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Potential conflict factors in learning exact differential equations: Analysis of students' work

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Mathematics courses constitute one of the important components in the university education of future engineers. Many engineering students find them challenging and may even view them as a serious obstacle to a successful engineering career. One of the difficulties is related to the efficient connection of new material to already acquired knowledge and skills. Using our praxeological analysis of two institutional approaches to the concept of exact differential equation in Calculus and Differential Equations indicating potential conflict factors, we analyse students' work on the assignment designed for a pilot project.

Keywords: Exact differential equations, conservative vector fields, anthropological theory of the didactic, praxeological analysis, concept image

Introduction

Engineering is one of the richest branches of science and technology playing a key role in social and economic development of a modern society. As technology advances, the demand for highly qualified engineers grows. Goold and Devitt (2012) reported that more than half of engineers use a high level of curriculum mathematics at the workplace: “mathematical thinking usage is ‘absolutely critical’ in the engineering practice, and it is the ‘value’ one engineer brings to his job” (p. 7). University courses in Differential Equations (DEs) constitute an indispensable part of the engineering curriculum. Although the subject helps to design mathematical models describing various natural phenomena and physical processes, students are often focused on technical details of specific solution methods and have difficulties with the conceptual understanding of the underlying mathematical ideas.

Students may also face difficulties connecting material learned in different mathematics disciplines due to the insufficient coordination between university courses which ideally should be arranged progressively to optimally support students' learning. It is especially important to create sustainable links between basic courses in the first years of studies (Calculus/Analysis sequence and Linear Algebra) and more advanced ones (Differential Equations, Numerical Methods, Partial Differential Equations, Applied Mathematics). Rogovchenko and Rogovchenko (2022) combined anthropological theory of didactic (ATD) (Chevallard, 2019) with the constructs of concept definition and concept image (Tall & Vinner, 1981) to explore potential conflict factors in the learning and teaching of exact differential equations (EDEs) to future engineers in Calculus and Differential Equations courses. The main concern is that the distinctions between two institutional praxeologies, both within the mathematics domain but hosted by different institutions, may create learning difficulties. This paper reports the results of a pilot study exploring students' work on one of the problems in a graded assignment designed to test this conjecture.

Theoretical framework

We very briefly present some main ideas of ATD and refer the reader to Chevallard (2019) for more details. ATD offers a flexible epistemological framework to describe mathematical knowledge as a form of human activity. Its major tenet is that “all ‘knowledge’ can be modelled in terms of praxeologies” (Chevallard et al., 2016, pp. 2615-6). Praxeologies are used as building blocks for didactic systems understood as “any created reality of which people can be members” (Chevallard & Bosch, 2019, p. xxxi). A praxeology consists of a task T , a technique τ (tau), a technology θ (theta), and a theory Θ (big theta). It is often viewed as the union of two “blocks,” the *praxis part* $\Pi = [T/\tau]$ and the *logos part* $\Lambda = [\Theta/\theta]$. The praxis part is needed for performing the tasks, whereas the logos part is important for preserving the activity and sharing it with others thus enabling them to participate in the activity (Hardy, 2009). Because didactic systems are rooted in institutions, institutional approaches significantly impact students’ inducting into mathematical practices (Hardy, 2009; Winsløw et al., 2014). In this paper, praxeologies (praxeological organisations which we term *mathematical organisation (MO)* after the subject area) are used for modeling a human activity aimed at learning EDEs.

The construct of concept image complements ATD in our analysis. A description of the mathematical notion accepted by the professional community usually forms the concept definition. However, all humans learn concepts differently, and “for each individual a concept definition generates its own concept image” (Tall & Vinner, 1981, pp. 152-153), which is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.” Because the concept image is built over years and its distinct parts can be activated at certain times under different circumstances, a sense of confusion may appear; a potential conflict factor is defined as “a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition” (Tall & Vinner, 1981, pp. 152-153). For instance, learners may have concept images for function as a graph, “black box,” algebraic expression, diagram relating elements of two sets, set of pairs of elements, etc. The role of the concept image is especially important because it often replaces the concept definition during informal learning of concepts: “the concept image is utilised instead of the concept definition and also when the concept definition has been constructed” (Gascón, 2003, p. 47). In our data, students created their own concept images of an EDE, associating them with different mathematical objects, ideas, and notions which do not fall into a single rubric of technique or technology and cannot be fully analysed within the ATD framework. Lack of coherence between a student’s concept images may create difficulties for the learner who may correctly comprehend parts of the concept without relating them to each other.

