top)^\top$ . It is known (see [1], [4]) that if there exists a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^{n_x+n_\eta} \mapsto \mathbb{R}^{n_x+n_\eta} \times \mathbb{R}^{n_x+n_\eta}$  with  $P = \mathcal{P}^\top > 0$  such that

$$L_F \mathcal{P}(\mathbf{x}) \leq -\underline{p}I, \quad \underline{p}I \leq \mathcal{P}(\mathbf{x}) \leq \bar{p}I \quad (13)$$

for all  $\mathbf{x}$ , for some strictly positive real numbers  $\underline{p}, \bar{p}, p > 0$  then (12) is globally exponentially incrementally stable. We look for a metric of the form

$$P(\mathbf{x}) := \begin{pmatrix} P(x) & 0 \\ 0 & \mu(H + b\bar{S}) \end{pmatrix} \quad (14)$$

with  $b, \mu$  being strictly positive real numbers to be defined,  $P$  taken as in Assumption 1,  $H$  as in Assumption 2 and  $\bar{S}$  being a strictly positive definite matrix to be defined. The main intuition behind this choice is that in view of Assumption 2 the matrix  $H$  by itself doesn't provide negativity in all the components of  $z$ . In order to "strictify" the metric, we rely on a design inspired by [33] (also used in [24]) by means of an observer. Indeed, by item 2 of Assumption 3, the couple  $(\Phi, (H\Lambda)^\top)$  is detectable, and therefore there exist two matrices  $S = S^\top > 0$  and  $K$  solving

$$\text{He} \left\{ S(\Phi - K(H\Lambda)^\top) \right\} \leq -2I. \quad (15)$$

Hence let  $\bar{S} = S$  and consider the matrix function  $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^{n_\eta} \mapsto \mathbb{R}^{(n+n_\eta) \times (n+n_\eta)}$

$$\mathcal{L}(\mathbf{x}) := L_F \mathcal{P}(\mathbf{x}) + \begin{pmatrix} \underline{p}_1 I & 0 \\ 0 & \underline{p}_2 I \end{pmatrix} \quad (16)$$

for some  $\underline{p}_1, \underline{p}_2$  strictly positive real numbers to be chosen. If  $\mathcal{L}(\mathbf{x}) \leq 0$  for all  $\mathbf{x}$  and for some  $\underline{p}_1, \underline{p}_2$ , then (13) holds with  $\underline{p} = \min\{\underline{p}_1, \underline{p}_2\}$ . Thanks to (8) and to the Killing Vector property  $L_g P(x) = 0$  in Assumption 1, we have that

$$\mathcal{L}(\mathbf{x}) = \begin{pmatrix} \ell_1(\mathbf{x}) & \ell_2(\mathbf{x}) \\ \ell_2^\top(\mathbf{x}) & \ell_3(\mathbf{x}) \end{pmatrix} \quad (17)$$

where

$$\begin{aligned} \ell_1(\mathbf{x}) &= L_f P(x) + \text{He} \left\{ P(x)g(x) \frac{\partial \varrho}{\partial x}(x) \right\} + \underline{p}_1 I \\ \ell_2(\mathbf{x}) &= \kappa P(x)g(x)(H\Lambda)^\top \\ \ell_3(\mathbf{x}) &= \mu \text{He} \left\{ (H + b\bar{S})(\Phi - \kappa\Lambda(H\Lambda)^\top) \right\} + \underline{p}_2 I. \end{aligned}$$

By adding and subtracting the term  $\mu b \text{He}\{SK(H\Lambda)^\top\}$ , by using equation (13) and by Assumption 2 we get

$$\begin{aligned} \ell_3(\mathbf{x}) &\leq \mathfrak{p}_2 I - \mu \left[ 2\kappa H\Lambda(H\Lambda)^\top + 2bI \right. \\ &\quad \left. - b \text{He}\{S(K - \kappa\Lambda)(H\Lambda)^\top\} \right] \\ &\leq \mathfrak{p}_2 I - \mu \left[ 2\kappa H\Lambda(H\Lambda)^\top + 2bI \right. \\ &\quad \left. - \frac{b^2}{2\beta} S(K - \kappa\Lambda)(K - \kappa\Lambda)^\top S - \frac{\beta}{2} H\Lambda(H\Lambda)^\top \right] \end{aligned}$$

