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# **Silent videos in heterogeneous classrooms as impulse for the development of mathematical notions exemplified on “What is a tangent?”**

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*The notion of heterogenous classrooms is en vogue and typically related to certain challenges in communication. Combining teaching experience in an international class and a special education class for hearing-impaired children, we want to propose a certain type of silent video. The special format puts strict conditions on the representation of mathematical notions and objects. We want to highlight these consequences thoroughly with the example of a tangent to a graph of a function by suggesting three story boards with different contexts.*

*Keywords: Videos, tangent, heterogeneity.*

## **Setting the stage for silent videos**

The crucial influence of language for the learning of mathematics in all classes is internationally accepted nowadays, not only restricted to certain special contexts such as international classes, bi- and/or multilingual classrooms or special education settings for hearing-impaired children (Erath et al., 2021).

In Sauerwein (2019), a special approach to elementary algebra for international classes was presented. Its key idea was, that the main focus lied on iconic representations instead of symbolic ones in the sense of Bruner (1966). Thus, the requirements for symbolic language were initially decreased with the intention that the use of iconic representations enables classroom discourse (without a common language) and offers rich potentials for the development of mathematical notions. This line of thought was continued and adapted in a developmental research project in a special education grade 11 of a grammar school for hearing-impaired children, where for various reasons, again, the communication was partly dysfunctional. Seizing advantages of learning videos such as individual learning rates, independence of time, etc. for heterogeneous classes, we developed a certain type of silent video for the introduction of the notion of a function (Sauerwein, 2022). The main idea can be summarized as a language reduction (as a reduction of symbolic representations) to make the video more accessible for heterogeneous classes while presenting certain representations (non-verbally) embedded in processes in contexts which can function as catalyst for seminal language development. In the video, the process (and involved objects) should come to the fore in such a way, that it can serve as an initiation for further discussions and thus foster the development of mathematical language and notions. This stands in line with the traditions of Winter (1978) and Moschkovich (1999), both encouraging mathematical discourse independent of the explicit lingual manifestation:

[...] language learners can and do participate in discussions where they grapple with important mathematical content. [...] But it is only possible to uncover the mathematical content in what students say if students have the opportunity to participate in a discussion and if this discussion is a mathematical one. (Moschkovich, 1999, pp. 18-19)

## Origins

This project has its roots in CoP MINT (part of Com<sup>e</sup>In, funded by the German Federal Ministry of Education and Research). Within the project group at the university of Bonn, it was discussed to use a checklist in order to evaluate learning videos. However, for the context of hearing-impaired students, this merely descriptive approach did not turn out to be a feasible and finally helpful way. Instead, we tried to constructively adapt the aforementioned idea of Sauerwein (2019): switching the mode of representation and focusing on well-defined topics in the mathematics curriculum. The decision against the use of a soundtrack tried to reduce potential lingual difficulties in the reception of the text, to avoid simultaneity of visual and auditive presentation and to inspire students to find their own verbal expression for the observed mathematical concepts. Kristinsdottir et al. (2022) developed “silent video tasks”: when their pupils had to add a soundtrack to a given silent video task, their videos were used for formative assessment. Neither the use for formative assessment nor the restriction to spoken language is in line with our goals.

As far as the author knows, there has been no empirical research on this kind of silent videos at all, but definitely not in the context of special need education. A first try in a grade 11 with hearing-impaired students, which is elaborated on in the next section, looked promising.

### Example video: “Otto grows”

In the video “Otto wächst” (= Otto grows), the dependence of the height on the age of a person is presented using a carpenter’s rule. By picking up on the measuring process repeatedly, we wanted to establish a context for the video. We highlighted the role of the measuring process by leaving out a table of values (see Figure 1). One carpenter’s rule was kept to act as the axis of ordinates and it was shown how to represent the measured pairs of magnitudes within a coordinate system.



**Figure 1: Will Otto reach 340 cm when he is 28 years old?**

In later discussions in the classroom the context turned out to be fruitful, because

- The measuring made it clear, that this dependence is right-unique. The height of Otto is uniquely defined (= measured) at each point in time.
- There were good arguments why one shouldn't connect the points in the coordinate system with a line (see Figure 2). The pupil's answer gave rise to discuss interpolation and extrapolation of points, which can be revisited later for example in the context of growth functions.



**Figure 2: One pupil gives two reasons why one shouldn't connect the dots**

Our approach consists of two steps:

- Step 1: Analyze common introductions (schoolbook) of the concerned mathematical notion under the extreme condition that all (or almost all) symbolic representations are to be suppressed; find out whether this can be achieved without undue simplifications or even producing misconceptions.
- Step 2: Find alternative enactive or iconic representations with a twofold aim: on the one hand to avoid the aforementioned obstacles and on the other hand to foster meaningful mathematical discourse and the development of mathematical notions and language by connecting the new content and its properties to the experience of the learners.

