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Meta-Schrödinger and meta-conformal symmetries in the non-equilibrium dynamics of the directed spherical model

Stoimen Stoimenov and Malte Henkel

Abstract New scaling regimes exist, for large distances and sufficiently long-ranged initial conditions, in the kinetics of directed spin models. The spatial length scales are larger than those found in spatially isotropic dynamic scaling. Their dynamical symmetries are given by the meta-conformal and the meta-Schrödinger algebras. These facts are illustrated through the exact solution of the directed spherical model at $T \leq T_c$ and compared with the 1D directed Glauber-Ising model at $T = 0$.

1 Introduction

Physical ageing in many-body systems can arise after a quenching from some initial (disordered) state, either onto a critical point or else into a two-phase coexistence regime [55, 9, 14, 32, 56]. This leads to slow relaxation processes, with formally infinite relaxation times. In addition, ageing is characterised in the macroscopic observables by dynamical scaling and by the breaking of time-translation invariance. A central quantity is the time-dependent length scale $L = L(t)$, which for the systems of interest here grows for large times algebraically, according to $L(t) \sim t^{1/z}$ where z is the *dynamic exponent*. Relevant physical observables include two-time *correlators* C and two-time *responses* R . In the limit of large times $t, s \gg \tau_{\text{micro}}$ (with a microscopic reference time τ_{micro}) such that $t/s > 1$, these are constructed from the time-space-dependent order-parameter $\phi(t, \mathbf{r})$ and the conju-

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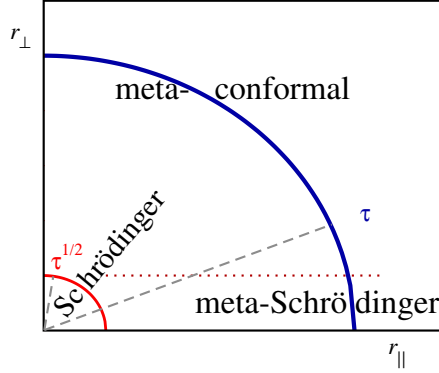


Fig. 1 Scaling regimes in the directed spherical model for spatial distances r_{\parallel} and r_{\perp} as compared to the characteristic length scales $\tau^{1/2}$ and τ . Conveniently long-ranged initial correlations in the preferred direction are necessary for the existence of the meta-conformal scaling regime.

gate symmetry-breaking external field $h(s, \mathbf{r})$ and are expected to admit dynamical scaling forms

$$C(t, s; \mathbf{r}) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle = s^{-b} F_C \left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{t+s} \right) \quad (1a)$$

$$R(t, s; \mathbf{r}) := \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} = s^{-1-a} F_R \left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{t-s} \right) \quad (1b)$$

where a, b are system-dependent, but universal, *ageing exponents*. The time t is called the *observation time* and s is the *waiting time*. In almost all cases, one finds an asymptotic algebraic behaviour of the form $F_C(y, 0) \sim y^{-\lambda_C/z}$ and $F_R(y, 0) \sim y^{-\lambda_R/z}$ for $y = t/s \gg 1$ and where λ_C and λ_R are the (universal) auto-correlation and auto-response exponents, respectively. Almost always $\lambda_C = \lambda_R = \lambda$ are identical. However, further dynamical symmetries are required in order to fix the form of scaling functions F_C and F_R in (1). These and other generic properties of the scaling functions follow from a new interpretation of broken time-translation-invariance [43].

In the scaling forms (1), spatial isotropy and spatial rotation-invariance are implicit. However, one may ask what occurs if in addition to the usual diffusive and spatially isotropic transport, an extra ballistic transport in a preferred direction arises. For simplicity of terminology, we shall restrict to the case of simple diffusive dynamics where the dynamic exponent $z = 2$.¹ Figure 1 outlines the generic scenario. First, for relatively short spatial distances and times, with $r^2/t = \text{cste.}$ being kept fixed and $z = 2$, the ballistic anisotropy is not relevant and the scaling form (1) are the well-known one of diffusive scaling. The underlying dynamical symmetry is the one of the Schrödinger group. On the other hand, for larger spatial separations

¹ This value of z arises in the phase-ordering kinetics of a spin system quenched to below criticality, $T < T_c$, and with a dynamics such that the order-parameter is not conserved [7, 8, 9].

one must distinguish spatial coordinates r_{\parallel} and \mathbf{r}_{\perp} parallel and perpendicular to the preferred direction of ballistic transport. Indeed, if the scaling limit is taken in such a way that

$$|r_{\parallel}|/t = \text{cste}, \quad |\mathbf{r}_{\perp}|/t = \text{cste}, \quad (2)$$

the scaling behaviour remains spatially isotropic with a dynamic exponent $z = 1$ in all directions. It turns out this in this case one has a new dynamical symmetry, the so-called *meta-conformal symmetry* [29, 40]. Alternatively, one may take the scaling limit also in a spatially anisotropic way such that

$$|r_{\parallel}|/t = \text{cste}, \quad |\mathbf{r}_{\perp}|^2/t = \text{cste}. \quad (3)$$

which implies a spatially anisotropic dynamic exponent, namely $z_{\parallel} = 1$ in the preferred direction whereas $z_{\perp} = 2$ in the transverse directions. The dynamical symmetry is the *meta-Schrödinger symmetry* [54].

What are the physical conditions to be satisfied that these new scaling regimes with their as yet largely unexplored dynamical symmetries are realised? Here, we shall review recent work in exactly solvable models which illustrate the physical conditions required. This might serve as a guide in future simulational studies.

