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Set-Valued Koopman Theory for Control Systems

Benoît Bonnet-Weill* and Milan Korda*,†

January 21, 2024

Abstract

In this paper, we introduce a new definition of the Koopman operator which faithfully encodes the dynamics of controlled systems, by leveraging the grammar of set-valued analysis. We likewise propose meaningful generalisations of the Liouville and Perron-Frobenius operators, and show that they respectively coincide with proper set-valued analogues of the infinitesimal generator and dual operator of the Koopman semigroups. We also give meaning to the spectra of these set-valued operators and prove an adapted version of the spectral mapping theorem. In essence, these results provide theoretical justifications for many existing approaches that consist in bundling together the Liouville operators associated with different control parameters to produce Koopman eigenvalues and eigenfunctions for control systems.

Keywords : Koopman Operators, Control Systems, Differential Inclusions, Set-Valued Analysis.

MSC2020 Subject Classification : 28B20, 34A60, 47N20, 93C15.

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1 Introduction

The Koopman operator provides an equivalent representation of a nonlinear dynamical system by an infinite-dimension linear operator and is now a well-established framework for analysis of dynamical systems. Originating with seminal works of Koopman [35] and Koopman and von Neumann [34] in the early 1930s, it has enjoyed a renewed interest pioneered by the works of Mezić and Banaszuk [43]

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and Mezić [42]. Heuristically, given a one-parameter semigroup $(\Phi_t)_{t \geq 0}$ representing e.g. the integral curves of a dynamical system, the *Koopman operators* are linear operators whose action is defined by

$$\mathcal{K}_t(\varphi) := \varphi \circ \Phi_t \quad (1.1)$$

for each (typically real or complex valued) function $\varphi \in \mathcal{X}$ belonging to some relevant subspace of function observables that denoted by \mathcal{X} . Besides its appealing theoretical properties, allowing for the use of the whole corpus of spectral theory available for linear operators to understand nonlinear systems, the Koopman operator has served as a foundation for numerical methods, most notably the (Extended) Dynamic Mode Decomposition [50, 58].

More recently, the Koopman operator framework was used for systems with external inputs, first regarded as exogenous inputs [47] that one cannot manipulate and later, starting with the work [36], regarded as control inputs that one chooses in order to achieve specific control objectives. When used within a model predictive control framework, the approach proposed in the latter work leads to a convex optimization problem, contrary to traditional nonlinear model predictive schemes. This very appealing feature has lead to a number of follow up works aiming at improving the practical aspects of the method, see e.g. [19, 37, 45, 46, 52], as well as a number of applications including soft robotics [31], power grid stabilisation [38], fluids [3, 46] or quantum control [30]. These are by no means exhaustive lists, and we point the interested reader to the surveys [10, 17] for more references. However, contrary to the abundance of methodological advances and applications of the Koopman framework, a sound theoretical footing for the Koopman operator with control is still missing at present. Indeed, the work [36] defined the Koopman operator with control on the so-called tensor-product system but did not leverage this definition for theoretical analysis. In [47], the authors considered the Koopman operator with one fixed value of the control input or with control subject to a dynamical evolution (e.g., determined by a feedback law), again without providing theoretical insights, while the subsequent work [46] focused on the Koopman operators associated with each individual control input and interpolated between them for control design purposes.

In this article, we develop a theoretical framework for the Koopman operator in the presence of controls, based on the theory of set-valued analysis. Since the seminal works of Filippov [25] and Wazewski [57] at the turn of the 1960s, it has been known that rephrasing control problems in terms of differential inclusions provided powerful analytical tools while bringing key insights on the optimal sets of assumptions needed to establish positive control-theoretic results. Indeed, the methods of set-valued analysis have been successfully applied to a large breadth of control problems throughout the past decades, ranging from the well-posedness of constrained dynamical systems both in the classical [9] and hybrid [29] settings, to Pontryagin [21, 55] and Hamilton-Jacobi [27, 28] optimality conditions as well as Lyapunov stability methods [53]. This list of references is far from complete, and we point the interested reader to the reference monographs of Aubin, Cellina, Clarke, Frankowska and Vinter [5, 7, 20, 54] for further details. Based on these observations, we propose a comprehensive adaptation of the main concepts of Koopman theory for general time-invariant control systems of the form