Methodology

The authors conducted analysis of two MOs in the courses Mathematics 2 (Calculus, MO1) and Mathematics for Mechatronics (DEs, MO2), taught respectively to first-year engineering students and master’s students at the University of Agder (summarized in Table 1). We anticipated that potential conflict factors stemming from the differences between the two logos blocks $\Lambda_1 = [\Theta/\theta_1, \theta_2]$ and $\Lambda_2 = [\Theta/\theta_3]$ in MO1 and MO2 might lead to significant variations of students’ concept images for an EDE, making the connection to a concept of an EDE obscure or invisible. Even though the

technique of partial integration τ_2 in MO1 and MO2 is the same, technologies θ_1 and θ_2 furnish different justification. Furthermore, the lack of justification for the technology due to the omission of the theory Θ in both MOs may lead to learning difficulties because an incomplete logos part makes the preservation and communication of the practice difficult (Hardy, 2009).

Table 1: Praxeologies MO1 and MO2 (adapted from Rogovchenko & Rogovchenko, 2022)

MO	Task, T	Technique, τ	Technology, θ	Theory, Θ
MO1	Find a potential of a conservative vector field T_1 .	Test for conservative vector fields τ_1 . Partial integration τ_2 .	Necessary condition θ_1 for conservative fields without proof. Partial integration procedure θ_2 .	Theorem 3
MO2	Solve an exact DE T_2 .	Exactness test τ_3 . Partial integration τ_2 .	The exactness test and partial integration procedure θ_3 in the proof of Theorem 1.	Theorem 4

We reported the following four potential conflict factors between MO1 and MO2 (Rogovchenko & Rogovchenko, 2022). (1) An MO2 praxeology defines EDEs assuming that $y = y(x)$ whereas MO1 praxeology treats both variables equally. (2) An MO2 praxeology views solutions of an EDE as implicitly defined functions represented geometrically by integral curves, whereas MO1 praxeology defines them as equipotential curves. (3) An MO2 praxeology considers EDEs in a rectangular domain, and it is mentioned that the region should be simply connected. MO1 praxeology stresses that the existence of a potential for a vector field depends not only on the topology of the domain of the vector field, but also on the structure of the components of the field. (4) An MO2 praxeology accounts only for a 2D case, whereas MO1 praxeology allows extension of the notion an EDE to higher dimensions. Furthermore, thanks to a differential form used for a DE, the exactness test is easy to formulate for a 3D case and even for higher dimensions.

We conjectured that students may not recognise an EDE they met for the first time in the calculus sequence when they meet it once again in a master's course on DEs. To verify the validity of our conjecture, two tasks fitting corresponding descriptions for T in the 'task' column of Table 1 were selected for a pilot study. Because students' written work did not mention Theory Θ (which is not at all unusual for engineering students), we expected to see at least manifestations of conflict factors (1) and (4), and we analyse students' work within the framework in Table 1 focusing on the following aspects of Technique τ and Technology θ .

For Technique in MO1: a) test τ_1 for conservative vector fields - find the corresponding partial derivatives and show that they are equal, b) partial integration τ_2 : find the required partial integral.
 For Technique in MO2: a) exactness test τ_3 - find the corresponding partial derivatives and show that they are equal, b) partial integration τ_2 : find the required partial integral.