Extended dynamical scaling is common for equilibrium phase transitions. Indeed, extensions to conformal invariance, especially for two-dimensional systems, leads to spectacular results [5, 17, 28, 33, 50]. Out of equilibrium, systems undergoing ageing are ideal candidates for analogous extensions, because of their natural dynamical scaling [55]. For example, in the case of phase-ordering kinetics at $T < T_c$ and a microscopic dynamics without any conservation laws ('model A'), one has $z = 2$ and the Schrödinger algebra [44, 45] is a good candidate for a dynamical symmetry algebra [27]. It is the maximal finite-dimensional Lie sub-algebra of the infinite-dimensional Schrödinger-Virasoro Lie algebra [30, 57]. Schrödinger-invariance furnishes the best known description of recent experiments on the phase-ordering kinetics in liquid crystals [2], see [32, 39] for reviews.

Other possibilities exist for non-trivial dynamical symmetries, see [16, 42]. These include dynamical scaling, with either $z = 1$ (for ortho- [5] and meta-conformal [52, 40, 53] or conformal-galilean algebras [26, 4, 3]) or else $z = 2$ (for the Schrödinger-related algebras). Dualisation techniques of string theory clarify the nature of n -point functions as either correlators or response functions [30, 34, 35, 38, 41]. Here, we are interested in systems with directed dynamics [51, 47]. Their non-equilibrium dynamics has been studied intensively, notably in the directed Ising [20, 22, 23, 24] and the directed spherical models [21]. As we shall see, long-ranged spatial initial correlations are essential for the physical realisation of meta-conformal or meta-Schrödinger symmetries [40, 42].

This work is organised as follows. Section 2 recalls the exact solution of the directed spherical model from which section 3 finds the two-time correlators and responses, with long-range initial correlations. Section 4 lists the meta-conformal and meta-Schrödinger generators in Fourier space, used in section 5 to predict the co-variant two-point correlation/response functions and compare them with the directed spherical model, and the 1D Glauber-Ising model. We conclude in section 6.

2 Directed spherical model

The directed spherical model will serve as illustration for the realisation of dynamical symmetries in directed spin models, at temperatures $T \leq T_c$. We begin by recalling the main lines of the computation of two-time correlators and responses [21].

Consider continuous spin variables $S_{\mathbf{n}}(t) \in \mathbb{R}$ on the sites \mathbf{n} of a lattice, subject to the spherical constraint $\sum_{\mathbf{n} \in \Lambda} \langle S_{\mathbf{n}}(t)^2 \rangle = |\Lambda|$, the number of sites of the hyper-cubic lattice $\Lambda \subset \mathbb{Z}^d$. In the $|\Lambda| \rightarrow \infty$ limit, these become $S = S(t, \mathbf{r}) \in \mathbb{R}$ and obey the equations of motion (with nearest-neighbour interactions and the habitual scalings)

$$\partial_t S(t, \mathbf{r}) = \sum_{a=1}^d \left[(1+v_a)S(t, \mathbf{r} - \mathbf{e}_a) + (1-v_a)S(t, \mathbf{r} + \mathbf{e}_a) - 2S(t, \mathbf{r}) \right] + \mathfrak{z}(t)S(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (4)$$

with the centred white noise $\eta = \eta(t, \mathbf{r})$ and the Lagrange multiplier $\mathfrak{z}(t)$ to enforce the spherical constraint. The directional bias is given by \mathbf{v} and \mathbf{e}_a is the unit vector in direction $a = 1, \dots, d$. From now on, one rotates axes such that $\mathbf{v} = v\mathbf{e}_1$.

The formal solution is in Fourier space, generalising the non-biased case [13, 19]

$$\widehat{S}(t, \mathbf{q}) = e^{-\Omega(\mathbf{q})t - Z(t)} \left(\widehat{S}(0, \mathbf{q}) + \int_0^t d\tau e^{\Omega(\mathbf{q})\tau + Z(\tau)} \widehat{\eta}(\tau, \mathbf{q}) \right) \quad (5a)$$

$$\Omega(\mathbf{q}) = \omega(\mathbf{q}) + 2iv\sin q_1 = 2 \sum_{a=1}^d (1 - \cos q_a) + 2iv\sin q_1, \quad (5b)$$

with $\mathbf{q} = (q_1, \mathbf{q}_\perp) \in \mathbb{R} \otimes \mathbb{R}^{d-1}$. The two-time correlator $\widehat{C}(t, t'; \mathbf{q})$ is found from

$$\langle \widehat{S}(t, \mathbf{q}) \widehat{S}(t', \mathbf{q}') \rangle = (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \widehat{C}(t, t'; \mathbf{q}) \quad (6a)$$

$$\langle \widehat{\eta}(t, \mathbf{q}) \widehat{\eta}(t', \mathbf{q}') \rangle = 2T(2\pi)^d \delta(t - t') \delta(\mathbf{q} + \mathbf{q}') \quad (6b)$$

along with $\langle \widehat{\eta}(t, \mathbf{q}) \rangle = 0$. Let $\ln g(t) = 2 \int_0^t d\tau \mathfrak{z}(\tau)$ to find the single-time correlator

$$\widehat{C}(t, \mathbf{q}) := \widehat{C}(t, t, \mathbf{q}) = \frac{e^{-2\omega(\mathbf{q})t}}{g(t)} \left(\widehat{C}(0, \mathbf{q}) + 2T \int_0^t d\tau e^{2\omega(\mathbf{q})\tau} g(\tau) \right) \quad (7)$$

The spherical constraint reads $\frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{q} \widehat{C}(t, \mathbf{q}) = 1$ with the Brillouin zone $\mathcal{B} = [-\pi, \pi]^d$. It turns into a Volterra integral equation for $g(t)$, together with [48]

$$g(t) = A(t) + 2T \int_0^t d\tau f(t - \tau) g(\tau) \quad (8a)$$

$$f(t) = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{q} e^{-2\omega(\mathbf{q})t}, \quad A(t) = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{q} \widehat{C}(0, \mathbf{q}) e^{-2\omega(\mathbf{q})t} \quad (8b)$$

From this, the critical temperature is readily found to be bias-independent

conditions	F
$T = T_c$ $2 < d < 2 - \alpha$	$-1 - \alpha/2$
$T = T_c$ $2 - \alpha < d < 4$	$d/2 - 2$
$T = T_c$ $4 < d$	0
$T < T_c$ $2 < d$	$-(d + \alpha)/2$

Table 1 Exponent F in the directed spherical model, for temperatures $T \leq T_c$.

$$T_c = T_c(d) = 2 \int_0^\infty du (e^{-4u} I_0(4u))^d \quad (9)$$

such that $T_c(d) > 0$ for $d > 2$ [19, 21], where $I_0(u)$ is a modified Bessel function [1].