$$\dot{x}(t) = f(x(t), u(t)),$$

where $f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ is a sufficiently nice controlled vector field. Then, one can associate to each admissible control signal $u(\cdot) \in \mathcal{U} \subset L^\infty(\mathbb{R}_+, U)$ a unique family of flow maps $(\Phi_{(\tau,t)}^u)_{\tau,t \geq 0}$ satisfying

$$\Phi_{(\tau,t)}^u(x) = x + \int_\tau^t f(\Phi_{(\tau,s)}^u(x), u(s)) ds$$

for all times $\tau, t \geq 0$ and every $x \in \mathbb{R}^d$, which leads us to defining the *set-valued Koopman operators* as the collection of evaluations of a given observable $\varphi \in \mathcal{X}$ along all possible controlled flows, that is

$$\mathcal{K}_{(\tau,t)}(\varphi) := \left\{ \varphi \circ \Phi_{(\tau,t)}^u \text{ s.t. } u(\cdot) \in \mathcal{U} \right\}.$$

In other words, $\mathcal{K}_{(\tau,t)}(\varphi)$ is the set of all observables reachable by admissible control trajectories starting from the observable $\varphi \in \mathcal{X}$. We discuss some of the fundamental topological properties of

these objects in Section 3, and provide a general representation formula for arbitrary time-dependent Koopman observables $(\tau, t) \in [0, T] \times [0, T] \mapsto \psi_{(\tau, t)} \in \mathcal{K}_{(\tau, t)}(\varphi)$, involving measurable families of control signals. It should be noted that while autonomous controlled systems can be described without loss of generality by a one-parameter semigroup, up to a time reparametrisation, we chose to define the Koopman operators as a two-parameter family following to [40]. The main incentive for doing so, besides being somewhat more precise from a mathematical standpoint, lies in the fact that the dynamics of Koopman observables depends in a key way on the starting time of the controlled flows.

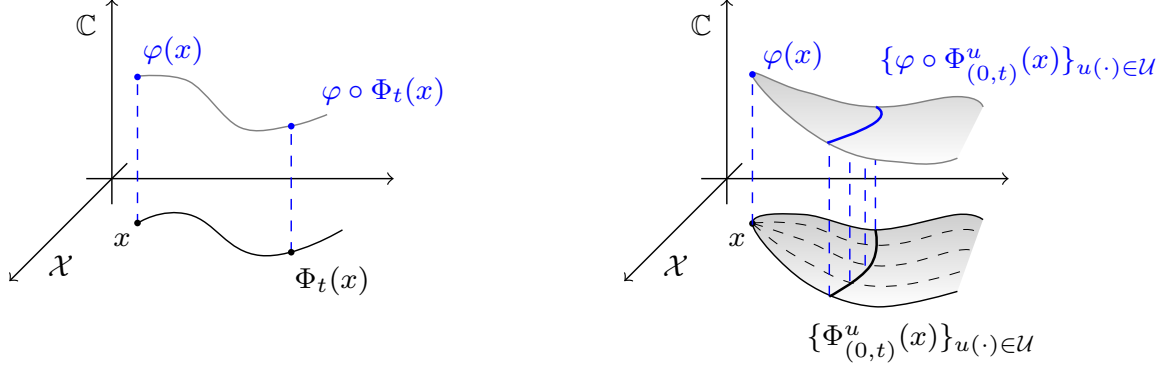


Figure 1.1: The classical Koopman operator associates to an observable $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$ the measurement $\varphi \circ \Phi_t(x)$ along a single flow $\Phi_t(x)$ evaluated from every possible $x \in \mathbb{R}^d$ at time $t \in [0, T]$ (right), whereas the set-valued Koopman operator outputs the whole reachable set $\{\varphi \circ \Phi_{(0,t)}^u(x)\}_{u(\cdot) \in \mathcal{U}}$ of observables along the collection of controlled flows $\{\Phi_{(0,t)}^u(x)\}_{u(\cdot) \in \mathcal{U}}$ starting from any given $x \in \mathbb{R}^d$ and evaluated at time $t \in [0, T]$.