for any real number  $\beta > 0$ , where we used the generalized Young's inequality. Therefore, we select  $\beta = 4\kappa$ ,  $b < \frac{2\beta}{|S(K - \kappa\Lambda)|^2}$  and  $\mathfrak{p}_2 < \frac{\mu b}{2}$  and we get  $\ell_3(\mathbf{x}) < 0$  for all  $\mathbf{x}$ , for any  $\kappa, \mu > 0$ . In order to have  $\mathcal{L} < 0$ , it remains to check the sign of its Schur complement  $\mathcal{S}_{\mathcal{L}}(\mathbf{x})$ . In particular, we need  $\mathcal{S}_{\mathcal{L}}(\mathbf{x}) := \ell_1(\mathbf{x}) + \ell_2(\mathbf{x})\ell_3^{-1}(\mathbf{x})\ell_2^\top(\mathbf{x}) < 0$ . Keeping in mind item 4 of Assumption 3 and the upper bounds of  $P$  and  $g$ , we get

$$\mathcal{S}_{\mathcal{L}}(\mathbf{x}) \leq -(\lambda - \mathfrak{p}_1)I + \frac{4\kappa^2 \bar{p}^2 \bar{g}^2}{\mu b} (H\Lambda)^\top (H\Lambda).$$

Hence, for any  $\kappa > 0$ , set

$$\mathfrak{p}_1 \leq \frac{\lambda}{4}, \quad \mu \geq \frac{16\kappa^2 \bar{p}^2 \bar{g}^2 |H\Lambda|^2}{\lambda b}$$

so that  $\mathcal{S}_{\mathcal{L}}(\mathbf{x}) < 0$  and therefore  $\mathcal{L}(\mathbf{x}) < 0$  for all  $\mathbf{x}$ . Note that the metric  $\mathcal{P}$  has been obtained in the  $\mathbf{x} = (x, z)$ -coordinates. In order to complete the proof, we need to come back to the original coordinates  $(x, \eta)$ . This can be done with a globally Lipschitz diffeomorphism. In particular, the metric  $\bar{\mathcal{P}}$  in the original coordinates is defined as

$$\bar{\mathcal{P}}(x, \eta) := E(x)^\top \mathcal{P}(\mathbf{x}) E(x), \quad E(x) := \begin{pmatrix} I & 0 \\ -\frac{\partial \bar{M}}{\partial x}(x) & I \end{pmatrix}, \quad (18)$$

namely

$$\bar{\mathcal{P}} = \begin{pmatrix} P(x) + \mu(H + bS) \frac{\partial \bar{M}^\top}{\partial x}(x) \frac{\partial \bar{M}}{\partial x}(x) & \star^\top \\ -\mu(H + bS) \frac{\partial \bar{M}}{\partial x}(x) & \mu(H + bS) \end{pmatrix}.$$

Note that

$$E(x)^{-1} = \begin{pmatrix} I & 0 \\ \frac{\partial \bar{M}}{\partial x}(x) & I \end{pmatrix},$$

and, since  $\bar{M}$  is Lipschitz,

$$|E(x)| \leq 1 + L_M, \quad |E(x)^{-1}| \leq 1 + L_M, \quad \forall x \in \mathbb{R}^n.$$

Hence, for all vectors  $v$ ,

$$\begin{aligned} v^\top \bar{\mathcal{P}}(x, \eta) v &\geq \min\{\underline{p}, \mu |H + bS|\} |E(x)v|^2 \\ &\geq \min\{\underline{p}, \mu |H + bS|\} \frac{|v|^2}{|E(x)^{-1}|^2} \\ &\geq \frac{\min\{\underline{p}, \mu |H + bS|\}}{(1 + L_M)^2} |v|^2. \end{aligned}$$

On another hand,

$$\begin{aligned} v^\top \bar{\mathcal{P}}(x, \eta) v &\leq \max\{\bar{p}, \mu |H + bS|\} |E(x)v|^2 \\ &\leq \max\{\bar{p}, \mu |H + bS|\} (1 + L_M)^2 |v|^2. \end{aligned}$$

Hence, the closed-loop system is incrementally globally exponentially stable with respect to the contraction metric  $\bar{\mathcal{P}}$  satisfying

$\mathfrak{p}_0 I \leq \bar{\mathcal{P}} \leq \bar{\mathfrak{p}}_0 I$  and  $L_{\bar{\mathcal{P}}}(x, \eta) \leq -\mathfrak{p}_0 I$  with  $\bar{\mathcal{F}}(x, \eta)$  being the closed-loop (12) in the original  $(x, \eta)$ -coordinates,  $\mathfrak{p}_0 = \min\{\mathfrak{p}_1, \mathfrak{p}_2\}$  and

$$\begin{aligned} \bar{\mathfrak{p}}_0 &:= \max\{\bar{p}, \mu |H + bS|\} (1 + L_M)^2, \\ \mathfrak{p}_0 &:= \frac{\min\{\underline{p}, \mu |H + bS|\}}{(1 + L_M)^2}. \end{aligned}$$

□

To conclude, note that the proposed control design may be applied recursively. As such, the feedback design procedure (and the verification of the main assumptions) can be split into multiple (easier) steps, in order to deal with (possibly) lower-dimensional systems, greatly simplifying the more general feedback design problem.

