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# Inclusive teaching for part-whole understanding: A case study and related reflections on desirable frameworks

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*The paper first discusses the dilemma that, due to the hierarchical order of arithmetic content, substantial content-related participation in inclusive classrooms will become increasingly difficult over the years if mathematics instruction does not succeed in imparting basic knowledge and skills such as part-whole understanding of numbers and adding and subtracting without counting, and this at an early stage. Against this background, an individual case study is reported. The girl in question started school with a very limited set of prerequisites for the learning of arithmetic. By the end of Year 1, she had made considerable progress, but the study suggests that this was greatly facilitated, if not made possible, by the massive additional support of a second teacher, which is extremely rare in South Tyrol, Italy, where the girl lives. The implications of this for the school organisational framework required for inclusion are discussed.*

*Keywords: Inclusive education, hierarchy of arithmetic, basic skills, participation.*

## Introduction

There seems to be consensus among scholars of inclusive mathematics education that “*participation and inclusion are strictly connected*” (Scherer et al., 2023, p. 4) and that *ideally* participation should not be limited to spatial and social aspects but include the very mathematical content (Roos, 2015, p. 115). A promising, well-tested possibility for this is offered by what Wittmann (2001) calls “substantial learning environments” (SLE): tasks that are mathematically rich, often open-ended, and that can be worked out at different levels. One important feature of SLE is that they are “flexible and can be adapted to the special conditions of a classroom” (Wittmann, 2001, p. 2). In our context, this means that children can participate in an appropriately selected SLE even with relatively few prerequisites, such as others can go much deeper into the task. This allows that “the students can choose their level of working by themselves, work on several levels of the task and be successful at their level rather than being assessed against one that is predetermined” (Scherer et al., 2016, p. 614).

However, even if access to a particular SLE is possible with relatively low levels of prior learning, a child must have a *minimum* level of prior learning in order to engage *meaningfully* in mathematical activity in any SLE and to *benefit* from it in their *further development* of competence. In this context, Wittmann points at the “learning hierarchy that lies in the nature of mathematics”, i.e. that “in mathematics, new knowledge can only be acquired if the necessary prerequisites are fulfilled” (Wittmann, 2015, p. 210; own translation). Wittmann states that, as a result, children who lack such prerequisites in relation to a particular learning content “are prevented from realising their full potential”, and further that “it is not possible to accommodate an arbitrarily large spectrum of heterogeneity in a learning group without jeopardising the learning of content that is built on top of another in a hierarchical fashion” (Wittmann, 2015, p. 211; own translation). Wittmann therefore calls for the creation of smaller learning groups within a classroom, with at least two teachers assigned to each maths lesson – mind you not as an *alternative* to all children working together, but as a

*supplement*, and as a *prerequisite* for enabling children with learning difficulties to participate at a higher level.

Of course, mathematical learning difficulties (MLD) exist in all gradations, caused by a variety of reasons. Given Wittmann's emphasis on the hierarchy of learning, it is reasonable to assume that, at least in some cases, it might be possible to prevent difficulties from becoming so serious as to jeopardise further participation if only children could be supported more intensively *at the very beginning* of formal education. Therefore, if we accept that "different children cannot and do not have to achieve the same goals" and that "inclusive education demands from all children exactly what they can achieve" (Häsel-Weide & Nührenbörger, 2023, p. 1), it must also be said that *what they can achieve* depends on whether they acquire in the first year the foundations that are indispensable for the further development of competence in arithmetic; and this is likely to depend on whether inclusive mathematics education succeeds in providing these foundations. That *part-whole thinking of numbers* is an important element of these foundations, seems to be yet another broad consensus within the relevant literature. The next section is devoted to this fundamental concept.