In Section 4, which is the core of the manuscript, we put forth relevant set-valued analogues of the *Liouville* and *Perron-Frobenius* operators. These objects are known to play a pivotal role in Koopman theory as well as many of its applications, the former by being the infinitesimal generator of the Koopman semigroup and therefore encoding the dynamics of Koopman observables, and the latter by being the topological dual of the Koopman operators, thus providing alternative angles and methods to address dynamical systems problems. In Section 4.1, we define the *set-valued Liouville operators* as

$$\mathcal{L}(\varphi) = \left\{ \nabla_x \varphi \cdot f_u \text{ s.t. } u \in U \right\}$$

for each continuously differentiable observable $\varphi \in \mathcal{D}$, and prove that when the set of controlled velocities is convex in a suitable sense, the latter is the infinitesimal generator of the Koopman semigroup. More precisely, we show in this context that

$$\text{Lim}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau, t)}(\varphi) - \varphi}{t - \tau} = \mathcal{L}(\varphi),$$

where limit is understood in the sense of Kuratowski-Painlevé. In addition, when the controlled vector fields are continuously differentiable in the space variable, we prove that time-dependent Koopman observables of the form $(\tau, t) \in [0, T] \times [0, T] \mapsto \psi_{(\tau, t)} := \varphi \circ \Phi_{(\tau, t)}^u \in \mathcal{D}$ for some fixed control signal $u(\cdot) \in \mathcal{U}$ coincide exactly with the strong solutions of the functional differential inclusion

$$\begin{cases} \partial_\tau \varphi_{(\tau, t)} \in -\mathcal{L}(\varphi_{(\tau, t)}), \\ \varphi_{(t, t)} = \varphi, \end{cases}$$

posed in the space of observables \mathcal{X} . In Section 4.2, we shift our focus to the investigation of duality results for the set-valued Koopman operators. In this context, we propose the following definition for the *set-valued Perron-Frobenius operators*

$$\mathcal{P}_{(\tau, t)}(\mu) := \left\{ \Phi_{(\tau, t)}^u \# \mu \text{ s.t. } u(\cdot) \in \mathcal{U} \right\},$$

and show that these latter are indeed the adjoints of the Koopman operators in the suitable set-valued sense, which is inspired by the concept of adjoint differential inclusion studied in [26]. We also leverage

this notion to prove that the infinitesimal generator of the Perron-Frobenius semigroup is the adjoint of the Liouville operator, namely

$$\text{Lim}_{t \rightarrow \tau} \frac{\mathcal{P}_{(\tau,t)}(\mu) - \mu}{t - \tau} = \mathcal{L}^*(\mu)$$

in the weak-* topology of \mathcal{X}^* . To our surprise, we discovered that the underlying functional inclusion

$$\begin{cases} \partial_t \mu_{(\tau,t)} \in \mathcal{L}^*(\mu_{(\tau,t)}), \\ \mu_{(\tau,\tau)} = \mu, \end{cases}$$

coincided – at least at a formal level – with the notion of *continuity inclusion* introduced by the first author and Frankowska in [11, 13] in the context of mean-field control. Lastly, in Section 4.3, we prove a set-valued counterpart to the usual spectral mapping theorem, which relates the point spectra of the Liouville operator with that of the Koopman semigroup. We show in particular that

$$e^{(t-\tau)\sigma_p(\mathcal{L})} \subset \sigma_p(\mathcal{K}_{(\tau,t)}),$$

which provides grounding to preexisting works such as [45, 46] in which spectral properties of Koopman operators for control systems are investigated by exponentiating the spectra of the collection of Liouville operators associated with each admissible control value.

The manuscript is organised as follows. We start by exposing preliminary notions of functional and set-valued analysis as well as control theory in Section 2, all of which will be used in our subsequent developments. In Section 3, we define the set-valued Koopman operators and study their main properties. We then move to the investigation of the matching set-valued Liouville and Perron-Frobenius operators in Section 4, wherein we show that the latter are respectively the infinitesimal generator and dual of the Koopman semigroups. We then close this section with the statement of a set-valued counterpart of spectral mapping theorem, and provide the proof of a technical regularity result for controlled flows in Appendix A.

2 Preliminaries

In this first section, we collect preliminary material on integration theory, set-valued analysis and controlled systems, for which we refer the readers to the monographs [33], [7] and [20] respectively.