### III. HARMONIC REGULATION

#### A. Sufficient conditions for global harmonic regulation

An interesting application context for the results in Section II regards the problem of harmonic output regulation [29]. Assume to have a nonlinear system of the form

$$\dot{x} = f(x) + g(x)(u + d(t)), \quad (19a)$$

$$e = h(x) - r(t) \quad (19b)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control action,  $f : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x}$  and  $g : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_u}$  coupled with a regulation error  $e$  where  $h : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  with  $f, g, h$  to be  $C^2$  and  $f(0) = 0$ ,  $h(0) = 0$  with  $n_e \leq n_u^2$ . Let  $r : \mathbb{R} \mapsto \mathbb{R}^{n_e}$ ,  $d : \mathbb{R} \mapsto \mathbb{R}^{n_u}$  being smooth time-varying external references with finite  $\mathcal{L}_\infty$ -norm which are  $T$ -periodic, i.e. there exists  $T \geq 0$  such that

$$r(t) = r(t + T) \quad \text{and} \quad d(t) = d(t + T). \quad (20)$$

Consider the problem of designing a control law such that system (19) has bounded trajectories in forward time and the regulated error (19b) asymptotically converges to a periodic trajectory that has no harmonic content at some desired frequencies whose basic period  $T$  is given from the signals  $(r, d)$ . In other words, the objective is to guarantee that the error (19b) is a  $T$ -periodic signal which should not present a harmonic content at the frequencies  $\omega_\ell = \ell \frac{2\pi}{T}$  for  $\ell = 0, 1, \dots, L$  for some  $L \geq 0$ . Namely, the first  $L$ -Fourier coefficients of  $e$  are zero, i.e.

$$c_\ell := \frac{1}{T} \int_0^T e(t) \exp\left(2i\ell\pi \frac{t}{T}\right) dt = 0, \quad \forall \ell = 0, \dots, L. \quad (21)$$

A common approach to solve such a problem is to use an internal-model based control design ([24], [27]), that is, to extend the plant (19) with a dynamic system processing the regulation error and containing harmonic oscillators at the desired frequencies, together with a stabilizer for the closed-loop system. In particular, we consider a dynamical controller of the form

$$\dot{\eta} = \Phi\eta + \Gamma e \quad (22)$$

where the matrices  $\Phi, \Gamma$  are selected as

$$\begin{aligned} \Phi &= \text{blkdiag}(0 \quad w_1\Phi_1 \quad \dots \quad w_L\Phi_1), \\ \Gamma &= \text{blkdiag}(\Gamma_0 \quad \Gamma_1 \quad \dots \quad \Gamma_L) \end{aligned} \quad (23)$$

where  $\Phi_1 \in \mathbb{R}^{2n_e \times 2n_e}$  and  $\Gamma_\ell \in \mathbb{R}^{2n_e}$  are selected as

$$\Phi_1 = \text{blkdiag}(\phi, \dots, \phi), \quad \phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (24)$$

<sup>2</sup>Such a condition is known to be necessary for the linear systems case.

and  $\Gamma_\ell = (\gamma, \dots, \gamma)$  such that each couple  $(w_\ell \phi, \gamma)$  is controllable, together with a stabilizer for the closed-loop (19),(22) (see [23], [28]). In the considered framework, we allow the external signals  $(r, d)$  to span the whole  $\mathbb{R}^{n_e} \times \mathbb{R}^{n_u}$  and we look for a global result in the domain of attraction. The existing results have been developed only for the class of nonlinear systems having a globally defined normal form and possessing a minimum phase property [24], [34]. On the other hand, for systems working in the “original” coordinates the existing results are only local in the amplitude of the external signals ([23], [29]), that is, with the external signal’s amplitude being sufficiently small. Without normal forms but still providing a global result, we rely on incremental properties of nonlinear systems [6], [7]. In such a case, the idea is to rely on the results developed in the previous Section to make the closed-loop system incrementally globally exponentially stable, uniformly with respect to the external signals. In fact, system (19), (22) is of the form (1) with  $v(x) = \Gamma h(x)$  and Assumption 2 is automatically satisfied with  $H = I$ . Hence the following holds.