The new ingredient [42] are long-range initial correlations, in the auxiliary function $A(t)$. We admit a disordered initial state in the transverse directions but long-range initial correlations in the preferred direction, such that for small momenta

$$\widehat{C}(0, \mathbf{q}) = \widehat{C}(0, q_1, \mathbf{q}_\perp) = c_\alpha |q_1|^\alpha, \quad -1 < \alpha < 0 \quad (10)$$

with a positive constant $c_\alpha > 0$. This goes beyond the case $\alpha = 0$ studied before where $A(t) = f(t) = (e^{-4t} I_0(4t))^d$ [21]. An asymptotic analysis for the constraint (8a) shows that for $T \leq T_c$ one has the long-time behaviour $g(t) \sim t^F$, where the values of F are given in table 1 [42]. It is independent of the bias \mathbf{v} [21], but depends through α on the initial correlations (10). Then the Lagrange multiplier $\mathfrak{z}(t) \simeq \frac{F}{2} \frac{1}{t}$. Then the leading scaling behaviour of eq. (4) can be derived from the equation

$$\partial_t S(t, \mathbf{r}) = \Delta_r S(t, \mathbf{r}) - 2\mathbf{v} \cdot \nabla_r S(t, \mathbf{r}) + \frac{F}{2t} S(t, \mathbf{r}) + \eta(t, \mathbf{r}). \quad (11)$$

We shall investigate below its dynamical symmetries.

Observables are now immediately found. The two-time correlator is [21]

$$\widehat{C}(t, s; \mathbf{q}) = \frac{\exp(-\Omega(\mathbf{q})t - \Omega(-\mathbf{q})s)}{\sqrt{g(t)g(s)}} \left(\widehat{C}(0; \mathbf{q}) + 2T \int_0^s d\tau e^{2\omega(\mathbf{q})\tau} g(\tau) \right) \quad (12)$$

in Fourier space. The response function is found by adding a magnetic perturbation $\delta \mathcal{H} = \sum_r h(t, \mathbf{r}) S(t, \mathbf{r})$ to the hamiltonian which amounts to replacing $\eta \mapsto \eta + h$ in (4). Adapting (5a), this gives [19, 21, 48, 25, 37, 42]

$$\widehat{R}(t, s, \mathbf{q}) = \left. \frac{\delta \langle \widehat{S}(t, \mathbf{q}) \rangle}{\delta \widehat{h}(s, \mathbf{q})} \right|_{\widehat{h}=0} = e^{-\Omega(\mathbf{q})(t-s)} \sqrt{\frac{g(s)}{g(t)}} \quad (13)$$

which depends on the initial conditions (10) only indirectly through the value of α .

3 Time-space responses and correlators

Working out observables in the scaling limits (2,3) is rendered more simple in Fourier space. The dispersion relation (5b) is in the limit of low momenta

$$\Omega(\mathbf{q}) = 2 \sum_{a=1}^d (1 - \cos q_a) + 2iv \sin q_1 \simeq (q_1^2 + \mathbf{q}_\perp^2) + 2ivq_1 \quad (14)$$

and we decompose $\mathbf{q} = (q_1, \mathbf{q}_\perp) \in \mathbb{R}^d$ into preferred and transverse components.²

First, consider the two-time response function. We study the scaling limits:

(A) The *meta-conformal scaling limit* (2) when $\tau = t - s \rightarrow \infty$ and $q_a \rightarrow 0$ such that $(t - s)q_a$ is kept fixed in all spatial directions $a = 1, 2, \dots, d$. Then $\Omega(\mathbf{q})(t - s) \simeq 2ivq_1(t - s) + \dots$, up to corrections to scaling, and we find

$$\widehat{R}(t, s; \mathbf{q}) = e^{-2ivq_1(t-s)} \left(\frac{s}{t}\right)^{F/2} \quad (15a)$$

$$R(t, s; \mathbf{r}) = \frac{1}{2v} (t - s)^{-d} \left(\frac{t}{s}\right)^{-F/2} \delta\left(\frac{r_1}{2v(t-s)} - 1\right) \delta^{(d-1)}\left(\frac{\mathbf{r}_\perp}{t-s}\right) \quad (15b)$$

where δ denotes the Dirac distribution [18]. The cases $T < T_c(d)$ and the different regimes at $T = T_c(d)$ are merely distinguished by the value of F , see table 1.

