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# How can the double number line promote students with mathematical learning difficulties to conduct and explain proportional reasoning?

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*Proportional reasoning includes flexibly reasoning up and down in proportional situations. As unexplained algorithms are error-prone, all students should be able to connect and explain their reasoning by the underlying concept elements, also students with mathematical learning difficulties. The paper reports from a design research study that aims at designing and investigating learning opportunities on proportional reasoning, based on the double number line as the conceptually focused graphical representation. The qualitative analysis of two cases of students with mathematical learning difficulties reveals that the double number line can substantially support students' proportional reasoning, but for promoting their ability to explain their reasoning, additional conceptual and language-related learning opportunities might be required.*

*Keywords: Proportional reasoning, mathematical learning difficulties, double number line, explain.*

## Introduction and research question

Proportional reasoning is an essential topic in the secondary curriculum with strong everyday relevance and importance for further mathematical learning (Lamon, 2007), so it should be taught with a conceptual base, that means with opportunities to understand the strategies and algorithms by explaining them through connections to the underlying conceptual ideas (Lobato et al., 2010). For lower-achieving students, however, learning opportunities are often restricted to algorithms without underlying conceptual base, and many misconceptions have been documented (Im & Jitendra, 2020).

This paper reports on a study aiming to contribute to design and investigate learning opportunities for *concept-based proportional reasoning also for students with mathematical learning difficulties* (MLD), a target group that has been characterized as students with special needs in relation to mathematics, not only due to learning disabilities or other formally attested special needs, but also due to restricted mathematical learning opportunities in families and schools (Scherer et al., 2016). For the design of our learning opportunities, we draw upon a graphical representation that was identified as potential for supporting students in conducting proportional reasoning, the double number line (Hino & Kato, 2019). We explore in how far this potential can also unfold for students with MLD and whether it also promotes their ability to explain their proportional reasoning strategies, with the following design research question: *How can the double number line promote students with mathematical learning difficulties to conduct and explain proportional reasoning?*

## Theoretical background: concept-based proportional reasoning

### Reasoning up and down as strategies of proportional reasoning and double number line

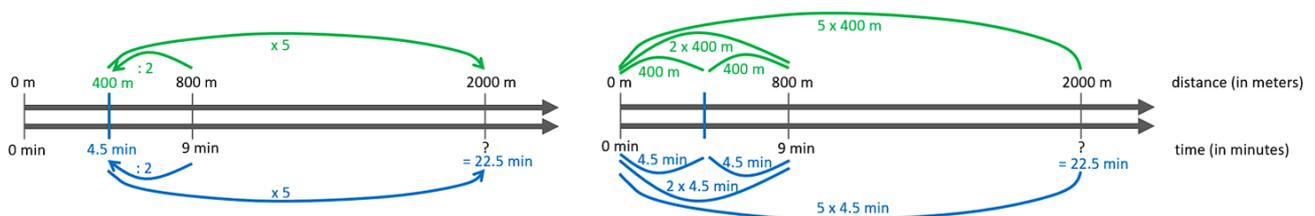
Proportional reasoning is an umbrella term for a complex ability with many aspects, and with strong connections to various other areas of rational numbers and algebra such as ratio, rate, fractions, func-

tions, etc. (Lobato et al., 2010; Lamon, 2007). In a narrower sense, Lamon (2007) characterizes proportional reasoning as the ability to solve comparison problems (searching for equivalent ratios) or missing value problems (when three out of four values in two equivalent ratios are given and the fourth is to be searched). These problems can be solved by (a) formal procedures such as cross multiplication, (b) semi-formal procedures such as the rule of three or the functional procedure of multiplying with a fixed factor (between-ratio multiplication), or by (c) informal and flexible strategies of *reasoning up and down*, also called scalar strategies with within-ratios (Lamon, 2007; Steinhorsdottir & Sriraman, 2009). Hoyles et al. (2001) showed that formal strategies are more prone to errors in their application, hence we focus on the *informal strategy of reasoning up and down*.

Although semi-formal and informal strategies have been shown to be nearer to the contextual meaning, not all students and adults can explain and use them without errors (Hoyles et al., 2001), especially when introduced only in ratio tables (Steinhorsdottir & Sriraman, 2009). The *graphical representation of the double number line* was reported to be more fruitful than ratio tables for developing conceptual understanding of proportional reasoning strategies (Hino & Kato, 2019). As difficulties in correctly conducting and explaining semi-formal and informal strategies have also been documented for students with MLD (Im & Jitendra, 2020), the double number line seems to be promising for supporting the reasoning up and down of this target group.