**Proposition 1** (Global Harmonic Regulation). *Consider system (19) extended with the dynamical system (22), (23), (24) and let Assumptions 1 and 3 hold with  $v(x) = \Gamma h(x)$  and  $H = I$ . Then for any  $(r, d)$  satisfying (20) and any initial conditions  $(x_0, \eta_0)$ , system (19) in closed-loop with the dynamic control law (22), (11) with any gain  $\kappa > 0$  has bounded trajectories and the error (19b) asymptotically converges to a  $T$ -periodic solution satisfying (21).*

*Proof.* We consider the change of coordinates  $\eta \mapsto z := \eta - \bar{M}(x)$  with  $\bar{M}$  solving (6). By making use of (7), the closed-loop system can be then written in form

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \Omega(\mathbf{x})R(t), \quad \Omega(\mathbf{x}) = \begin{pmatrix} g(x) & 0 \\ \Lambda & I \end{pmatrix}, \quad (25)$$

with  $\mathbf{x} = (x^\top, z^\top)^\top$ , the vector field  $F$  defined as in (12) and  $R(t) = \begin{bmatrix} d^\top(t) & r^\top(t) \end{bmatrix}^\top$ . From Theorem 1, the matrix function  $\mathcal{P}$  defined in (14) is a Riemannian metric for the vector field  $F$ . Moreover  $\Omega(\mathbf{x})$  possesses the Killing vector property with respect to such  $\mathcal{P}(\mathbf{x})$ , i.e.  $L_{\Omega} \mathcal{P}(\mathbf{x}) = 0$  for all  $\mathbf{x}$ . Since  $\Omega(\mathbf{x})$  is bounded for all  $\mathbf{x}$ , by [32, Theorem 2], this implies that the closed-loop is incremental-ISS (see [3, Definition 2.1]) with respect to  $R(t)$ . Since  $(r, d)$  satisfy (20) with period  $T$ , then the trajectories of the system and of the error  $e$  converge to a bounded and periodic solution with same period  $T$ . This can be obtained by combining [3, Proposition 4.4] and [3, Proposition 4.5]. By construction of the matrices  $\Phi, \Gamma$  as in (23), (24) and by [29, Proposition 1], the first  $L$ -Fourier coefficients of the error  $e$  are zero, i.e. (21) holds. To conclude, note that the Killing vector property is invariant with respect to the globally Lipschitz diffeomorphism (18), and hence the result holds also in the coordinates  $(x, \eta)$ .  $\square$

## B. A test design for a class of nonlinear systems

In this section, we propose a possible design to apply the results in Proposition 1. We consider a nonlinear system of the form (19) where

$$\begin{aligned} f(x) &= Ax + N\varphi(\zeta), & g(x) &= B, \\ h(x) &= Cx + D\varphi(\zeta), & \zeta &= Jx \end{aligned} \quad (26)$$

where  $A, N, J, B, C, D$  are constant matrices of suitable dimension and  $\varphi(s)$  is a scalar  $C^1$  nonlinearity with  $\varphi(0) = 0$  without loss of generality. In this case, many practical examples can be found whose plant belongs to the classes of systems described by (26). Some examples are the mechanical ventilation machine in [28], a flexible

link manipulator in [35] and the surge system considered in [36], [37]. For such a class of systems, we first assume that Assumption 1 is satisfied with respect to some constant metric  $P = P^\top > 0$ . In order to rely on Proposition 1, we propose the following design. Let  $\mathcal{Q}$  be defined as the following matrix parametrized by  $a$  and  $\omega$

$$\mathcal{Q}(a, \omega) := \begin{pmatrix} A^\top \otimes I - I \otimes \Phi(\omega) & -C^\top \otimes I \\ J^\top (aB^\top - N^\top) \otimes I & -J^\top D^\top \otimes I \end{pmatrix} \quad (27)$$

where  $\Phi$  is as in (23) where we explicitly expressed the dependency on the parameter  $\omega = (w_1, \dots, w_L)$  for some  $L \geq 1$  and let  $\mathcal{A} \subset \mathbb{R}$  be the set defined as

$$\mathcal{A}(\lambda) := \left\{ a \in \mathbb{R} \mid \forall x \in \mathbb{R}^n \right. \\ \left. \text{He}\{P(A + (N + Ba)\frac{\partial \varphi}{\partial \zeta}(Jx)J)\} \leq -\lambda I \right\}. \quad (28)$$

Then the following holds.