2.1 Functional analysis and integration

In what follows, we denote by \mathcal{L}^d the standard d -dimensional Lebesgue measure over \mathbb{R}^d and denote by $I \subset \mathbb{R}$ a generic subinterval of the real line. Following [7, Section 8.1], we recall that a map $f : I \rightarrow \mathcal{X}$ valued in a separable Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is \mathcal{L}^1 -measurable provided that

$$f^{-1}(\mathcal{O}) := \{t \in I \text{ s.t. } f(t) \in \mathcal{O}\}$$

is \mathcal{L}^1 -measurable for each open set $\mathcal{O} \subset \mathcal{X}$. Below, we recall the concept of integrability in the sense of Bochner for maps valued in separable Banach spaces, for which we refer to [23, Chapter II].

Definition 2.1 (Bochner integrable maps). *An \mathcal{L}^1 -measurable maps $f : I \rightarrow \mathcal{X}$ is said to be Bochner integrable if it satisfies*

$$\int_0^T \|f(t)\|_{\mathcal{X}} dt < +\infty.$$

The collection of all such maps is a separable Banach space that we denote by $L^1(I, \mathcal{X})$. Likewise, we denote by $L^\infty(I, \mathcal{X})$ the Banach space of \mathcal{L}^1 -measurable maps satisfying $\sup_{t \in I} \|f(t)\|_{\mathcal{X}} < +\infty$.

Similarly to the Lebesgue integral, the integral in the sense of Bochner of a map against a probability measure defines an application whose range is convex, as evidenced by the following proposition.

Proposition 2.2 (Convexity of the range of the Bochner integral). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a separable Banach space, $K \subset \mathcal{X}$ be a closed set and $f(\cdot) \in L^1(I, \mathcal{X})$ be such that $f(t) \in K$ for \mathcal{L}^1 -almost every $t \in I$. Then, it holds that*

$$\frac{1}{t - \tau} \int_{\tau}^t f(s) ds \in \overline{\text{co}} K$$

for each $\tau, t \in [0, T]$.

Proof. This result follows from a straightforward combination of Hahn-Banach's separation theorem (see e.g. [16, Theorem 1.7]) together with the linearity and homogeneity of the Bochner integral. \square

Given a normed vector space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, we denote by \mathcal{X}^* its topological dual, i.e. the collection of all bounded complex-valued functionals over \mathcal{X} , and write $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ for the underlying duality pairing. In the next proposition, we recall a fine compactness criterion for the weak topology of $L^1(I, \mathcal{X})$, which is excerpted from [22].

Proposition 2.3 (A weak L^1 -compactness criterion for the Bochner integral). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a separable Banach space and $(f_n(\cdot)) \subset L^1([0, T], \mathcal{X})$. Suppose that there exists a map $m(\cdot) \in L^1(I, \mathbb{R}_+)$ along with a convex and compact set $K \subset \mathcal{X}$ such that*

$$\|f_n(t)\|_{\mathcal{X}} \leq m(t) \quad \text{and} \quad f_n(t) \in K$$

for \mathcal{L}^1 -almost every $t \in I$. Then, there exists a subsequence $(f_{n_k}(\cdot))$ that converges weakly to some $f(\cdot) \in L^1(I, \mathcal{X})$ satisfying $f(t) \in K$ for \mathcal{L}^1 -almost every $t \in I$. In particular, it holds

$$\int_I \langle \mu(t), f(t) - f_{n_k}(t) \rangle_{\mathcal{X}} dt \xrightarrow{k \rightarrow +\infty} 0$$

for each $\mu(\cdot) \in L^\infty(I, \mathcal{X}^*) \subset L^1(I, \mathcal{X}^*)$.

In the sequel, given some $k \in \{0, 1\}$, we denote by $(C_c^k(\mathbb{R}^d, \mathbb{C}), \|\cdot\|_{C^k(\mathbb{R}^d, \mathbb{C})})$ the separable vector space of k -times differentiable maps with compact support, endowed with the relevant supremum norm. We recall then that by Riesz's representation theorem (see e.g. [49, Theorem 6.19]), one has

$$C_c^0(\mathbb{R}^d, \mathbb{C})^* \simeq \mathcal{M}(\mathbb{R}^d, \mathbb{C})$$

where $\mathcal{M}(\mathbb{R}^d, \mathbb{C})$ is the vector space of finite complex-valued Radon measures. In this context, given an element $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{C})$ and some $f \in L^1(\mathbb{R}^d, \mathbb{C}; \mu)$, we define the so-called *divergence distribution* $\text{div}_x(f\mu) \in C_c^1(\mathbb{R}^d, \mathbb{C})^*$ via its duality action as