(B) The *meta-Schrödinger scaling limit* (3) when $\tau = t - s \rightarrow \infty$ and $q_a \rightarrow 0$ such that $(t - s)q_1$ as well as $(t - s)q_a^2$ are kept fixed for $a = 2, \dots, d$. Then $\Omega(\mathbf{q})(t - s) \simeq 2ivq_1(t - s) + \mathbf{q}_\perp^2(t - s) + \dots$, up to corrections to scaling. The two-time response is

$$\widehat{R}(t, s; \mathbf{q}) = e^{-2ivq_1(t-s) - \mathbf{q}_\perp^2(t-s)} \left(\frac{s}{t}\right)^{F/2} \quad (16a)$$

$$R(t, s; \mathbf{r}) = \frac{\pi^{1/2}}{v} R^{[\text{ms}]}(t, s) \delta\left(\frac{1}{2v} \frac{r_1}{t-s} - 1\right) \exp\left[-\frac{\mathbf{r}_\perp^2}{4(t-s)}\right] \quad (16b)$$

with the autoresponse $R^{[\text{ms}]}(t, s) = (4\pi s)^{-(d+1)/2} \left(\frac{t}{s} - 1\right)^{-(d+1)/2} \left(\frac{t}{s}\right)^{-F/2}$ which is of the same form as for un-biased case [19, 48], but with different scaling dimensions. Again, the different cases of $T \leq T_c(d)$ are distinguished by the value of F .

Next, we analyse the two-time correlator, which in Fourier space is given by (12) with its contributions of the initial noise and thermal noise, respectively. Using the low-energy dispersion (14) and the initial condition (10) we find

$$\begin{aligned} \widehat{C}(t, s; \mathbf{q}) &= \frac{\exp[-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)]}{g_\infty(t-s)^{F/2}} \left(c_\alpha |q_1|^\alpha + 2T \int_0^s d\tau e^{2\mathbf{q}_\perp^2 \tau} \left(\frac{\mathcal{A}_0}{M^2} \delta(\tau) + g_\infty \tau^F \right) \right) \\ &= \frac{\exp[-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)]}{(ts)^{F/2}} \left(\frac{c_\alpha}{g_\infty} |q_1|^\alpha + \frac{2T}{1+F} s^{1+F} {}_1F_1(1+F, 2+F; 2\mathbf{q}_\perp^2 s) \right) \end{aligned} \quad (17)$$

² Formally un-bounded two-point functions in direct space in the meta-conformal and meta-Schrödinger cases are regularised by dualising the generators, see [38, 41, 42] for details.

where ${}_1F_1$ is a Kummer function [1]. In the first line, we recognise that the singular term $\sim \mathcal{A}_0 \delta(\tau)$ produces for $\alpha < 0$ merely a correction to scaling. As to the two scaling limits, in the meta-conformal limit, we have $\mathbf{q}_\perp^2 s \rightarrow 0$ and in the meta-Schrödinger limit, $\mathbf{q}_\perp^2 s$ is finite. In (17), the initial correlator is more relevant, if

$$\alpha \stackrel{!}{<} -1 - F \quad (18)$$

Using table 1, we can distinguish the following situations for $T \leq T_c(d)$ [42]:

1. For $T < T_c(d)$, we should have $\alpha \stackrel{!}{<} d - 2$. Since $d > 2$ is needed to have a non-vanishing $T_c(d)$, and $\alpha < 0$ that condition is automatically satisfied and thermal noise is irrelevant in the entire ordered phase.
2. For $T = T_c(d)$ and $2 < d < 2 - \alpha$, the condition (18) leads to $\alpha \stackrel{!}{<} \alpha/2 < 0$. Hence at criticality with $2 < d < 2 - \alpha$ thermal noise is always irrelevant.
3. For $T = T_c(d)$ and $2 - \alpha < d < 4$, we find from (18) that $\alpha \stackrel{!}{<} 1 - d/2$. Only for these values of α , with an initial spatial correlator decaying more slowly than $C_n(0) \sim |n|^{-(2-d/2)}$, the initial noise is relevant and the thermal noise generates a correction to scaling. But if $0 > \alpha > 1 - d/2$, the thermal noise is relevant.
4. For $T = T_c(d)$ and $d > 4$, we have $F = 0$, hence $\alpha \stackrel{!}{<} -1$. This is impossible. Hence thermal noise is always relevant: the usual mean-field behaviour.

This leads to the following scaling forms of the two-time correlator:

1. First, if the initial noise is relevant, the momentum-space two-point correlator is

$$\widehat{C}(t, s; \mathbf{q}) = \frac{c_\alpha}{g_\infty} \frac{\exp[-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)]}{(ts)^{F/2}} |q_1|^\alpha \quad (19)$$

but the form in direct space does depend on the scaling limit to be taken.

(A) In the *meta-conformal limit* (2), the term $\mathbf{q}_\perp^2(t+s) \rightarrow 0$, so that the scaling form in time-space of the two-point correlator is

$$\begin{aligned} C(t, s; \mathbf{r}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{c_\alpha}{g_\infty} \frac{\exp[-2ivq_1(t-s)]}{(ts)^{F/2}} |q_1|^\alpha \\ &= C_{(0)} (ts)^{-F/2} |r_1 - 2v(t-s)|^{-\alpha-1} \delta^{(d-1)}(\mathbf{r}_\perp) \end{aligned} \quad (20)$$

with the constant $C_{(0)} := \frac{c_\alpha}{2\pi g_\infty} \int_{\mathbb{R}} dq_1 e^{iq_1} |q_1|^\alpha$ (this is a distribution [18]).

(B) In the *meta-Schrödinger limit* (3), we rather have

$$\begin{aligned} C(t, s; \mathbf{r}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{c_\alpha}{g_\infty} \frac{\exp[-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)]}{(ts)^{F/2}} |q_1|^\alpha \\ &= C_{[0]} (ts)^{-F/2} |r_1 - 2v(t-s)|^{-\alpha-1} (t+s)^{-(d-1)/2} \exp\left[-\frac{1}{4} \frac{\mathbf{r}_\perp^2}{t+s}\right] \end{aligned} \quad (21)$$

with a constant $C_{[0]}$ which we do not specify.

Relevant initial noise occurs (i) either if $T < T_c(d)$, or else (ii) if $T = T_c(d)$ if either $d < 2 - \alpha$ or else for $2 - \alpha < d < 4$ if $\alpha < 1 - d/2$.