For Technology in MO1: a) necessary condition θ_1 for conservative vector fields: explain how to use the equality of corresponding partial derivatives in case of 3-dimensional vector field, b) partial integration procedure θ_2 : apply the partial integration to find the potential function. For Technology in MO2 - unpack the procedure θ_3 in the constructive proof of Theorem 1: a) the exactness test as a necessary and sufficient condition – explain how the equality of the corresponding partial derivatives leads to exactness of the DE, b) partial integration procedure – finding the general solution of the DE.

Organisation of a pilot study

The test problem was suggested in a regularly graded assignment to a group of senior students in mechatronics enrolled in the course Mathematics for Mechatronics. Students worked in eleven small groups of two-three students on the home assignment for one week and could also discuss the assignment with their peers from other small groups. The following problem was suggested:

Problem (a) The vector function $E = (E_x, E_y, E_z)$ where $E_x = 6xy, E_y = 3x^2 - 3y^2, E_z = 0$, represents possible electrostatic field. Calculate the line integral of E from the point $(0,0,0)$ to the point $(x_1, y_1, 0)$ along the path which runs straight from $(0,0,0)$ to the point $(x_1, 0, 0)$ and then to $(x_1, y_1, 0)$. Do a similar calculation for the path which runs along the two other sides of the rectangle, via the point $(0, y_1, 0)$. You get the same answer, if E indeed represents the electrostatic field. Find the equation describing surfaces of constant potential of this field.

(b) Find all solutions to a differential equation $6xydx + (3x^2 - 3y^2)dy = 0$.

(c) Compare your solutions in parts (a) and (b). What is common and what is different between the two solutions? Explain your answer.

Analysis of students' work

First we briefly describe students' work in ten small groups (Group 6 did not submit the report and thus was excluded from the analysis). *Group 1* acknowledged that the conservative vector field is two-dimensional and deduced the necessary condition for a 2D vector field to be conservative. However, their reasoning that if the result is the same for two different paths, the field is conservative was wrong. They concluded that "since the answers from 2A and 2B are similar, it can be seen that they are two different methods of solving the same problem, one using vector fields, and the other using differential equations."

Group 2 posed that the equality of two values of the line integral from one point to another along different paths is sufficient for a vector field to be conservative. They argued that "the same function is calculated as a solution both for the differential equation in problem 2b, as well as the potential function of the vector field given in problem 2a. In both methods, partial integration and derivation are used interchangeably in order to determine the constants. In 2b, the differential function is presented with two variables (x, y) . In problem 2a however, the potential function as well as the vector field represent points and vectors in the 3D space, and therefore three coordinates (x, y, z) are used. The line integrals investigated, however, are two-dimensional as their z coordinate is 0 in both cases. Therefore, the problem could be investigated as a two-dimensional problem in the xy -plane, as was the case with problem 2b." Remarkably, students' attention was focused on the reduced dimension of the vector field and its advantages. *Group 3* provided no explanations to their solutions, just calculations. Their conclusion was "both methods resulted in the exact same answer, but method b was a lot easier and faster. In addition, the solution of the first integral in method a) from $(0,0,0)$ to $(x_1, 0, 0)$ or $(0,0,0)$ to $(0, y_1, 0)$, is the same as $\int M dx$ or $\int N dy$."

Group 4 explained the result of calculations in part a) as follows. One can see by comparing I_1 and I_2 that they are giving the same answer. Meaning that E indeed represents an electrostatic field. The equation that describes the surface of constant potential of this field, could be described as "earlier

mentioned in Eq. 2.1 known as the negative gradient of phi.” They concluded that “comparing the two solutions, the obvious first thing that is noticed is that they are exactly similar. The line integrals solved in part a) are equal to the general solution (C) found in b). We can also see that the representation of the electrostatic field is similar to the differential equation stated in b). One key difference between the two solutions are the calculations done to produce the solution. In a), the line integrals were calculated based on definite integrals, whereas in b), the solution was found using indefinite integrals. This means that the integrals in a) have an infinite number of antiderivatives that give a valid answer. In b) we need to find the value of the antiderivative (C) to find the solution, as the DE was proven to be exact with only one solution. This means that the two tasks a) and b) are just two diverse ways of obtaining the same answer, where one is by using definite integrals and one using an exact differential equation with indefinite integrals.” *Group 5* provided accurate calculations concluding that “the answers we get in a) and b) are the same. This means that a conservative electric field can be seen as an exact differential equation and the general solution of the differential equation represents the potential of the electric field.”