$$\langle \text{div}_x(f\mu), \zeta \rangle_{C_c^1(\mathbb{R}^d, \mathbb{C})} := - \int_{\mathbb{R}^d} \nabla_x \zeta(x) \cdot f(x) d\mu(x) \quad (2.1)$$

for all $\zeta \in C_c^1(\mathbb{R}^d, \mathbb{C})$. We also recall that the *image* – or *pushforward* – of a finite Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{C})$ through a Borel map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the unique measure $f_{\#}\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{C})$ which satisfies

$$f_{\#}\mu(B) := \mu(f^{-1}(B))$$

for each Borel set $B \subset \mathbb{R}^d$. Moreover, the latter is characterised by the change of variable formula

$$\int_{\mathbb{R}^d} \varphi(x) d(f_{\#}\mu)(x) = \int_{\mathbb{R}^d} \varphi \circ f(x) d\mu(x) \quad (2.2)$$

which holds for every $\varphi \in C_c^0(\mathbb{R}^d, \mathbb{C})$.

In the next definition, we recollect the notion of *weak- $*$ convergence* for finite Radon measures (see e.g. [2, Definition 1.58]). Note that this latter induces the same topology over measure sets with uniformly bounded mass as the so-called vague convergence, which is defined in duality with the completion of $(C_c^0(\mathbb{R}^d, \mathbb{C}), \|\cdot\|_{C^0(\mathbb{R}^d, \mathbb{C})})$, see the discussion in [4, Section 4.2.2] for more details.

Definition 2.4 (Weak-* convergence of measures). *We say that a sequence $(\mu_n) \subset \mathcal{M}(\mathbb{R}^d, \mathbb{C})$ converges towards some $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{C})$ for the weak-* topology provided that*

$$\int_{\mathbb{R}^d} \zeta(x) d\mu_n(x) \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} \zeta(x) d\mu(x) \quad (2.3)$$

for each $\zeta \in C_c^0(\mathbb{R}^d, \mathbb{C})$.

In what ensues, we will also consider elements of the space $C^0(\mathbb{R}^d, \mathbb{R}^d)$ of continuous functions defined over the whole space whose support may not be compact. In that case, the adequate topology to consider is that of *local uniform convergence*, whose definition is recalled below.

Definition 2.5 (Topology of uniform convergence on compact sets). *A sequence of maps $(f_n) \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ converges uniformly on compact sets – or locally uniformly – towards $f \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ if*

$$\|f - f_n\|_{C^0(K, \mathbb{R}^d)} \xrightarrow{n \rightarrow +\infty} 0$$

for each compact set $K \subset \mathbb{R}^d$. The underlying topology is induced by the translation invariant metric

$$d_{cc}(f, g) := \sum_{k=1}^{+\infty} 2^{-k} \min \left\{ 1, \|f - g\|_{C^0(B(0,k), \mathbb{R}^d)} \right\},$$

which is defined for each $f, g \in C^0(\mathbb{R}^d, \mathbb{R}^d)$.

Following the general results from [33, Chapter 7, Theorems 12 and 13] and [56, Theorem 6], it can be shown that the space $(C^0(\mathbb{R}^d, \mathbb{R}^d), d_{cc}(\cdot, \cdot))$ of continuous functions endowed with the topology of local uniform convergence is a complete separable metric space. More classically, it is well known that $(C^0(K, \mathbb{R}^d), \|\cdot\|_{C^0(K, \mathbb{R}^d)})$ is a separable Banach space for each compact set $K \subset \mathbb{R}^d$.

2.2 Set-valued analysis

In this section, we recall preliminary material pertaining to set-valued analysis, for which we point the reader to the reference monographs [7, 8]. In what follows, we suppose that we are given two metric spaces $(\mathcal{X}, d_{\mathcal{X}}(\cdot, \cdot))$ and $(\mathcal{Y}, d_{\mathcal{Y}}(\cdot, \cdot))$.