2. Second, if the thermal noise is relevant, we rather find

$$\begin{aligned}\widehat{C}(t, s; \mathbf{q}) &= \frac{\exp[-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)]}{(ts)^{F/2}} 2T_c(d) \int_0^s d\tau e^{2\mathbf{q}_\perp^2 \tau} \tau^F \\ &= \frac{2T_c(d)}{1+F} s \left(\frac{s}{t}\right)^{F/2} e^{-2ivq_1(t-s) - \mathbf{q}_\perp^2(t+s)} {}_1F_1(1+F, 2+F; 2\mathbf{q}_\perp^2 s)\end{aligned}\quad (22)$$

reduces to the un-biased correlator, up to a small damping term. This is the situation which has been discussed in detail in the littérature [21]. This occurs at criticality if either $d > 4$ or else for $2 - \alpha < d < 4$ if $\alpha > 1 - d/2$ and notably for all dimensions $d > 2$ for short-ranged initial correlations where $\alpha = 0$.

Proposition 1: [42] *In the directed spherical model at $T \leq T_c$, with long-ranged initial correlations (10) in the preferred direction, there exists a ballistic scaling regime, for distances larger than for diffusive scaling. The dynamic exponent is isotropically $z = 1$ for a meta-conformal scaling limit (2) or $z_\parallel = 1$ and $z_\perp = 2$ in the preferred/transverse directions for a meta-Schrödinger scaling limit (3).*

4 Representations in Fourier space

We need the representations of meta-conformal [40] and meta-Schrödinger algebras [54] in Fourier space. For notational simplicity, we use one ‘parallel’ or preferred direction $x \rightarrow q_\parallel =: q$ and one ‘perpendicular’ or transverse direction $y \rightarrow q_\perp =: p$ in Fourier space. For the meta-Schrödinger algebra, scaling operators transform as

$$\Phi(t, x, y) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} dq dp e^{i(xq+yp)} \widehat{\Phi}(t, q, p), \quad \widehat{\Phi}(t, q, p) = \int_{\mathbb{R}^2} dx dy e^{-i(xq+yp)} \Phi(t, q, p)\quad (23)$$

and from this follows the correspondence between differential operations³

$$\partial_x \rightarrow iq, \quad x \rightarrow i\partial_q, \quad x^2 \rightarrow -\partial_q^2, \quad \partial_x^2 \rightarrow -q^2, \quad x\partial_x \rightarrow -q\partial_q - 1$$

Furthermore, for application to non-equilibrium dynamics, it turns out that a further change of representation is necessary [54, 42, 43]

$$X_n \mapsto e^{\xi \ln t} X_n e^{-\xi \ln t}, \quad Y_n \mapsto e^{\xi \ln t} Y_n e^{-\xi \ln t}\quad (24)$$

which also implies a corresponding change in the Schrödinger operator

$$\mathcal{S} \mapsto e^{\xi \ln t} \mathcal{S} e^{-\xi t} = \mathcal{S} - \xi/t\quad (25)$$

³ Similarly for y . This also implies rules such as $\mathbf{r}^2 \partial_r = \mathbf{r} \cdot (\mathbf{r} \partial_r) \rightarrow i\partial_q \cdot (-\mathbf{q} \partial_q - d)$, where d is the number of the space dimensions.

since \mathcal{S} is a first-order differential operator in t . In the spherical model, with the effective long-time equation of motion (11), we can identify $\xi = F/2$.

Now we are able to write the generators of meta-conformal and meta-Schrödinger algebra⁴, in Fourier space, that is acting on $\widehat{\Phi}$ defined in (24).

4.1 The meta-conformal algebra in Fourier space

In order to find the n -point functions of quasi-primary scaling operators, the generators of the finite-dimensional algebras are required. We have using (24) [54, 42]

$$\begin{aligned}
X_{-1} &= -\partial_t + \xi/t, & X_0 &= -t\partial_t + q\partial_q - \delta + \xi + 1 \\
X_1 &= -t^2\partial_t + i\beta q\partial_q^2 + 2tq\partial_q - 2i(\gamma - \beta)\partial_q - (2\delta - \xi - 2)t \\
Y_{-1} &= -iq, & Y_0 &= -itq + \beta q\partial_q - (\gamma - \beta) \\
Y_1 &= -it^2q + 2\beta tq\partial_q - 2i\beta(\gamma - \beta)\partial_q + i\beta^2q\partial_q^2 - 2t(\gamma - \beta)
\end{aligned} \tag{26}$$

where we set $d_\perp = 1$ for simplicity. Meta-conformal generators do not contain the transverse momentum p . The scaling operators on which these generators act are each characterised by the scaling dimensions δ, ξ and the rapidity γ . The equilibrium ('standard') representation is recovered by setting $\xi = 0$.

4.2 The meta-Schrödinger algebra in Fourier space

For quasi-primary scaling operators in Fourier space, we have for $d_\perp = 1$ [54]:

$$\begin{aligned}
X_{-1} &= -\partial_t + \xi/t, & X_0 &= -t\partial_t + q\partial_q + \frac{p}{2}\partial_p - \delta + \xi + \frac{3}{2} \\
X_1 &= -t^2\partial_t + i\beta q\partial_q^2 + 2tq\partial_q - 2i(\gamma - \beta)\partial_q + tp\partial_p + \frac{\mathcal{M}}{2}\partial_p^2 - (2\delta - \xi - 3)t \\
Y_{-1}^{(\parallel)} &= -iq, & Y_0^{(\parallel)} &= -itq + \beta q\partial_q - (\gamma - \beta) \\
Y_1^{(\parallel)} &= -it^2q + 2\beta tq\partial_q - 2i\beta(\gamma - \beta)\partial_q + i\beta^2q\partial_q^2 - 2t(\gamma - \beta) \\
Y_{-\frac{1}{2}}^{(\perp)} &= -ip, & Y_{\frac{1}{2}}^{(\perp)} &= -itp - i\mathcal{M}\partial_p, & M_0 &= -\mathcal{M}
\end{aligned} \tag{27}$$

Spatial rotation-invariance allows a trivial generalisation to $d_\perp \geq 1$ transverse dimensions. Notice that the transverse momentum p not only appears in the generators X_n but also leads to the appearance of the additional generators $Y_{\pm\frac{1}{2}}^{(\perp)}$ and M_0 . The equilibrium ('standard') representation is recovered by setting $\xi = 0$.