Group 7 argued that “Throughout this assignment two different methods were used for calculating the same thing, one easier than the other. The solutions display the exact same outcome, calculated with two quite different approaches. The common thing states that from two different approaches, one can determine a problem using a significantly more complicated method, here the part a), and get the same result by solving a differential equation. In this scenario the same expression outcomes when calculating the partial function, compared to solving the DE. Using the approach with DE is a more sophisticated way of handling the problem, and throughout this assignment the knowledge about not needing the full line integral has emerged.” *Group 8* submitted inaccurate calculations with MATLAB concluding that “Both methods end up with the same answer. However, B) calculation is easier than calculation A). The solution of the first integral in method (from (0,0,0) to $(x_1, 0, 0)$ or $(0,0,0)$ to $(0, y_1, 0)$), is the same as $\int M dx$ or $\int N dy$. Further, in part A) a partial derivation was used, where we first moved in x and then y , and the opposite way. In B) we take both ways at the same time ending up with the same answer. For this we can conclude that the field is conservative in every direction and not just in the edges of the ‘square.’ This improves the validity of calculations and also shows how partial derivation in a conservative field is a valid solution.”

Group 9 concluded that “the given differential equation is in the same form. That means that $6xydx + (3x^2 - 3y^2)dy = 0$ describes all the paths along which the net work done on a particle moving through the field is zero. This is the same as finding level curves on the potential function found in 2a).” *Group 10* made the following conclusion: “The two methods give the same result, so in practice, solving the differential equation is the same as finding the potential function.” *Group 11* concluded that “the potential function from the vector field in task (a) is the same function as the function representing all solutions to the differential equation in part (b). The computation to find a potential function is also the same as the computation to solve exact DE in this case. Meaning, instead of parameterising like it is done in part (a), the solution from (b) could have been used to find the work done between the two points.”

Table 2 collects information regarding steps performed by students in MO1 and MO2. We see that in MO1 students were most successful in the use of the partial integration technique τ_2 (ten groups) and

partial integration procedure θ_2 in the technology part (ten groups). The most difficult part was the use of the necessary condition θ_1 in the technology part managed only by five groups. In MO2, students were most successful with partial integration technique τ_2 (eight groups) and partial integration procedure θ_3 in the technology part (ten groups). The exactness test in the technology part was the most difficult for students (only two groups). Interestingly, the use of the same partial integration technique τ_2 was successful for eight groups in both MO1 and MO2, and two groups were also successful with its application in MO1 but not in MO2.

Table 2: Students' work in terms of praxeologies MO1 and MO2

	Task, T	Technique, τ		Technology, θ	
MO1	Find a potential of a conservative vector field T_1 .	Test for conservative vector fields τ_1 .	G1, G2, G4, G7, G9-G11	Necessary condition θ_1 for conservative fields without proof.	G4, G7, G9-G11
		Partial integration τ_2 .	G1-G5, G7-G11	Partial integration procedure θ_2 introduced in Example 1.	G1-G5, G7-G11
MO2	Solve an exact DE T_2 .	Exactness test τ_3 .	G1, G2, G4, G5, G11	The exactness test as a necessary and sufficient condition.	G4, G11
		Partial integration τ_2 .	G1-G4, G8-G11	Partial integration procedure θ_3 in the proof of Theorem 1.	G1-G5, G7-G11

We classify students' conclusions in part (c) of the problem into four categories induced by the praxeologies MO1 and MO2. Our criteria for defining categories are based on students' recognition of similarities and distinctions between techniques and technologies in two praxeologies. Praxeological analysis (Rogovchenko & Rogovchenko, 2022) in Table 1 distinguishes the test for conservative vector fields τ_1 in MO1 from the exactness test τ_3 in MO2 and views partial integration τ_2 in MO1 and MO2 as the same technique but these conclusions were not easy for students. Students' difficulties with the comparison of two distinct formulations of the same task stem from their inability to connect relevant (parts of) concept images of an EDE into a coherent full vision of the concept. A 3D approach to an EDE with equal status of three variables in MO1 does not fit well a 2D approach with dependence of y on x in MO2 and this requires additional cognitive efforts from learners.