Definition 2.6 (Set-valued maps). *A set-valued map – or multifunction – is a function $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ valued in the subsets of \mathcal{Y} , namely $\mathcal{F}(x) \subset \mathcal{Y}$ for each $x \in \mathcal{X}$. The domain of a set-valued mapping*

$$\text{dom}(\mathcal{F}) := \left\{ x \in \mathcal{X} \text{ s.t. } \mathcal{F}(x) \neq \emptyset \right\}$$

is then defined as the set of all points whose images are nonempty.

In the following definitions, we recall several useful regularity notions for set-valued mappings, starting with the standard concepts of continuity and Lipschitz regularity.

Definition 2.7 (Continuity of set-valued maps). *A multifunction $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is said to be continuous at $x \in \text{dom}(\mathcal{F})$ if both the following conditions hold.*

(i) \mathcal{F} is lower-semicontinuous at x , i.e. for any $\varepsilon > 0$ and all $y \in \mathcal{F}(x)$, there exists $\delta > 0$ such that

$$\mathcal{F}(x') \cap \mathbb{B}_{\mathcal{Y}}(y, \varepsilon) \neq \emptyset,$$

for each $x' \in \mathbb{B}_{\mathcal{X}}(x, \delta)$.

(ii) \mathcal{F} is upper-semicontinuous at x , i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathcal{F}(x') \subset \mathbb{B}_{\mathcal{Y}}(\mathcal{F}(x), \varepsilon)$$

for each $x' \in \mathbb{B}_{\mathcal{X}}(x, \delta)$.

Definition 2.8 (Lipschitz regularity of set-valued maps). *A multifunction $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is Lipschitz continuous with constant $L > 0$ provided that*

$$\mathcal{F}(x') \subset \mathbb{B}_{\mathcal{Y}}(\mathcal{F}(x), L d_{\mathcal{X}}(x, x')) \quad (2.4)$$

for all $x, x' \in \text{dom}(\mathcal{F})$. In particular, \mathcal{F} is continuous at every point $x \in \text{dom}(\mathcal{F})$.

Below, we recall the standard concept of Lebesgue measurability for set-valued maps defined over subintervals of the real line.

Definition 2.9 (Measurable set-valued maps and selections). *A multifunction $\mathcal{F} : I \rightrightarrows \mathcal{X}$ valued in a complete separable metric space $(\mathcal{X}, d_{\mathcal{X}}(\cdot, \cdot))$ is \mathcal{L}^1 -measurable – or Lebesgue measurable – provided that for any given open set $\mathcal{O} \subset \mathcal{X}$, the preimage*

$$\mathcal{F}^{-1}(\mathcal{O}) := \left\{ t \in I \text{ s.t. } \mathcal{F}(t) \cap \mathcal{O} \neq \emptyset \right\}$$

is Lebesgue measurable. Moreover, a Lebesgue measurable map $f : I \rightarrow \mathcal{X}$ is called a measurable selection of $\mathcal{F} : I \rightrightarrows \mathcal{X}$ provided that $f(t) \in \mathcal{F}(t)$ for \mathcal{L}^1 -almost every $t \in I$.

As a sanity check, one may note that whenever $\mathcal{F}(t) = \{f(t)\}$ is single-valued, this definition boils down to requiring that $f : I \rightarrow \mathcal{X}$ be Lebesgue measurable in the sense introduced above, since then

$$\mathcal{F}^{-1}(\mathcal{O}) = \left\{ t \in I \text{ s.t. } \{f(t)\} \cap \mathcal{O} \neq \emptyset \right\} = f^{-1}(\mathcal{O}).$$

In the following theorem, we recall a variant of the instrumental selection principle of Filippov whose statement is borrowed from [7, Theorem 8.2.10]. Therein, we suppose that the metric spaces $(\mathcal{X}, d_{\mathcal{X}}(\cdot, \cdot))$ and $(\mathcal{Y}, d_{\mathcal{Y}}(\cdot, \cdot))$ are both complete and separable.

Theorem 2.10 (Filippov's measurable selection principle). *Suppose that $\mathcal{F} : I \rightrightarrows \mathcal{X}$ is an \mathcal{L}^1 -measurable set-valued map with nonempty closed images, and let $\Psi : I \times \mathcal{X} \rightarrow \mathcal{Y}$ be \mathcal{L}^1 -measurable in $t \in I$ and continuous in $x \in \mathcal{X}$. Then for every \mathcal{L}^1 -measurable map $\psi : I \rightarrow \mathcal{Y}$ satisfying*

$$\psi(t) \in \Psi(t, \mathcal{F}(t)) := \left\{ \Psi(t, f) \text{ s.t. } f \in \mathcal{F}(t) \right\},$$

there exists a measurable selection $t \in I \mapsto f(t) \in \mathcal{F}(t)$ such that $\psi(t) = \Psi(t, f(t))$ for \mathcal{L}^1 -almost every $t \in I$.