⁴ Technically, we restrict to case $\alpha = 0$ for both algebras, see [40, 54] for details.

5 Two-point functions and comparison with the spherical model

In Fourier space one has for the covariant two-point functions

$$F = F^{[2]}(t_a, t_b, q_a, q_b, p_a, p_b) = \langle \widehat{\Phi}(t_a, q_a, p_a) \widehat{\Phi}(t_b, q_b, p_b) \rangle \quad (28)$$

where the scaling operators $\widehat{\Phi}$ are quasi-primary, i.e. transform covariantly with respect to the maximal finite-dimensional subalgebra of the meta-conformal algebra (26) or of the meta-Schrödinger algebra (27). The Ward identities readily follow since the transformation terms are already included in the generators.

5.1 Meta-conformal covariant two-point functions

We first look for the form of two-point function covariant under representation (26) of meta-conformal algebra. Here time-translation-invariance is explicitly broken via the time-dependence of the generator $X_{-1} = e^{\xi \ln t} (-\partial_t) e^{-\xi \ln t} = -\partial_t + \xi/t$, obtained from the standard time-translations $-\partial_t$. The co-variance under X_{-1} leads to the following equation for the two-point function $F = F(\tau, u, q_a, q_b)$, after a change of variables $\tau = t_a - t_b$ and $u = t_a/t_b$

$$((u-1)u\partial_u + \xi_a + u\xi_b)F(\tau, u, q_a, q_b) = 0 \quad (29)$$

Integrating (29), the u -dependence is determined

$$F(\tau, u, q_a, q_b) = u^{\xi_a} (u-1)^{-\xi_a - \xi_b} \delta(q_a + q_b) f(\tau, q_a) \quad (30)$$

where spatial translation-invariance is also taken into account. Herein, the function $f(\tau, q_a)$ satisfy the system of covariance conditions under X_0 and Y_0 ;

$$(\tau\partial_\tau - q_a\partial_{q_a} + \delta_a + \delta_b - 1 - \xi_a - \xi_b - 1)f(\tau, q_a) = 0 \quad (31a)$$

$$(i\tau q_a - \beta_a q_a \partial_{q_a} + \gamma_a + \gamma_b - \beta_b)f(\tau, q_a) = 0 \quad (31b)$$

together with $\beta = \beta_a = \beta_b$. The solution of the above system is

$$f(\tau, q_a) = F_0 \tau^{-\delta_a - \delta_b + (\gamma_a + \gamma_b)/\beta} q_a^{(\gamma_a + \gamma_b)/\beta - 1} \exp\left(\frac{i}{\beta} \tau q_a\right) \quad (32)$$

Finally, the covariance under X_1 and Y_1 merely gives the constraints

$$\delta_a = \delta_b = \delta, \quad \gamma_a = \gamma_b = \gamma. \quad (33)$$

and the final result for the two-point function covariant under representation (26) of meta-conformal algebra is (F_0 is an undetermined normalisation constant)

$$F^{[2]}(t_a, t_b, q_a, q_b) = F_0 \delta(q_a + q_b) t_a^{\xi_a} t_b^{\xi_b} (t_a - t_b)^{-\delta_a - \delta_b + 2\gamma/\beta} q_a^{2\gamma/\beta - 1} \exp\left(\frac{iq_a}{\beta}(t_a - t_b)\right). \quad (34)$$

The analogous result of equilibrium dynamics is recovered by setting $\xi_a = \xi_b = 0$.

5.2 Correlators and meta-conformal invariance

Meta-conformal invariance predicts the form of the two-point *correlator* $C = \langle \Phi \Phi \rangle$ [34, 35, 36, 38, 41, 53] of quasi-primary scaling operators Φ . Therefore, we compare the explicit spherical model correlator (17) with the prediction (34) of meta-conformal invariance. Since the spherical model result (17) has two contributions, one from the initial noise and one from the thermal noise, two cases must be distinguished, depending on the relevance of both terms, according to the criterion (18).

In the first case, when the initial noise is relevant, it is enough to compare (19) with the meta-conformal prediction (34). From the exponential factors and the momentum-dependence, we read off

$$\frac{1}{\beta} = -2\nu, \quad \frac{2\gamma_1}{\beta} = 1 + \alpha \quad (35)$$

and comparison of the exponents of the time-dependent factors leads to

$$\xi_1 = \xi_2 = -\frac{F}{2}, \quad \delta_1 + \delta_2 - \frac{2\gamma_1}{\beta} = 0 \quad (36)$$

which gives for the scaling dimension and the rapidity

$$\delta_1 = \delta_2 = \frac{1 + \alpha}{2} > 0, \quad \frac{\gamma_1}{\beta} = \frac{\gamma_2}{\beta} = \frac{1 + \alpha}{2} \quad (37)$$

It is satisfying that the scaling dimensions $\delta_1 = \delta_2$ come out to be positive. We also find that the second scaling dimension $\xi_1 = \xi_2 = -F/2 > 0$ does not vanish. Its value depends in general on the initial correlator through α , with the only exception at $T = T_c(d)$ and $2 - \alpha < d$.