### Conceptual base of reasoning up and down on the double number line

The double number line is no guarantee for developing conceptual understanding, e.g., when reasoning up and down is conducted completely detached from the meanings (as one could do with the left example in Figure 1) without the *ability* to provide a conceptual justification. We use the term *concept-based strategy* (CBS) when students (who might usually use the strategy in detached ways) are able to explain and justify the strategy if asked for (e.g., in concept-based interpretations as in the right example in Figure 1). Thus, a concept-based strategy is a *justifiable* strategy, not necessarily a justified one, and also the left example might be justifiable. Hence, understanding proportional reasoning entails not only *conducting* formal and informal strategies for solving comparison problems and missing value problems, but also all concept elements (CE) needed to *understand, explain and justify* these strategies, starting from the often-identified concept elements CE0 (coordinating two quantities) and CE1c (ratios as composed units) (Lobato et al., 2010; Lamon, 2007).



**Figure 1: Reasoning down and up on the double number line: detached (left) or concept-based (right)**

In Figure 2, we list strategies and concept elements into which the detached strategy S4 (of reasoning up and down) can be unfolded for explaining and justifying it. In contrast to a detached strategy S4, the concept-based strategy CBS4 is composed of four strategies CBS0–3, with underlying concept elements CE0–3 for each of them (some for them from Grade 2–4). Although these concept-based

strategies and concept elements for reasoning up and down comprise only a small extract of all identified concept elements for proportional reasoning at large (Lamon, 2007; Lobato et al., 2010; Hino & Kato, 2019), the complexity of Figure 2 indicates why proportional reasoning is difficult to acquire (Steinthorsdottir & Sriraman, 2009), in particular for students with MLD (Im & Jitendra, 2020). That is why it is crucial to explore in how far the double number line can support students with MLD in constructing, articulating and connecting some (not necessarily all) of these components. This requires basic conceptual understanding of the number line and multiplication; both are (re)-introduced in initial introductory context-based tasks with transition opportunities from CBS1 to CBS2.