It should be clear, that this approach cannot be extended to a general strategy, which then could be easily applied to different mathematical contexts. Every mathematical object requires a new in-depth subject-matter analysis. Foundation of the above-mentioned approach is the research of Fetzer and Tiedemann (2015, p. 1392):

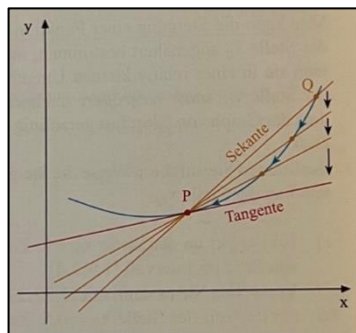
If human actors invite the object to be a part of the discourse, objects can challenge the students to find a new language [...]. This aspect is of special interest from a didactical point of view. Objects cannot only relieve students on the language level, but also challenge them. They can be 'supporters' in the development of mathematical thinking and of a precise mathematical language.

### What is a tangent?

In the following we want to present our approach in an exemplary way with respect to the notion of a tangent, which is crucial for the notion of a derivative of a function in high school. In Step 1 we highlight selected obstacles with the notion of a tangent in high school. We conclude by proposing three storyboards for such videos with rich contexts.

#### Step 1

The notion of a tangent to a graph of a function in German school is typically introduced via the limit of secants (see Figure 3).



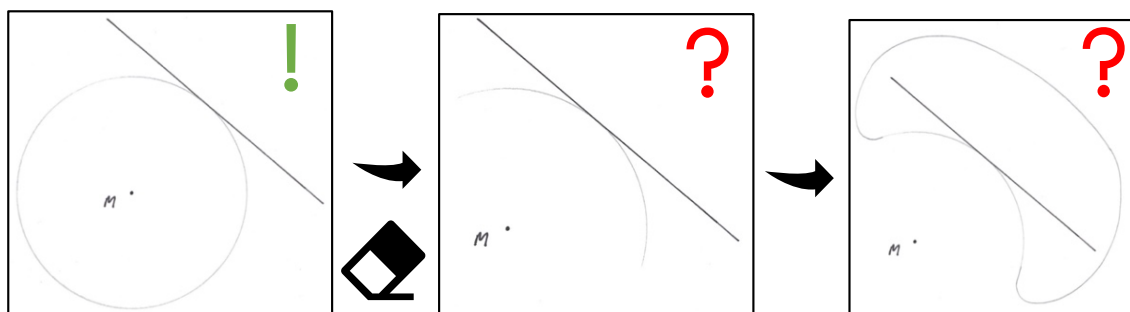
**Figure 3: The limit process secant to tangent (Bigalke & Köhler, 2019, p. 294)**

Either the notion of sequences and limits have been discussed before, which would yield a constructional point of view to the process “secants to tangent” (see Figure 3) or no notion of limits and/or convergence have been developed before this introduction of the tangents. Hence, the pictured process is only an illustration and the argumentation builds on a naïve geometric interpretation of a tangent whose existence is taken for granted although it differs crucially from the prior experience of the learners when dealing with tangents to circles (intersection in only one point).

If one tried to develop a language reduced representation of the above approach, it becomes obvious that the limit process from the slope of secants to that of tangents necessarily leads to the meaningless task of dividing by zero and thus would not be understandable if, as it is common in German schools, limits have not been introduced in previous lessons. It is not clear whether the picture is a mere illustration of an analytical process (containing the above mentioned “meaningless” step) or whether it illustrates the geometrical notion of a tangent. So, this introduction might result in a vicious circle.

## Step 2: Two contexts and three storyboards

Context 1 starts with the practical problem of drawing a tangent to a given curve, parabola, etc. Prior to the introduction of calculus in school, most pupils only know tangents to circles. They learned that these can be constructed as the perpendicular onto the radius and are completely determined by the number of intersection points with the circle. Both properties may turn out to be misconceptions and impede the development of a sustainable notion of a tangent to an arbitrary graph or curve. The two storyboards (Figure 4 and 5) aim to tackle these obstacles.



**Figure 4: Storyboard for “Being tangent is a *local* property”**

The storyboard in Figure 4 builds upon the notion of a tangent to a circle. By erasing parts of the circle, it is now questioned if the former tangent to the circle should be still considered a tangent to the remaining curve. In a second step, the curve is continued to a closed curve in such a way that (after extending the former tangent) two additional intersection points with the former tangent would exist. The intention of this storyboard is to challenge the prior notion of a tangent as a global property of a line and a given curve, where the existence of tangents to non-circular curves is not problematized and taken for granted intuitively.