**Corollary 1.** *Consider system (26) and assume that Assumption 1 holds for some constant matrix  $P = P^\top > 0$ . Given  $\omega > 0$ , let  $\lambda > 0$  and suppose there exists  $\bar{a} \in \mathcal{A}(\lambda)$  such that  $\det(\mathcal{Q}(\bar{a}, \omega)) = 0$ . Let  $M, \Gamma$  be any solution to*

$$\mathcal{Q}(\bar{a}, \omega) \begin{pmatrix} \text{vec}(M) \\ \text{vec}(\Gamma) \end{pmatrix} = 0.$$

*If  $(\Phi, B^\top M^\top)$  is detectable and  $(\Phi, \Gamma)$  is controllable, then Assumption 3 holds with  $\bar{M}, \Delta, \varrho, \Lambda$  given as*

$$\begin{aligned} \bar{M}(x) &= Mx, & \Delta(x) &= (MN - \Gamma D)\varphi(Jx), \\ \varrho(x) &= \bar{a}\varphi(Jx), & \Lambda &= MB. \end{aligned} \quad (29)$$

*Proof.* First, note that by Assumption 1 and by continuity, the set  $\mathcal{A}$  is non-empty. Then, let  $\bar{M}, \Delta, \varrho, \Lambda$  be defined as in (29). For the considered class of systems (26), the main idea behind this choice is to pick the function  $\bar{M}(x)$  as a linear function which satisfies the linear part of (6) (and therefore also (7)), and to stick all the nonlinearities in the term  $\Delta(x)$ , which will be handled by the robustness of the open-loop system with (9) through the existence of a mapping  $\varrho(x)$  satisfying (8). With this choice, the conditions (6), (8) reduce to the existence of constant matrices  $M, \Gamma$  and a real number  $a$  solution of the matrix equalities

$$MA = \Phi(\omega)M + \Gamma C \quad (30a)$$

$$aMBJ = (MN - \Gamma D)J \quad (30b)$$

where (30a) comes from the linear terms of (6) and (30b) from the definition of  $\Delta$  in (29) and (8). Recalling the definition of Sylvester equation, the conditions (30a), (30b) can be rewritten with the Kronecker operator  $\otimes$  as a linear problem of the form

$$\mathcal{Q}(a, \omega)\mathcal{Y} = 0 \quad (31)$$

where  $\mathcal{Y}$  is a  $2k \times n_x + 2k$  column vector of unknowns defined as  $\mathcal{Y} = (\text{vec}(M), \text{vec}(\Gamma))^\top$  and  $\mathcal{Q}(a, \omega)$  is the matrix defined as in (27). For fixed  $\omega$ , if there exists  $\bar{a} \in \mathcal{A}$  such that the matrix  $\mathcal{Q}(\bar{a}, \omega)$  has a non-null kernel, then there exists at least one non-null vector  $\bar{\mathcal{Y}}$  such that  $\mathcal{Q}(\bar{a}, \omega)\bar{\mathcal{Y}} = 0$ . In such case, the matrices  $M, \Gamma$  can be constructed from the vector  $\bar{\mathcal{Y}}$  and the function  $\varrho$  is selected as  $\varrho(x) = \bar{a}\varphi(Jx)$ . From such choice, (6), (8) are satisfied. Moreover, since  $\bar{a} \in \mathcal{A}$ , then also (9) holds. Since  $(\Phi, B^\top M^\top)$  is detectable and  $(\Phi, \Gamma)$  is controllable by assumption, then Assumption 3 holds.  $\square$

**Remark 1.** *Note that, for a given system and a fixed set  $\mathcal{A}$ , Corollary 1 shows the frequencies  $w$  that can be naturally regulated*

with the proposed design. In this case, the set of  $w$  that can be regulated is a subset of the  $w$  satisfying the non-resonance condition

$$\text{rank} \begin{pmatrix} A - \sigma I & B \\ C & 0 \end{pmatrix} = n_x + n_e$$

with  $\sigma$  being any eigenvalue of the matrix  $\Phi$ , see [29].

**Remark 2.** Note that the design proposed in Corollary 1 provides a test that can be performed on the system: for a given set of frequencies  $\omega = (w_1, \dots, w_N)$ , compute the matrix  $\mathcal{Q}(a, \omega)$ . Then compute its determinant which is a polynomial in the variable  $a$ . Find the (finite) values of  $a$  that nullify the determinant and check if such values are in the set  $\mathcal{A}$ . If this is the case, then it's sufficient to check the detectability of  $(\Phi, B^\top M^\top)$  and the controllability of  $(\Phi, \Gamma)$ .