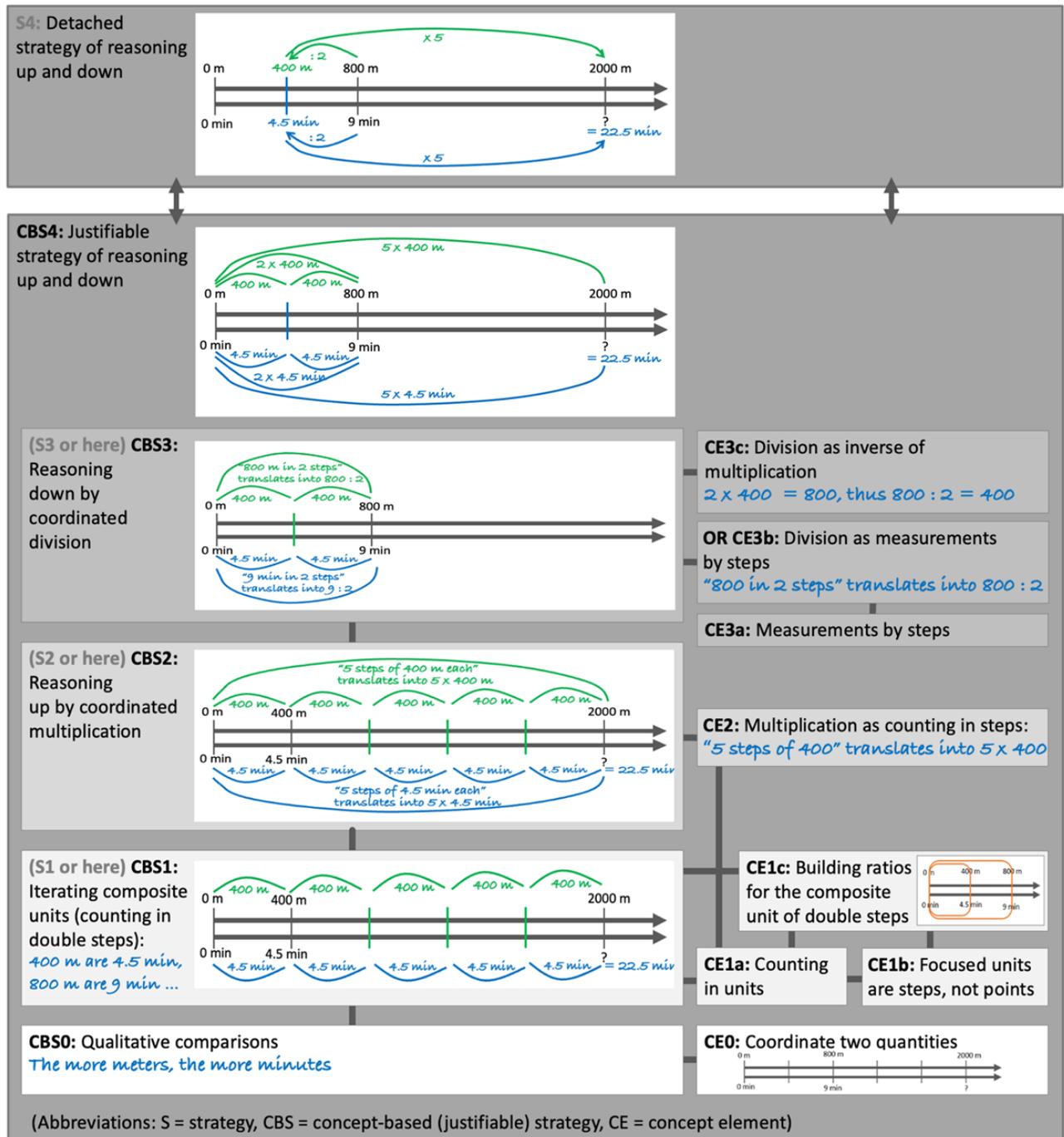


Figure 2: Unfolding the strategy of reasoning up and down into its underlying components

## Methods

**Methods for data collection in design experiments and sampling.** This study stems from a larger design research project (Gravemeijer & Cobb, 2006) on proportional reasoning for students with and without MLD. Design research is chosen as methodology because the project combines two major goals: (a) the design of an empirically grounded and conceptually focused learning environment as well as (b) the empirical investigation of students' individual learning pathways regarding concept-based strategies for proportional reasoning. In Cycle 1 and 2, the first author worked as design experiment (briefly, DE) leader with ten pairs of students with MLD (identified by their teachers), some of them with additional learning or social-behavioral challenges, which were not so relevant for the mathematics learning in our design experiments. Each design experiment series spanned 2-6 sessions of 40 minutes each (some interrupted by pandemic school closures). In total, 33 sessions were video-recorded (~1320 minutes).

**Methods for qualitative analysis.** To pursue the research question, the transcripts of students' learning processes with the double number line were qualitatively analyzed with an inductive-deductive coding procedure. First, codes for concept-based strategies (CBS) and concept elements (CE) were derived from the literature, these were inductively completed by further concept elements that were identified as relevant for students' (successful or incomplete) explanations in the analysis, resulting in the coding scheme printed in Figure 2. Each utterance and thematically relevant gesture on the double number line was coded by the addressed, implicitly used, or missing CBS and CE on a turn-by-turn base (with sub-turns inserted when new ideas occurred). As the explanation of a concept-based strategy can stretch over several turns by successively unfolding into underlying CBS and CE, sequences of connected utterances were also coded by the CBS in view of an explanation. In this way, the use of the double number line for conducting and explaining strategies could be traced.

## Empirical insights into students' ways of conducting and explaining strategies

### Episode 1: Max explains his strategy of reasoning down and up (resembling CBS4)

Prior to the Walking Task in Figure 3, Max and Henry, two seventh graders with MLD, have already worked on the double number line. So, they know CBS1 (counting in double steps), the underlying CE0 (coordinating two quantities), CE1c (building ratios for composite units of double steps), CE2 (multiplication as counting in steps), and CE3b (division as measurement in steps). When individually solving the Walking Task, Max uses CE3a to decide to multiply 4.5 min x 40 to reach 180 min and analogically 400 m x 40, resulting in 16,000 m, a strategy roughly resembling CBS4 without rescaling both units of the composite unit (see scan in Fig. 3). The transcript starts after Max is asked to explain how he came up with his strategy by the DE leader.