### **Part-whole thinking as a starting point of arithmetic**

Part-whole (or part-part-whole) thinking about numbers, i.e., for example, thinking of eight as a composition of five and three, enables the child to understand addition and subtraction as two related operations within a composed whole and to deduce at least the directly related addition and subtraction operations without resorting to counting strategies (in the case of our example:  $8-5$ ,  $8-3$ ,  $3+5$ ,  $5+3$ ; Gaidoschik, 2019). Part-whole thinking is the basis for more complex problem solving (Björklund et al., 2021) as well as for less salient derived facts strategies. Derived facts strategies (e.g.,  $8-5=3$ , *therefore*  $8-6=2$ , *one less* than  $8-5$ ) for themselves are, besides fact retrieval, the only sustainable alternative to calculation by counting (Gaidoschik, 2019). Moreover, empirical evidence as well as subject matter analysis suggest that deriving facts indeed helps children to automate facts (Gaidoschik, 2012). Finally, the importance of children learning to solve addition and subtraction problems without counting, either by recalling facts or by reasoning from known facts, and doing so at an early age, can be justified not least by the risk that computational strategies based on counting become entrenched and thus a dead end for further mathematical development. Indeed, calculating by counting is a key feature of severe and prolonged MLD (Gaidoschik, 2019).

However, as Kullberg et al. (2020, p. 168) note in relation to the mathematics education literature: "Working with part-part-whole relations has been seen as a component in children's learning of addition and subtraction skills, but has not, however, been viewed as the starting point."

Instead, eminent researchers have advocated "counting and the advancement of counting strategies, as the main path for development" (Kullberg et al., 2020, p. 168). This is theoretically underpinned by stage models that postulate a kind of "natural development progression" (Clements & Sarama, 2009, p. 2) of computational strategies, starting with counting all, which would eventually be replaced by less tedious counting strategies such as counting on, and finally leading to direct and derived fact strategies. Gaidoschik (2019) points to the danger of such stage models mistaking for necessity what can be better understood as the consequences of the dominant way in which pre-schoolers currently seem to engage with numbers, namely using them primarily for counting.

Björklund, Kullberg, and colleagues have recently published a series of papers on an 8-month intervention with Swedish preschool children, demonstrating and explaining how an approach that *focuses* on part-whole relations supports preschool children in solving arithmetic problems without resorting to counting strategies. The Swedish team based much of their intervention on activities involving the use of fingers. However, the fingers were not used to solve addition and subtraction problems by counting, but rather to represent numbers in a structured way and thus as a basis for reflecting on the part-whole relationships involved in, for example, dividing eight fingers into two hands of five and three fingers, respectively (Björklund et al., 2021; Kullberg et al., 2020).

Targeting not preschool but initial primary school arithmetic instruction, Gaidoschik (2019) outlines a design for classroom activities in the first weeks of the first year that also relies heavily on the use of fingers to elaborate and consolidate part-whole thinking. In the meantime, this design has been implemented within the first circle of a small design research study. The present paper tries to give an insight into the individual case of one of the participating children who started her first year of school with rather little knowledge and competence in the field of quantities and numbers. Against the background of what has been said so far, the case study presented in this paper is intended to contribute to a better understanding of the possibilities, but also the limits, of preventive early education, which aims to lay the foundations for as many children as possible to participate in inclusive mathematics lessons at *their* highest possible level of content in the following school years.

### **Cornerstones of the teaching design underlying the case study**

The design, the implementation of which forms the basis of the case study, involves classroom activities organised around three interrelated “construction sites” A), B), and C), during the first weeks of the first school year (for a fuller description, see Gaidoschik, 2019): First, the targeted working out and/or consolidation of A) *counting connected with cardinal understanding*, i.e. as a *means of determining a specific quantity* (Björklund et al., 2021). Counting in this way is not only a fundamental mathematical activity in itself, but also a prerequisite for exploring part-whole relationships for those quantities that cannot be determined by subitising. *Subitising*, as a developmentally independent form of determining small numbers, is then part of the activities in B) on how to *answer the question “How many?” without counting*. However, little is gained for further steps if the *perception* of small quantities without counting is not accompanied by an *awareness* of the underlying *structures*, such as four being composed of two and two. Therefore, the design includes activities such as examining and reconstructing the pattern of dots on some dice; or using the fingers to show the same part-whole relations to be found on a dice. Such part-whole explorations are then extended to all numbers up to ten, mainly by returning to the fingers structured by five on each hand. Finally, the activities in the first weeks revolve around C) *comparing quantities* by one-to-one correspondence, with the aim of establishing a solid understanding of “equality” as a binary relation between two quantities that does not require counting to establish.