### C. The case of minimum phase systems

The main drawback of the design provided by Theorem 1 when applied to the case of harmonic regulation, is that in the most general case a constructive design which is uniform on the number of oscillators  $L$  and independent on the considered frequency  $w_\ell$  to check the validity of Assumption 3 might not be always possible but instead depends on the considered system. In this section, we show that this is not the case if we consider the special case of nonlinear systems which admit a globally defined normal form ([26, Section 4]) with a zero-dynamics which is incrementally stable. We want to stress that the result in Proposition 1 does not assume a priori the existence of a globally defined normal form. We consider, for the sake of simplicity, a single-input single-output (SISO) system with unitary relative degree (the extension to higher relative degree can be dealt with canonical tools, see, e.g., [27, Section V]) of the form

$$\dot{z} = \psi(z, y) \quad \dot{y} = q(z, y) + u - r(t) \quad (32)$$

where  $x = (z^\top, y)^\top \in \mathbb{R}^{n_x-1} \times \mathbb{R}$  is the state, and the vector fields  $\psi : \mathbb{R}^{n_x-1} \times \mathbb{R} \mapsto \mathbb{R}^{n_x-1}$  and  $q : \mathbb{R}^{n_x-1} \times \mathbb{R} \mapsto \mathbb{R}$  are both  $C^1$  (we take  $d(t) = 0$  for all  $t$ ). Most of the existing results on output regulation for nonlinear systems focus on systems having a minimum phase zero-dynamics. As our approach is to cast the problem in the contraction framework, we will assume that system (32) possesses a contractive zero-dynamics as follows.

**Assumption 4 (Minimum Phase).** Consider system (32). There exist positive real numbers  $\bar{q}, \bar{\psi}_y, \bar{p}_z, \underline{p}_z, \lambda_z > 0$  and a  $C^1$  matrix function  $P_z : \mathbb{R}^{n_x-1} \mapsto \mathbb{R}^{(n_x-1) \times (n_x-1)}$  taking symmetric positive values such that the following inequalities<sup>3</sup> hold

$$\left| \frac{\partial q}{\partial x}(x) \right| \leq \bar{q}, \quad \left| \frac{\partial \psi}{\partial y}(x) \right| \leq \bar{\psi}_y, \quad (33)$$

$$\underline{p}_z I_{n-1} \leq P_z(z) \leq \bar{p}_z I_{n-1}, \quad L_\psi P_z(z) \leq -2\lambda_z I_n, \quad (34)$$

for all  $x \in \mathbb{R}^{n_x}$ .

**Remark 3.** Note that the Assumption 4 extends (for the global case) the assumption in [24, Theorem 1] on the zero-dynamics to have a (locally) exponentially attractive steady-state when  $y = 0$  (see also [34, Assumption 2], [27]).

Under such an assumption we have the following result.

**Proposition 2 (Global Harmonic Regulation for Minimum Phase Systems).** Consider system (32) extended with the dynamics (22), (23), (24) and let Assumption 4 hold. Then for any number of

<sup>3</sup>The notation  $L_\psi P_z(z, y)$  has to be understood as the Lie derivative of  $P_z$  along the vector field  $z \mapsto \psi(z, y)$  where  $y$  is fixed.

oscillators  $L \geq 0$ , Assumption 1 and 3 are satisfied with  $h(x) = y$ ,  $H = I$  and

$$\begin{aligned} P(x) &= \begin{pmatrix} P_z(z) & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \alpha_0(x) = -ky, \\ \bar{M}^\top(x) &= \begin{bmatrix} y & \bar{M}_1^\top(x) & \dots & \bar{M}_L^\top(x) \end{bmatrix}, \\ \bar{M}_\ell^\top(x) &= (y \quad y), \quad \varrho(x) = -q(z, y) + (k-1)y, \\ \Lambda^\top &= \begin{bmatrix} 1 & \Lambda_1^\top & \dots & \Lambda_L^\top \end{bmatrix}, \quad \Lambda_\ell^\top = (1 \quad 1), \\ \Gamma^\top &= \begin{bmatrix} -1 & \Gamma_1^\top & \dots & \Gamma_L^\top \end{bmatrix}, \quad \Gamma_\ell^\top = [-w_\ell - 1 \quad w_\ell - 1] \\ \Delta^\top(x) &= \begin{bmatrix} \Delta_0(x) & \Delta_1^\top(x) & \dots & \Delta_{2L}^\top(x) \end{bmatrix}, \\ \Delta_0(x) &= \Delta_\ell(x) = -\varrho(x) \end{aligned} \quad (35)$$

with  $\ell = 1, \dots, L$  for some gain  $k > 0$  sufficiently large and  $\varepsilon > 0$  sufficiently small, both independent on  $L$ .