Proposition 2: *In the directed spherical model with relevant long-ranged initial conditions (10), a meta-conformal scaling regime (2) exists (i) either for quenches to $T < T_c(d)$ for all $\alpha < 0$ (ii) or else for quenches to $T = T_c(d)$ where $d < 2 - \alpha$ and $-1 < \alpha < 0$ or for $2 - \alpha < d < 4$ and $-1 < \alpha < 1 - d/2$.*

In the second case, the thermal noise is relevant and we now compare (22) with the prediction (34), where we take into account that $q_{\perp}^2 s \rightarrow 0$ in the meta-conformal scaling limit. As before, comparison of the exponential factors and the momentum-dependence would lead to $1/\beta = -2\nu$ and $2\gamma_1/\beta = 1$. Then, identifying the power-law terms in t and s , we would read off $\xi_1 = -\frac{F}{2}$ and $2\delta = 1$. If true, this would

imply $\delta_1 = \delta_2 = \frac{1}{2}$ and $\xi_2 = 1 - \xi_1 = 1 + F/2$. The finding $\xi_1 \neq \xi_2$ is incompatible with a correlator of two identical scaling operators.

Proposition 3: *In the directed spherical model with relevant thermal noise, a meta-conformal scaling regime does not exist.*

5.3 Meta-Schrödinger covariant two-point functions

For simplicity, consider first the equilibrium case $\xi_a = \xi_b = 0$. Write the Ward identities for the two-point function

$$F^{[2]}(t_a, t_b, q_a, q_b, p_a, p_b) = \langle \widehat{\Phi}(t_a, q_a, p_a) \widehat{\Phi}(t_b, q_b, p_b) \rangle \quad (38)$$

Co-variance under $X_{-1}, M_0, Y_{-1}^{(\parallel)}, Y_{-\frac{1}{2}}^{(\perp)}$ gives (again with $\tau = t_a - t_b$)

$$F^{[2]}(t_a, t_b, q_a, q_b, p_a, p_b) = \delta(\mathcal{M}_a + \mathcal{M}_b) \delta_1(q_a + q_b) \delta(p_a + p_b) f(\tau, q_a, p_a) \quad (39)$$

Next, co-variance under the generators $X_0, Y_0^{(\parallel)}, Y_{\frac{1}{2}}^{(\perp)}$ produces the system

$$\left(\tau \partial_\tau - q_a \partial_{q_a} - \frac{p_a}{2} \partial_{p_a} + \delta_a + \delta_b + \frac{3}{2} \right) f(\tau, q_a, p_a) = 0 \quad (40a)$$

$$(i\tau q_a - \beta_a q_a \partial_{q_a} + \gamma_a + \gamma_b - \beta_b) f(\tau, q_a, p_a) = 0 \quad (40b)$$

$$(\tau p_a + \mathcal{M}_a \partial_{p_a}) f(\tau, q_a, p_a) = 0. \quad (40c)$$

where $\beta = \beta_a = \beta_b$ and which has the solution, unique up to a factor F_0 :

$$f(\tau, q_a, p_a) = F_0 \tau^{-\delta_a - \delta_b + (\gamma_a + \gamma_b)/\beta + \frac{1}{2}} q_a^{(\gamma_a + \gamma_b)/\beta - 1} \exp\left(\frac{i}{\beta} \tau q_a - \frac{p_a^2}{2\mathcal{M}_a} \tau\right) \quad (41)$$

Finally the covariance under X_1 and $Y_1^{(\parallel)}$ reproduces (33).

Going via (25) to the non-equilibrium representation with $\xi_a, \xi_b \neq 0$, we find [42]

$$F^{[2]}(t_a, t_b, q_a, q_b, p_a, p_b) = F_0 \delta(\mathcal{M}_a + \mathcal{M}_b) \delta(q_a + q_b) \delta(p_a + p_b) \times \quad (42)$$

$$\times t_a^{\xi_a} t_b^{\xi_b} (t_a - t_b)^{-\delta_a - \delta_b + 2\gamma/\beta + d_\perp/2} q_a^{2\gamma/\beta - 1} \exp\left(\frac{i q_a}{\beta} (t_a - t_b) - \frac{p_a^2}{2\mathcal{M}_a} (t_a - t_b)\right)$$

with the constraint (33). Assuming rotation-invariance in the transverse directions, we have also restored an arbitrary transverse dimension $d_\perp = d - 1 \geq 1$.

5.4 Responses and meta-Schrödinger invariance

Meta-Schrödinger invariance predicts the form of the two-point *response* [54], written as a correlator of the order parameter Φ with a response operator $\tilde{\Phi}$, and both assumed quasi-primary, such that $R = \langle \Phi \tilde{\Phi} \rangle$. The comparison must be done between the explicit spherical model result (16a) and the prediction (42) of meta-Schrödinger invariance. Comparing the exponential factors, we identify

$$\frac{1}{\beta} = -2\nu, \quad \mathcal{M}_1 = \frac{1}{2} \quad (43)$$

and comparison of the algebraic prefactors further leads to, using also (33)

$$\xi_1 = -\xi_2 = \frac{F}{2}, \quad \delta_1 = \delta_2 = \frac{1}{2} + \frac{d_{\perp}}{4} = \frac{d+1}{4}, \quad \frac{\gamma_1}{\beta} = \frac{\gamma_2}{\beta} = \frac{1}{2} \quad (44)$$

and using the values of F , listed in table 1. We finally have the scaling dimensions $\delta = \delta_1$ of the order parameter, $\tilde{\delta} = \delta_2$ of the response operator, the second scaling dimensions $\xi = -\xi = \xi_1$ (in agreement with eq. (11)) and the rapidities $\gamma/\beta = \tilde{\gamma}/\beta = \frac{1}{2}$. These identifications hold for all temperatures $T \leq T_c(d)$ and all dimensions $d > 2$ and it is satisfying that always $\delta > 0$ and $\tilde{\delta} > 0$. The identity $\delta = \tilde{\delta}$ is expected from local scale-invariance [29, 43].