It should be noted, that the storyboard in Figure 5 does not differ much from the typical schoolbook introduction in Figure 3 at first glance. The significant difference is the context of drawing a tangent. One way of trying to draw a tangent could be that one point P is fixed and the ruler (not necessarily scaled) is rotated after each unsuccessful try and the drawn secants edge closer to the curve. Each of the drawn secants intersects the curve in a second point and one can see these points approaching the

fixed point P. In the end, all drawn secants are marked red and the geometric result of the limit process (= tangent) is colored in green.

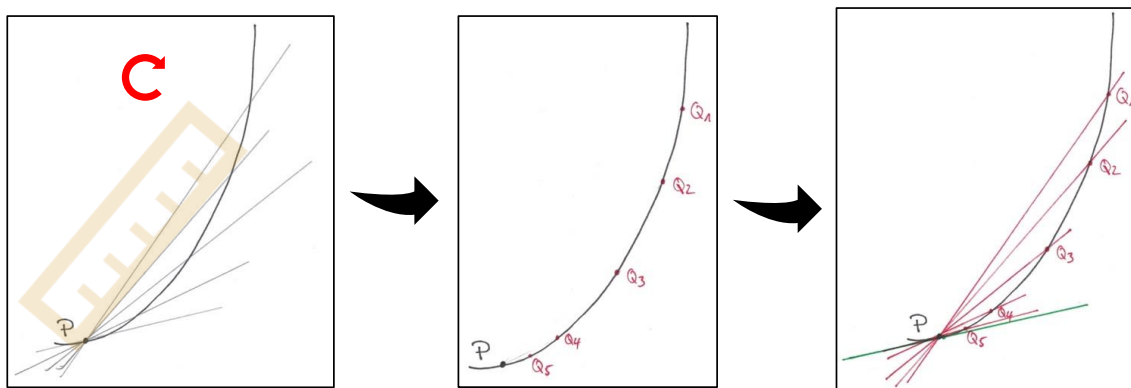


Figure 5: Storyboard for “Geometric limit process”

Context 2 builds upon a slot car race course<sup>1</sup> (see Figure 6). The vehicles will not change directions and will go on straight if no force (apart from friction) acts on them. However, on a race course the cars are guided (or better *forced*) through the course by grooves or slots. In the storyboard, one can see some built parts (= black) of a bigger planned race course (= grey); from take to take the built parts vary. The *forced* parts of the car’s route are marked red and the *unforced straight lines* are marked green. In the first take, one sees the slot car going straight through the planned curve. If one adds a “U-turn” then the car is forced to follow the curve and then goes straight back. If one detaches some sections of the curve as in take 3, we see that the car goes straight as soon as no force is acting on it anymore. This corresponds to Newton’s first law of motion.

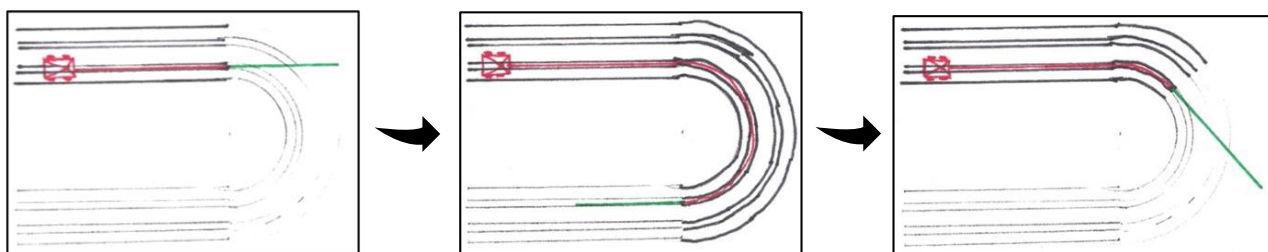


Figure 6: Storyboard for “Slot Car Racing”

The context represents not only parts of the concept of a tangent as the best linear approximation but also distinguishes three properties of a *smooth* function, which are rather similar and not easy to set apart from each other for beginners. When referring to natural movements, we stick to the classic principle *natura non facit saltus*. The aspects are: What a function gets at certain argument, what a function wants to get there and where a function tends to go to. In mathematical notions these correspond to  $f(x_0)$ ,  $\lim_{x \rightarrow x_0} f(x)$  and  $f'(x_0)$ .

<sup>1</sup> Other wind-up-toys and appropriate route margins could be a suitable substitute for the context. Here, slot cars were chosen since with their guiding pin they prompt the idea that we replace the car with a dot (in later stages).