*Proof.* We first show that Assumption 1 holds. After the pre-stabilizing action  $\alpha_0$ , the open-loop system (32) is of the form (19) where  $x = (z, y)$  and

$$f(x) = \begin{pmatrix} \psi(z, y) \\ q(x, y) - ky \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider the metric  $P(x)$  defined as in (35). First, note that the Killing Vector property is satisfied, i.e.  $L_g P(x) = 0$  for all  $x$ . We look for  $p > 0$  such that  $\mathcal{L}(x) < 0$  where

$$\begin{aligned} \mathcal{L}(x) &:= L_f P(x) - pI \\ &= \begin{pmatrix} L_{\psi_z} P_z(z) & P_z(z) \frac{\partial \psi}{\partial y}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \\ \star^\top & 2\varepsilon \left( \frac{\partial q}{\partial y}(x) - k \right) \end{pmatrix} - pI \end{aligned}$$

Fix any  $\varepsilon > \frac{\lambda_z}{2(\bar{p}_z^2 \bar{\psi}_y^2)}$  and set  $k \geq \bar{k} := \bar{q} + \frac{\lambda_k}{2\varepsilon \lambda_z} + \frac{(\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2}{2\varepsilon \lambda_z}$  for some  $\lambda_k > 0$ . From the bounds on  $q$  from Assumption 4, the bottom-right term of  $L_f P(x)$  is negative definite for any  $\varepsilon, \lambda_k > 0$ . Hence to show (3) we check the Schur's complement  $\mathcal{S}(x)$  of  $L_f P(x)$ . With the bounds on  $\psi, q$  and since there exists  $\lambda_z$  such that  $L_{\psi_z} P_z(z) \leq -2\lambda_z I$  from Assumption 4, we get

$$\mathcal{S}(x) \leq \left( -2\lambda_z + \frac{(\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2}{\frac{1}{\lambda_z} (\lambda_k + (\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2)} \right) I \leq -2\lambda_k I$$

Therefore, Assumption 1 holds with  $P$  defined in (35),  $\underline{p} = \min\{\underline{p}_z, \varepsilon\}$ ,  $\bar{p} = \max\{\bar{p}_z, \varepsilon\}$  and  $p = \min\{2\lambda_k, 2\lambda_z, \frac{2\lambda_k}{\lambda_z}\}$ . We then check each point of Assumption 3 separately. For item 1 note that (6) holds. Indeed for  $\ell = 0$  we get

$$\begin{aligned} \frac{\partial M_0}{\partial x}(x) f(x) &= h(x) + \Delta_0(x) \\ q(x) - \tau y &= -y + \Delta_0 \end{aligned}$$

and we recover the definition of  $\Delta_0(x)$  in (35). Similarly, for  $\ell = 1, \dots, L$  from the definitions of  $\Phi, \Gamma$  as in (23), (24), (35) we have

$$\frac{\partial M_\ell}{\partial x}(x) f(x) = \begin{pmatrix} 0 & w_\ell \\ -w_\ell & 0 \end{pmatrix} M_\ell(x) + \Gamma_\ell h(x) + \Delta_\ell(x)$$

which is a set of  $2L$  identities of the form

$$\begin{cases} (q(x) - \tau y) = w_\ell y + (-w_\ell - 1)y + \Delta_\ell(x) \\ (q(x) - \tau y) = -w_\ell y + (w_\ell - 1)y + \Delta_\ell(x) \end{cases}$$

from the definition of  $\Delta$  as in (35). About item 2, note that  $L_g M(x) = \Lambda$ . Since  $M$  is linear and  $\Phi$  is block diagonal, the observability matrix  $\mathcal{O}$  of  $(\Phi, \Lambda^\top)$  is

$$\mathcal{O} = \text{blkdiag}\{1, \mathcal{O}_1, \dots, \mathcal{O}_L\}, \quad \mathcal{O}_\ell = \begin{pmatrix} 1 & w_\ell \\ 1 & -w_\ell \end{pmatrix}$$

which has non-zero determinant for all  $w_\ell \neq 0$ . Hence  $(\Phi, \Lambda^\top)$  is detectable. About item 3, again, we recover a set of  $2L+1$  identities. Indeed for all  $\ell = 0, \dots, 2L$  we defined  $\varrho(x) = -\Delta_\ell(x)$ . About item 4, equation (9) reads

$$\begin{pmatrix} L_{\psi_z} P_z(z) & P_z(z) \frac{\partial \psi}{\partial e}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \\ \star^\top & 2\varepsilon \left( \frac{\partial q}{\partial e}(x) - k \right) \end{pmatrix} - \begin{pmatrix} 0 & \left( \varepsilon \frac{\partial q}{\partial z}(z, y) \right)^\top \\ \star^\top & 2\varepsilon \left( \frac{\partial q}{\partial y}(z, y) - k + 1 \right) \end{pmatrix} \leq -\lambda_z I$$