The design envisages working more or less *simultaneously* on all three sites A), B), and C), in the first few weeks and making the connections between them as transparent as possible, for example by emphasising that, when counting upwards, each natural number actually *contains* its predecessor as a part, plus one as a complement to the whole, and is therefore *one more* (not just *the next number*).

An essential principle of the design is inclusivity, i.e. it takes into account the heterogeneity of the school beginners and tries to use this as productively as possible for the further learning of as many children as possible. Important instruments in this endeavour are open-ended tasks that can be worked on at different levels of complexity and in different number ranges. Working simultaneously on the three “construction sites” facilitates this in that children who have greater needs in learning to count, for example, may spend more time in this area than others, exploring part-whole relationships (their zone of proximal development) first only up to five, then ten. Other children very soon work on part-whole relationships up to 20 or even 100 and are therefore not hindered in their further learning. Nevertheless, there is always a common ground for class discussions, for example, about what to look for when counting or how to compare two sets without counting, regardless of the number range in which different children have previously worked on these questions.

The work on the three sites A), B), and C), in the first few weeks is intended to lay the foundation for the next step, the introduction of addition and subtraction. This is closely linked to the part-whole thinking developed so far. In fact, linking part-whole thinking with ideas of taking away, putting together, adding, and comparing, is already an important part of the activities in B), for example by asking: How many of your eight fingers are left if you take away the five fingers of the whole hand, without counting? In addition, the children are repeatedly asked to invent word problems on their own, starting from a part-whole triple that they have just represented with their fingers, for example “There are seven children in the playground, five have to go, how many are left?”. At this first stage, however, all this is done by *acting* with the fingers, preferably without counting them one by one; by *reflecting* on these actions and their results; by *acting in the imagination* (in any case, the children are encouraged to do so). However, they do not have to *write* anything down. In the next step, of course, it is precisely this transfer of *actions* performed with quantities, into *symbols*, which in the best case evoke mental representations of such actions, that is at the heart of the activities involving all the children in the classroom. It is beyond the scope of this paper to continue with this rough sketch of the design. The case study presented below is of one girl’s learning progress and problems during the first ten weeks of the class activities described in the previous section.

## **Methods and framework conditions of the case study**

The teaching design was implemented in a German-speaking first grade class of 18 children in a small village in South Tyrol, Italy. The teacher who volunteered for this project had three half-day meetings with the author before the start of the school year to clarify the main ideas of the design and prepare its implementation. One boy in the class had been classified as “disabled”. This is why, in accordance with Italian legislation (Gaidoschik, 2022), a so-called “integration teacher” (IT) supported the class teacher in most lessons. According to the regulations, ITs should have additional training in inclusive pedagogy. This does *not* usually include advanced skills in dealing with MLDs. In the present case, however, the IT had participated on her own initiative in several training courses on the prevention of MLDs led by the author of this paper. The IT also took part in the above-mentioned meetings and was thus informed about the design, which she endorsed. From the second week in early September, the author visited the school every two weeks to observe a maths lesson, to interview, subject to parental consent, selected children individually outside the classroom (duration: 10 to 25 minutes) and to discuss the next steps with the two teachers. All interviews were videotaped and transcribed;

the transcripts were later subjected to qualitative content analysis to identify learning progress and persistent difficulties as a basis for further development of the design. As the design was intended to guide the *initial* teaching, the close collaboration described above ended in mid-November after five visits. From then on, the teachers worked on their own, but informed and consulted the author once a month. Two weeks before the end of the school year, the author interviewed selected children, again with parental consent, to identify their calculation strategies up to 20.