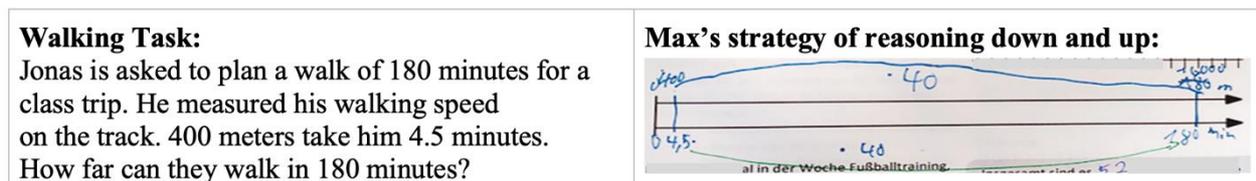


Figure 3: Walking Task treated in Episode 1 and Max's strategy

84a	Max:	Okay, so, first, I just have checked how many. Whether that even fits into [points at the double number line and the text of the task].	CE3a	CBS3
84b		4.5 times, err, divided by 180. Or, 180 divided by 4.5.	CE3b	
84c		And then I have just calculated, that makes 40.	CE1b	
85	Henry	Then I have put a line here [points at the 180 at the end of the number line], wrote 80 min, because in total he has 80 minutes [points at the text].		
86	Max:	[corrects Max's mispronunciation] 180.		
87	DE leader:	And then I just calculated this 4.5 [points at the 4.5 on the number line] times 40, that makes 180.	CE3c	CBS4
88a	Max:	Mhm [affirming].		
88b		And then, I check up here [points at the vertical line 4.5min / 400m]	CE0	CBS2
88c		there were, value um, 4.5 minutes, 400 kilome- er, meters,	CE1c	
88d		managed.	CE1b	
88d		And then, I have just got 400 times 40 [points at 400 and 40 on the number line] equals 16000.	CE2, CBS2	

(Note: CE = concept element, CBS = concept-based strategy, grey = implicitly used, black = articulated)

**Figure 4: Analysis of Max's explanation for his strategy (with codes for turns and sequences)**

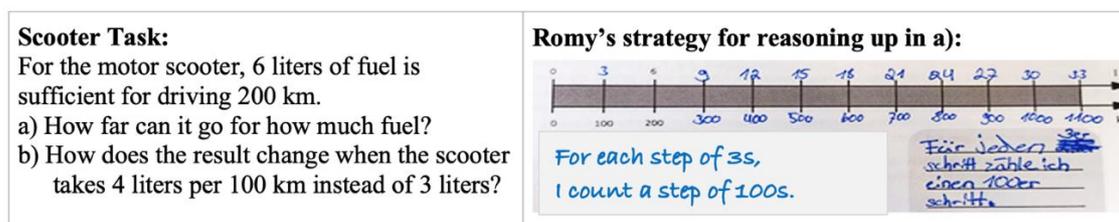
The analysis of the transcript is printed in Figure 4. Max starts his explanation by unfolding into CBS3, in the graphical and textual representation. He explains the reasoning down for the minutes, asking how many times the 4.5 fits into 180 (Turn 84a: CE3a), in a highly concise meaning-related language. By activating the measurement interpretation of the division on the number line (Turn 84b: CE3b), he derives his calculation  $180 : 4.5 = 40$ , and thereby implicitly indicates that not only the end point, but the whole interval is in view (Turn 84c: CE1b). Implicitly drawing upon the idea of multiplication being inverse to division (Turn 86: CE3c), he shifts from dividing to multiplication. Thus, in Turns 84–86, Max unfolds his strategy into the concept elements CE3b, CE3a (and implicitly uses CE3c), and combines them into the concept-based strategy CBS3 of reasoning down for the minutes, drawn in his sketch as multiplication up (see Fig. 3).

To explain the reasoning up part of CBS2 for the meters, Max continues to unfold his ideas on the double number line. With “and then”, he shifts from the minute scale to the meter scale (Turn 88a: CE0, coordinating two quantities) and builds a composite unit “4.5 minutes, 400 ... meters”, interestingly without articulating any explicit connection between the two quantities (Turn 88b: CE1c) with a focus on the covered distance, not the end point he addresses (Turn 88c: CE1b). Finally, he combines all components and condenses them back into a symbolic multiplication for reasoning up by calculating  $400 \times 40 = 16000$  (Turn 88d: CBS2) while implicitly referring to the conceptual meaning of multiplication by pointing them out on the number line (Turn 88d: CE2). In Turns 86–88d, Max shows a profound understanding in his explanation of CBS2.