where the right-hand-side follows a similar analysis as previously, from the definition of  $\varepsilon$  and Assumption 4. To conclude the proof, note that the controllability matrix  $\mathcal{C}$  of  $(\Phi, \Gamma)$  has a block-diagonal structure of the form

$$\mathcal{C} = \text{blkdiag}\{1, \mathcal{C}_1, \dots, \mathcal{C}_L\}, \quad \mathcal{C}_\ell = \begin{pmatrix} -w_\ell - 1 & w_\ell(w_\ell - 1) \\ w_\ell - 1 & w_\ell(w_\ell + 1) \end{pmatrix}$$

where each  $\mathcal{C}_\ell$  is full rank and hence  $\mathcal{C}$  is so.  $\square$

From Proposition 1, the following then holds.

**Corollary 2.** *Consider system (32) with any  $r \in \mathbb{R}$  satisfying (20) and let Assumption 4 hold. Then for any  $L \geq 0$ , and any initial condition  $(z_0, y_0, \eta_0)$  the system in closed loop with the dynamical control law (22), (11), (23), (24), (35) for any gain  $\kappa > 0$  has bounded  $T$ -periodic trajectories and the regulation error  $e = y - r(t)$  satisfies (21).*

**Remark 4.** *The control (11) recovers known designs used in nonlinear output regulation theory for minimum phase systems (see [27], [29]). The term  $\alpha_0 = -ky$  is a high-gain which acts as a pre-stabilizer for the  $y$ -dynamics where  $k$  is sufficiently large to handle the Lipschitz constant of  $q$  while the term  $\Lambda^\top(\eta - \bar{M}(x))$  provides negativity in the directions of the internal model. In the design (11) a particular feature is the term  $\varrho$  which behaves as a feedback linearization term in the I/O mapping between  $u$  and  $\eta$ . It allows obtaining an upper-triangular structure in the Jacobian of the closed-loop system under the change of coordinates  $\eta \mapsto z = \eta - \bar{M}(x)$ .*

#### D. Academic example

Consider system of the form (19) with

$$f(x) = \begin{pmatrix} -5x_1 + x_2 + \sin(x_1) \\ x_1 - 2x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

and  $h(x) = \sin(x_1) - x_2$ . We aim to cancel the harmonic content at the frequency  $w = \frac{2\pi}{T} = 1$ . The interest in such an example is that the system does not admit a globally defined normal form. Therefore, when global harmonic regulation is the goal, approaches based on the use of normal forms cannot be applied. Indeed  $L_g h(x) = \cos(x_1) + 1$ , which is not constant for  $x_1 = (2j+1)\pi$  for any integer  $j$ . Yet, a solution does exist. Indeed, the approach proposed in our work is a valuable tool. Assumption 1 is satisfied with  $P = I$  where the Killing Vector property holds as both  $P$  and  $g$  are constant. Then, following Proposition 1, we extend the plant with

the dynamics (22) where we pick  $\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $\Gamma = \begin{pmatrix} 0.478 \\ 0.433 \end{pmatrix}$ .

We select  $\bar{M}, \Delta, \varrho, \Gamma, \Lambda$  solution of (6), (7), (8) according to Corollary 1 where  $A = (-5, 1; 1, -2)$ ,  $B = (-1; -1)$ ,  $C = (0, -1)$ ,  $D = 1$ ,  $H = (1, 0)$ ,  $N = (1; 0)$  and  $\varphi(s) = \sin(s)$ . In such case, we can select  $\bar{a} = 0.4962 \in \mathcal{A}(1)$  and check the detectability of  $(\Phi, \Lambda^\top)$  and the controllability of  $(\Phi, \Gamma)$ . With such a choice, Assumption 3 holds, and global harmonic regulation is achieved with the dynamic controller (22), (11).

## IV. CONCLUSIONS

In this work, we provided sufficient conditions for the design of a control law for systems in feedforward form to be incrementally globally exponentially stable. For this, we presented an incremental version of the forwarding mod $\{L_g V\}$ . Then we applied our result for harmonic regulation by means of a dynamical controller made of  $L$  linear oscillators processing a regulation error and an incremental forwarding design for the closed-loop system to be globally exponentially incrementally stable, uniformly with respect to the external signals. With our design, the regulation error asymptotically converges to a periodic steady-state bounded trajectory for any initial condition and for any amplitude of the external signals. On such a steady-state trajectory, harmonic regulation is achieved, namely the first (desired)  $L$ -Fourier coefficients are zero. The design is also specialized to a class of linear systems with a scalar nonlinearity and minimum-phase systems with incrementally stable zero-dynamics.

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