### **The case of a first-year student starting with low learning prerequisites**

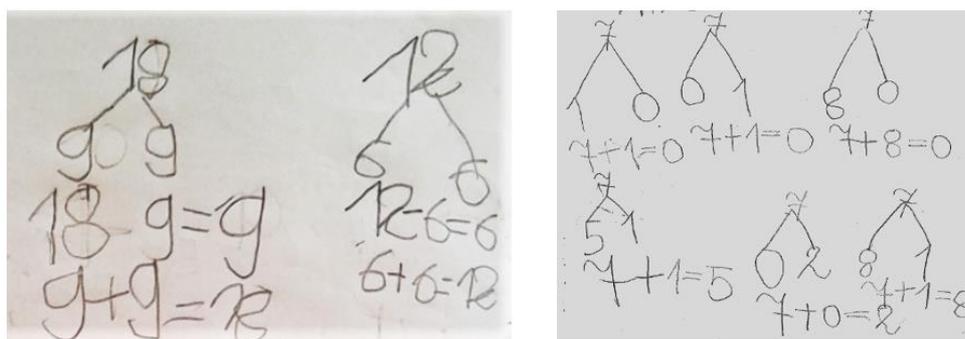
The girl described below, whom I will call Lina, was very shy and spoke in a low voice throughout our sessions but seemed to make every effort to complete the tasks set for her. In interview 1, Lina knew the series of number words up to twenty but had difficulty counting backwards or identifying the successor and predecessor of numbers up to ten. When I asked her how many cubes I had put in front of her, she came up with eight instead of nine, because she tapped on two cubes while saying the two syllables of “sieben” (seven in German). After we counted together and found eight to be the correct number, I rearranged the cubes and asked if she still knew how many there were; she immediately started counting again and did so again after I rearranged the cubes a second time in front of her eyes. She was able to show five and ten fingers without counting, but she had to count to show six, seven, eight, and nine fingers, respectively.

Between interviews 1 and 2, Lina participated in the classroom activities described above, which focused on counting with cardinal understanding, comparing, and reflecting on part-whole relations using fingers, if possible, *without counting*, and talking about underlying structures. In addition, the IT, informed by my observations, tried to support Lina (like other children, such as the boy classified as “disabled”) individually, both in class and in additional one-to-one sessions of a few minutes during the so-called “open learning time”. In interview 2 (which, like all interviews, is only reported here in a few selected excerpts), Lina showed progress with cardinal understanding. Again, she *counted all* the fingers needed to indicate seven or eight. In interview 3, Lina for the first time used *counting on* from five to show eight fingers. She tried to show seven fingers without counting, but ended up with five and three, without realising her mistake. But she showed nine fingers correctly without counting, explaining that her teachers had taught her a trick to show ten and take one away.

At the time of interview 4, the class teacher had already introduced plus and minus signs. Inventing word problems and noting the associated term had been an integral part of the last four lessons. I asked Lina to *explain how* to show eight fingers. She told me without hesitation that she would need five and three fingers for doing so. Then I asked her to tell me a story, just as she had learnt in class, starting with eight and using her knowledge that eight can be shown with five and three fingers. She suggested “Eight horses run away, only one is left”. With a lot of prompting, we ended up with “There are eight horses, then five run away, leaving three”. Asked to write down this story with numerals, she did not know what to do. With help, she finally wrote  $8-5=3$ . Immediately afterwards, she could not transfer this concept to a story starting with seven. Her first suggestion was “Seven horses run away, one stays”.

In interview 5, the last considered for this paper, I showed Lina, among other things, cards with subtraction terms as  $6-5$ ,  $8-5$ ,  $5-4$ . At that time, an open-ended task used repeatedly in the classroom

was to write down a known part-whole triple and next to it write addition and subtraction problems that could be easily solved by applying this decomposition. Figure 1 shows, on the left, an example of another child in Lina’s class doing this task from the same period, and on the right, what Lina herself had written in her workbook during a lesson the day before interview 5.



**Figure 1: Deriving addition and subtraction from self-selected part-whole triples; left, a classmate’s work, right, what Lina wrote in her workbook**

Here it seems that Lina was trying to *reproduce a scheme* without making sense of it. In addition, she seemed to be overwhelmed by having to distinguish between two ways of using such triples, one for addition, one for subtraction, using different operation signs. In the interview, she first did not know what to do with the written task 6–5 at all. I asked her to show six fingers, which she did very quickly, without counting, holding up one full hand and the thumb of the other. Then I asked her to take away five, which she did immediately by simply dropping the full hand. After that she had no problem solving 8–5 and 9–4, as written on the cards, by herself, using her hands, without counting. However, when I asked her to solve 5–4, she unfolded first all the fingers of one, then four fingers of the other hand, dropped the hand with the four fingers and said “five” as the solution to 5–4.