In total, Max provides an entire explanation of CBS4 by unfolding into CBS3 (Turns 84a–86) and CBS2 (Turns 86–88d) and these into all their underlying concept elements (except for explicitly using CE1a, counting in units). For his explanation, he uses the double number line in various functions: (1) as an *information provider* for reading off values (Turn 88a), (2) as a *communication scaffold* by pointing to elements on the double number line (Turns 84c, 86, 88a) and helping to structure his explanation (Turn 84c). Furthermore, (3) as an *epistemic scaffold* that helps to *structure the given information* in the thinking process (Turn 84c) and *conveys familiar meanings by activating already learned concept elements* such as CE1a and CE3a that the duration has to be measured in steps (Turns 84a, 84c).

## Episode 2: Romy successfully reasons up, yet with limited understanding

Romy and Anna, both seventh graders, work on the Scooter Task in Figure 5, one of the first tasks in the learning environment. Prior to this task, both students developed a first understanding of CBS1 and the double number line. The aim of the Scooter Task is to consolidate CBS1, the iteration of composite units without yet using multiplication. Romy completes task a) successfully, finding the maximal travel distance of 1100 km with 33 liters of fuel.



**Figure 5: Scooter Task treated in Episode 2 and Romy's strategy for a)**

The transcript starts when the DE leader asks orally b) how the result is affected when changing the value from 3 l per 100 km to 4 l per 100 km. After a moment of thinking, Romy answers “40” (Turn 6) and explains her ideas. Figure 6 depicts the analysis of her trial of explanation.

6	Romy:	40	S2 or /S2, CE0	Unjustified /S2
7	DE leader:	[...] how did you fin-, so what did you calculate? What did you do?		
8	Romy:	I have counted all these things [ <i>moves her finger along double number line</i> ], this many, so how many [ <i>moves her pen up and down the 2<sup>nd</sup> vertical line</i> ], how is this called, bar [ <i>moves her complete hand up and down</i> ] are there.	CE1a /CE1b	
10a	Romy:	And that were 10 pieces, so starting here [ <i>points at 2<sup>nd</sup> vertical line</i> ] and then I calculated 10 times 4.	/CE2 /CBS2	
10b			aims at CE1a and CE2	
11	DE leader:	Ah, okay! You have looked at the entire bar [ <i>moves the pen along the double number line</i> ] and determined how many there are. How many bars have you counted? Can you check once more?		
12	Romy:	[ <i>counts by tipping her pen at the vertical lines</i> ] 1,2,3,4,5,6,7,8,9,10,11!	/CE1b, /CE1c	
13	DE leader:	So, the solution should be what?		
14	Romy:	44.	/S2, /CE2, /CE1c	
17	DE leader:	Can you explain once more, how you calculated? Then, we try once more.	aims at CBS2	
22	Romy:	Because this, those ... I don't know.	unjustified /S2	

(Note: /CE = misunderstood concept element, CBS = concept-based strategy, grey = implicitly used, black = articulated)

**Figure 6: Analysis of Romy's trial of explanation for her misunderstood strategy**

For dealing with the composite unit of 4 l/100 m, Romy chooses a seemingly correct multiplicative strategy S2 for reasoning up in a coordinated multiplication (Turn 6: S2 or /S2, CE0). At a first glance, it looks as if she might draw upon multiplication as counting in units (Turns 10a/b: /CE2, /CBS2), but it is not based on an adequate idea of how to use the number line for reading off the number of units. Whereas the vague idea of counting in units is successfully activated (Turn 8: CE1a), her challenges stem from a misunderstanding of the nature of the units: she counts the vertical separation lines on the double number line rather than the steps between these vertical lines, although calling them “bars” (Turns 8, 12: /CE1b). The misunderstood strategy /S2 still works as she starts counting at the second vertical line rather than the zero line. When the DE leaders tries to engage her in a productive struggle by asking for a recount (Turn 11), she counts 11 vertical lines (Turn 12: /CE1b, /CE1c) and immediately corrects the product to 44 (Turn 14: /S2, /CE2, /CE1c), consistently using her wrong strategy /S2 based on the misunderstood /CE2 and /CE1c. In a second explanation between

Turns 17 and 22 she repeats her strategy /S2 in nearly the same words. Without understanding the relevant composite unit not being the vertical lines but steps (/CE1c based on /CE1b), she cannot correct her strategy or justify it further (Turn 22: unjustified /S2).

For her unsuccessful explanation, Romy uses the double number line in limited functions: (1) as an *information provider* for reading off values (Turn 10a, 12), and (2) as a *communication scaffold* by substituting speech by pointing gestures on the number line (Turns 8, 10a). But the double number line fails to be a suitable (3) epistemic scaffold for her as it does not *convey the relevant meanings* (Turns 8, 10a, 12). Instead of focusing the double steps as relevant composite units, she only concentrates on the vertical lines (/CE1b), by which she is unable to explain CE2 and CBS2.