## The usage of the silent videos

Consider the example “Otto wächst” from the beginning again: It was used as an incentive in the classroom with three other silent videos<sup>2</sup>. After watching the silent videos at home as homework, we discussed these videos with the class. The context of growing turned out to be the most fruitful for the following discussions in which the aforementioned important aspects of the mathematical notion of a function became accessible and nameable for the pupils. The other contexts were less authentic and seemed rather artificial to pupils. This may result from the fact that pupils can actively reenact the context based on the measurements and observations they made themselves. In the first cycle of the developmental research project though, pupils were not given sufficient time to take that step; this will be altered in a second cycle.

For the notion of a tangent one cannot expect that one video could incorporate enough aspects to stimulate the discourse in a sufficient way to raise a complete notion of a tangent: the notion is more complex as it consists of more interrelated aspects. Here again, the silent videos (based on the storyboards in Figure 4–6) are only meant to act as an incentive, each with its own direction:

Storyboard 1: *Being a tangent to a curve* is not a global property for a straight line.

Storyboard 2: Tangents to arbitrary curves cannot always be drawn easily. In a seminal discussion about tangents to parabolas, it can be worked out that the parallel to the symmetry axis intersects the parabola only in one point but should not be considered as a tangent anymore. Hence, the characterization of tangents via the number of intersection points turns out to be a misconception, too.

Storyboard 3: Tangents are the *best linear approximation*. This notion is inspiring, for example for an intuitive proof of the chain rule, for the generalisation of derivatives to higher dimensions and Newton’s method for the calculation of zeroes.

The three storyboards in both contexts “drawing a tangent” and “slot car racing track” address important aspects of the heart of the mathematical notion of a tangent and allow its development in such a way, that it can be formalized in seminal lessons into algebraic terms und thus, paving the way for the notion of a derivative and calculus in school.

## Other formative and withdrawn contexts

Due to a lack of geometrical background in school, the following contexts were not considered for storyboards, although they possess potential for the development of the notion of a tangent. However, the influence of those onto the above-mentioned storyboard should not be underestimated and will be sketched. The first experience of tangents in school are typically made in the context of circles. Considering the circle as a conic section, it joins the ranks of a zoo of classical curves such as parabolas, hyperbolas and ellipses. All those curves can be looked at in two ways: static or in motion. These two fundamental perspectives can also be interpreted within the framework of Schwank’s distinction between predicative and functional thinking:

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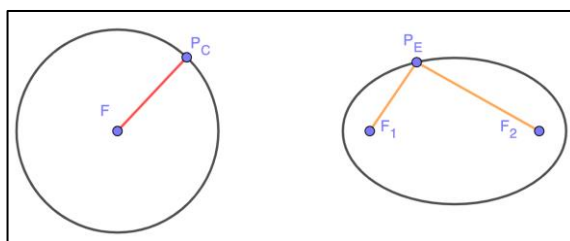
<sup>2</sup> The other contexts were buying pretzels, temperature profile during a day and the number of pips of a dice.



Predicative thinking emphasizes the preference for thinking in terms of relations and judgments; functional thinking emphasizes the preference for thinking in terms of courses and modes of actions. (Schwank, 1993, p. 249)

Following the *static* perspective, one could try to find a different characterization of tangents. For a circle, it is easily seen that the connecting line of the radius and the middle of an arbitrary secant is perpendicular to the secant. This invariant should now be inherited to the tangent, interpreted as a special secant, yielding the canonical notion of tangents to circles. Similar geometric constructions are given for all conic sections in Berendonk (2017). Besides that, it was pointed out that the notion of tangents does not need necessarily be grounded on the notion of limits and that this could be a common misconceptions of teacher education students (Berendonk, 2019, pp. 2106–2107). Although the static perspective could lead to a notion of tangents without the subtle notion of a limit, it requires a rather substantial geometric background and is not easily integrable into the typical calculus courses in school.

However, the *in motion* perspective fits much better to our proposed type of silent videos. The circle, the parabola, the hyperbola and the ellipse all allow kinematic generations (Lookwood, 1961). In particular, the one for the circle and the ellipse are very similar to our third storyboard (see Figure 7): the moving point (or slot car) is forced onto the circle/ellipse (or race course) by a string (or pin).



**Figure 7: Circle and ellipse generating via the “string method”**

Further elaboration on such in motion perspectives can be found in Hall (1961) from a kinematic perspective and in Haftendorn (2016) using dynamic geometry environments such as GeoGebra.

## Outlook

Based on the storyboards, the silent videos will be recorded and evaluated in class. It should be noted that the evaluation will also take place in regular classes as the videos are intended to be inclusive learning material. The approach to elementary algebra in Sauerwein (2019) turned out to be very fruitful for international classes *and* regular classes.

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