To summarise her development from interview 1 to 5, Lina made considerable progress over this period. When counting items in later interviews, she now showed cardinal understanding. She was finally able not only to represent *all* numbers up to ten with her fingers without counting, but also to verbally express the part-whole relationships involved. She showed part-whole thinking in problems such as 8–5 or 9–4 when she solved them using her fingers, without counting. At this stage, however, this was strictly limited to problems that can be solved by taking away the *totality* of the fingers extended on one hand. When confronted with 5–4, she did not identify the four as part of one hand but held up *another four* on the second hand. Even with those addition and subtraction problems that she could link to her part-whole knowledge, she was overwhelmed when confronted with their symbolic representations and needed more explicit prompting to translate them into word problems.

As said, I did a follow-up interview at the end of the first year. At this time, Lina did not use counting at all to solve the 7 addition and 7 subtraction problems up to 10 (such as 3+4, 9–6) and 8 problems up to 20 (such as 6+7, 14–9) that I asked her. However, only three of them she solved by fact retrieval, and none by derived facts. Instead, she used her fingers, but without *counting* them. For 9–8, for example, she held up at once nine fingers, then folded, quickly, two sets of four fingers each, stating “one” as the result. She seemed to have gained some confidence in her competences and even smiled a few times after completing a task, something she had never done in the first five interviews.

## Discussion

The teaching design aimed to enable all children to learn to add and subtract without recourse to counting strategies, based on part-whole thinking. In this context, the interim conclusion to be drawn from this case study for Lina is mixed. At the start of her first year of school, Lina had lacked many competencies that many other children in her class had already acquired. The aim of the design was not to slow these children down, but at the same time to allow others to develop and consolidate a part-whole understanding of numbers as a solid basis for further steps. For Lina, this goal was finally achieved, but with great effort and delay. It may be assumed that the additional support she received from the IT was a decisive contribution to making this success possible. In any case, both teachers stated in retrospect of the year that without the consistent, patient, and appreciative support of the IT inside and outside the classroom Lina would not have been able to benefit from many of the classroom activities in the second term and finally learn to calculate without counting, albeit with residual use of her fingers. However, a second teacher in the classroom (and the more so one with specific training in dealing with MLDs) is the exception rather than the rule in South Tyrol, at least in recent years, mandatory only if a child in the class has been classified “disabled” (Gaidoschik, 2022).

Returning to the original question, how the “internal hierarchy of learning” constitutive of arithmetic can be prevented from becoming an obstacle to substantial content-related participation: What can we learn from Lina’s case? There is reason to assume that without the extensive additional support of a second teacher with advanced competencies in dealing with MLDs, working in close coordination with the class teacher, Lina might have ended up as one of the many children known from research to be largely dependent on counting strategies at the end of the first year and well beyond. Subsequently, it would have become increasingly difficult for her to gain competence and experience self-efficacy by participating in SLE that required, for example, fluent addition and subtraction with two-digit numbers. Of course, such participation is also possible for children who use counting as their main strategy. However, if they do take part in this way, it is unlikely to help them to *overcome* computing by counting, quite the opposite: Counting as a problem-solving strategy is likely to become even more entrenched, with even greater consequential problems. There are promising didactical approaches to helping children learn to calculate without counting, but they take time to teach to a child with an MLD and require a level of one-to-one attention that a classroom teacher is unlikely to be able to provide, especially if left to her own devices.

Of course, didactical research should strive to develop and optimise teaching designs that work under “normal conditions”. However, Lina’s case suggests that what is currently *exceptional*, at least in Italy, should become *normal*, namely the cooperation of two committed and competent teachers in daily mathematics lessons. If this can be established, at least for the first grade, the dilemma described by Wittmann (2015), that inclusion could conflict with the realisation of the individual potential of all children to make progress in arithmetic, could be avoided, at least for some of the children.

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