### **Overview on all twenty students**

Like Romy and Max, also all other 18 students were able to *conduct* the reasoning up by coordinated multiplication on the double number line (S2), and those who were introduced to it (before pandemic school closures) also achieved to conduct the reasoning down by division (S3) and their flexible combination (S4). The double number line supports their informal reasoning by structuring the ideas and conveying some meanings for basic concept elements such as counting in steps (CE1a). However, they were challenged to explain and justify the strategies.

## **Discussion and outlook**

### **Summary of empirical insights**

The qualitative analysis of ten pairs of students with MLD provides interesting insights into how the double number line can indeed support students in working with informal strategies of reasoning up and down. It confirms that students with MLD not only bring difficulties but a wealth of productive ideas and approaches that we can be built upon (Im & Jitendra, 2020). Yet the findings suggest to clearly distinguish between *conducting* strategies (which all students achieved), and *explaining and justifying* strategies (which was challenging, as for the adults in Hoyles et al., 2001). *Explaining and justifying the strategies* involves the more ambitious learning goals of reaching concept-based strategies CBS2, CBS3, CBS4. For this goal, the double number line serves as an important tool, not only (1) as *information provider* when reading off values, but mainly (2) as a *communication scaffold* by allowing students to refer to its elements, and (perhaps even more importantly) structuring the given information and parts of their explanation (Hino & Kato, 2019). The presented analyses of two cases also revealed a first insight into the complexity of explanations as it requires to unfold the highly condensed ideas into many different concept elements either learnable in the learning environment (CE0–3), or presumed to be known from earlier learning processes (CE1a, CE3a). Max's achievement in his complex explanation is impressive and far beyond what students with MLD are often expected to achieve (Scherer et al., 2016). (3) Whereas the double number line can *support* the construction of meanings for these concept elements as an *epistemic scaffold*, it does not make communication unnecessary, on the contrary: The unnoticed misunderstanding of basic concept elements (such as Romy's focus on vertical lines rather than steps, /CE1b) can prevent the students from constructing adequate meanings for more elaborate concept elements (such as Romy's /CE1c and /CE2). This observation illustrates the high importance of deep, concept-based communication, especially for students with MLD, when working with the double number line, to overcome these conceptual obstacles.

The analysis confirms the state of research that building ratios of composite units is crucial for proportional reasoning (Lamon, 2007, Lobato et al., 2010). But the case of Romy clearly shows that the relevant units in view are not the points on the number line, but the steps (i.e., the intervals), which has not really been emphasized in the state of research yet.

### **Limitations and outlook on further research**

Of course, the findings must be treated with caution and without hasty generalizations because the data gathering was restricted to only 20 students, all from one school and working with particular tasks and in the privileged laboratory settings of one pair – one teacher. Further research will be required to extend the scope of the findings (a) to more students, (b) to more tasks, (c) to more different proportional reasoning strategies, and (d) to regular whole-class settings. However, the existing data also demands deepening the analysis of further cases, in particular on the more specific roles of the double number line and the possibilities of connecting CEs and CBSs during its use.

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### **References**

- Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. van den Akker et al. (Eds.), *Educational Design Research* (pp. 17–51). Routledge.
- Hino, K., & Kato, H. (2019). Teaching whole-number multiplication to promote children's proportional reasoning. *ZDM – Mathematics Education*, *51*(1), 125–137. <https://doi.org/10.1007/s11858-018-0993-6>
- Hoyles, C., Noss, R., & Pozzi, S. (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education*, *32*(1), 4–27. <https://doi.org/10.2307/749619>
- Im, S.-H., & Jitendra, A. K. (2020). Analysis of proportional reasoning and misconceptions among students with mathematical learning disabilities. *Journal of Mathematical Behavior*, *57*(100753), 1–20. <https://doi.org/10.1016/j.jmathb.2019.100753>
- Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629–667). Macmillan.
- Lobato, J., Ellis, A. B., & Charles, R. I. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8*. National Council of Teachers of Mathematics.
- Scherer, P., Beswick, K., DeBlois, L., Healy, L., & Moser Opitz, E. (2016). Assistance of students with mathematical learning difficulties. *ZDM – Mathematics Education*, *48*(5), 633–649. <https://doi.org/10.1007/s11858-016-0800-1>
- Steinhorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5<sup>th</sup> grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, *21*(1), 6–30. <https://doi.org/10.1007/BF03217536>