Category 1 (the same answer but different methods): students claimed that they obtained the same results by two different methods, first using the vector field and then with differential equations and noticed the similarity between the answers. “The answers we get in a) and b) are the same. This means that a conservative electric field can be seen as an exact differential equation and the general solution of the differential equation represents the potential of the electric field.” However, students failed to see different formulations of the same problem and the same method of partial integration used in both cases. They did not recognise that the method is the same arguing that “Since the answers from 2A and 2B are similar, it can be seen that they are two different methods of solving the same problem. One using vector fields, and the other using differential equations.” “The same function is calculated

as a solution both for the differential equation in problem 2b, as well as the potential function of the vector field given in problem 2a" (G1-G5, G7-G11).

Category 2 (similarities in Technique and Technology) Although some students noticed that "in both methods, partial integration and derivation are used interchangeably in order to determine the constants," they did not see the differences in Technology. This is not at all surprising given their background and traditional lack of attention of future engineers to theoretical results supporting mathematical procedures (G2-G4, G9).

Category 3 (methods are different but one is easier to implement): students described the DE approach as an "easier" one still arguing that these two methods are different. The claim that "both methods resulted in the exact same answer, but method b was a lot easier and faster" is also typical for engineering students who tend to value much more methods that are "easier and faster" (G3, G7, G8).

Category 4 (problems are different because the dimensions differ): students noticed that the vector field in part a) is two-dimensional: "in 2b, the differential function is presented with two variables (x, y). In problem 2a, however, the potential function as well as the vector field represent points and vectors in the 3D space, and therefore three coordinates (x, y, z) are used. The line integrals investigated, however, are two dimensional as their z coordinate is 0 in both cases. Therefore, the problem could be investigated as a two-dimensional problem in the xy -axis, as was the case with problem 2b." Therefore, in this case the problem in 3D was reduced to a 2D problem establishing the connection between parts (a) and (b) which otherwise would not have been possible (G1, G2).

Conclusions

The analysis of students' work reveals that the similarity of the techniques used in both parts of the problem was described as based on the "same answer," that is, the same function that was the potential function of the vector field in part a) and the general solution of the EDE in b). Engineering students often choose solution methods departing from the problem formulation. They felt more comfortable with MO1 rooted in Calculus in comparison with a more abstract praxeology MO2 rooted in DEs. Consequently, quite distinct concept images for EDEs were formed. For instance, MO1 applies to a 3D space where certain "symmetry" of the three components of the vector function $E = (E_x, E_y, E_z)$ is preserved. On the other hand, for most students MO2 induced an unequal relationship between x and y (x is independent and $y = y(x)$ is dependent variable), and thus students perceived them differently from x and y in MO1. Some students pointed out that techniques (partial differentiation and partial integration) were the same in both parts, but some claimed they were different. Students compared their 'efficiency' mentioning that "the method in part b) was easier than in part a)" and ignoring the fact that part a) included engineering context. Another important observation regards the dimension of the vector field in part a) and the dimension of the EDE in part b). Some students addressed the dimension issue but viewed it only as a possibility to simplify calculations of the integral reduced to 2D rather than a more complex one in a 3D space. Students did not mention at all the properties of the domains prescribed by the technology in MO1 and MO2 to support the techniques used in parts a) and b). Both in MO1 and MO2 students experienced troubles with the technology θ and completely missed Theory θ , it was particularly challenging for them to arrive at deeper conclusions about similarities and distinctions between the praxeologies MO1 and MO2. In

our search for an overarching praxeology, both MO1 and MO2 will be followed up separately in two forthcoming papers in accordance with their distinct raison d'être to explore possible connections.

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