Proposition 4: *In the directed spherical model, in the meta-Schrödinger scaling limit (3), the scaling two-response function perfectly reproduces for all initial conditions (10) the prediction of meta-Schrödinger invariance.*

6 Conclusions

We have been investigating the existence of new scaling regimes in the relaxational dynamics of spin models with a directional bias in the interactions. At the time of writing, information on this question only exists from two exactly solvable models, namely the $d > 2$ directed spherical model [21, 42] (studied here) and the directed 1D Glauber-Ising model quenched to the temperature $T = T_c = 0$ [20, 22, 40]. The admissible reaction rates in the latter model are illustrated in figure 2 and are symmetric under a global reversal of all spins. Without a bias, these models always have $z = 2$, with the Schrödinger group (and the representations (24)) as dynamical symmetry [32]. This is realised in the isotropic scaling regime where $t \rightarrow \infty$, $|\mathbf{r}| \rightarrow \infty$ and where $|\mathbf{r}|^2/t$ is kept fixed. In the presence of a directional bias either a meta-conformal or else a meta-Schrödinger Lie algebra are candidates for a new dynamical symmetry, which however can only be realised in one of the scaling limits (2) or (3), respectively, as is illustrated in figure 1.

The supplementary condition required for either meta-conformal or meta-Schrödinger-invariance is the presence of long-ranged initial correlations (10) in the pre-

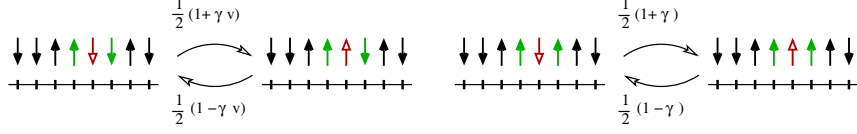


Fig. 2 Reaction rates in the biased 1D Glauber-Ising model, with $\gamma = \tanh \frac{2}{T}$ and the bias $0 \leq \nu \leq 1$. The flip rates of the central spin (in red) only depend on its two nearest neighbours (in green).

ferred direction, of the form $C(0; r_{\parallel}, r_{\perp}) \sim |r_{\parallel}|^{-\aleph}$, with $0 < \aleph = 1 + \alpha < 1$ [40, 42]. If the initial correlations are more relevant than thermal noise, then meta-conformal invariance arises for sufficiently long-ranged initial correlations, otherwise the dynamical symmetry is given by the meta-Schrödinger algebra. From the available evidence, we arrive at the following scenario: Meta-conformal invariance occurs

1. at $T < T_c(d)$ for any kind of initial correlations. The spherical model order-parameter scaling operator Φ is characterised by

$$\delta = \frac{1 + \aleph}{4} > 0, \quad \frac{\gamma}{\beta} = \frac{\aleph}{2}, \quad \xi = \frac{d - 1 + \aleph}{4} \quad (45)$$

2. at $T = T_c(d)$ for relevant initial correlations. In the spherical model, when

$$\text{if } 2 < d < 3 - \aleph : \quad \delta = \frac{\aleph}{2} = \frac{\gamma}{\beta} > 0, \quad \xi = \frac{1 + \aleph}{4} \quad (46a)$$

$$\text{if } 3 - \aleph < d < 4 - 2\aleph : \quad \delta = \frac{\aleph}{2} = \frac{\gamma}{\beta} > 0, \quad \xi = 1 - \frac{d}{4} \quad (46b)$$

but in the 1D Glauber-Ising model at $T = 0$, we rather find

$$\text{if } 0 < \aleph < \frac{1}{2} : \quad \delta = \frac{\aleph}{2} = \frac{\gamma}{\beta} > 0, \quad \xi = 0 \quad (46c)$$

with a similar result for δ in all known cases (46).

There is no meta-conformal invariance in these directed spin models if the above conditions on the initial correlations are not met. In that case, at very large distances meta-Schrödinger-invariance (3) will hold instead, which predicts the two-time response functions $R = \langle \Phi \tilde{\Phi} \rangle$ for all $T \leq T_c$. In the spherical model

$$\delta = \tilde{\delta} = \frac{d+1}{4} > 0, \quad \frac{\gamma}{\beta} = \frac{\tilde{\gamma}}{\tilde{\beta}} = \frac{1}{2}, \quad \xi = -\tilde{\xi} = -\frac{F}{2} \quad (47)$$

Figure 1 gives a schematic view of these new regimes of dynamical scaling. It illustrates the scaling limits to be taken in either the preferred spatial direction (r_{\parallel}) or else the transverse spatial direction (r_{\perp}) as compared to the characteristic timescales τ , see eqs. (2,3). If both $r_{\parallel} \sim r_{\perp} \sim \tau^{1/2}$ are relatively small, then the relaxation dynamics of the spherical model is described by Schrödinger-invariance.

If larger distances are considered such that anisotropically $r_{\parallel} \sim \tau$ but $r_{\perp} \sim \tau^{1/2}$, the dynamical symmetry become meta-Schrödinger-invariance. Finally, if both distances become isotropically large $r_{\parallel} \sim r_{\perp} \sim \tau$, together with sufficiently long-ranged initial correlation in the preferred direction, then the dynamical symmetry is meta-conformal invariance.

This physical picture should give indications where to look for these new scaling regimes in further spin models, notably in the two-dimensional directed Ising model [23, 24], and beyond those relatively rare cases where exactly solutions are readily available. Since the dynamic exponent $z = 1$ also arises in the non-equilibrium dynamics of quantum systems [10, 11, 15] or emergent hydrodynamics [12, 49, 46, 6] it would be interesting to see if in such systems evidence for meta-conformal or meta-Schrödinger invariance could be found.

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