In what follows, we recollect the definitions of standard limit concepts for sequences of sets. Therein, we shall use the notation

$$\text{dist}_{\mathcal{X}}(x; \mathcal{Q}) := \inf_{y \in \mathcal{Q}} d_{\mathcal{X}}(x, y)$$

for the distance between a point $x \in \mathcal{X}$ and a closed set $\mathcal{Q} \subset \mathcal{X}$.

Definition 2.11 (Kuratowski-Painlevé limit of sequences of sets). *Given a sequence of sets (K_n) , we define its lower limit as the set*

$$\text{Liminf}_{n \rightarrow +\infty} K_n := \left\{ x \in \mathcal{X} \text{ s.t. } \lim_{n \rightarrow +\infty} \text{dist}_{\mathcal{X}}(x, K_n) = 0 \right\},$$

as well as its upper limit by

$$\text{Limsup}_{n \rightarrow +\infty} K_n := \left\{ x \in \mathcal{X} \text{ s.t. } \liminf_{n \rightarrow +\infty} \text{dist}_{\mathcal{X}}(x, K_n) = 0 \right\}.$$

In this context, we say that $K \subset \mathcal{X}$ is the Kuratowski-Painlevé limit of the sequence (K_n) provided

$$\text{Lim}_{n \rightarrow +\infty} K_n := \text{Liminf}_{n \rightarrow +\infty} K_n = \text{Limsup}_{n \rightarrow +\infty} K_n = K.$$



Figure 2.1: Illustration of the concepts of lower and upper limits for sequences of sets. The lower and upper limits of the sequence with general term $K_n := \{\frac{1}{n}\} \times [0, (-1)^n]$ (left) are respectively $\{(0, 0)\}$ and $\{0\} \times [-1, 1]$, whereas those of the sequence $K_n := \{\frac{1}{n}\} \times [-\frac{1}{n}, \frac{1}{n}]$ (right) coincide and are equal to the singleton $\{(0, 0)\}$.

For the next definition, we assume that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space, and recall two standard notions of tangent and polar cones to some arbitrary closed set.

Definition 2.12 (Intermediate tangent and normal cones). *The adjacent cone – or sometimes intermediate tangent cone – to a closed set $K \subset \mathcal{X}$ at some $x \in K$ is defined by*

$$T_K^b(x) := \operatorname{Liminf}_{h \rightarrow 0^+} \frac{K - x}{h} = \left\{ v \in \mathcal{X} \text{ s.t. } \lim_{h \rightarrow 0^+} \frac{1}{h} \operatorname{dist}_{\mathcal{X}}(x + hv; K) = 0 \right\}, \quad (2.5)$$

and the corresponding intermediate normal cone is given by the polarity relation

$$N_K^b(x) := \left\{ p \in \mathcal{X}^* \text{ s.t. } \langle p, v \rangle_{\mathcal{X}} \leq 0 \text{ for all } v \in T_K^b(x) \right\}$$

for each $x \in K$.

Remark 2.13 (On the notions of adjacent and polar cones). *It is worth noting that when $K \subset \mathcal{X}$ is a convex set, these objects coincide with the usual tangent and polar cones of convex analysis, see e.g. [7, Section 4.2].*



Figure 2.2: Illustration of the adjacent cone (blue) and normal cone (red) taken at some point on the boundary of a nonsmooth (left) or smooth (right) convex body.

In what follows, we recall the definitions of closed processes, convex processes and fans between normed spaces. These latter were introduced respectively by Rockafellar at the end of the 1960s (see e.g. [48, Chapter 9]) and Ioffe in the 1980, see for instance [32], to generalise the main properties of closed and linear operators to the set-valued setting.

Definition 2.14 (Closed processes, convex processes and fans). *A set-valued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is called a closed process if its graph $\operatorname{Graph}(\mathcal{F}) \subset \mathcal{X} \times \mathcal{Y}$ is a closed cone, and a convex process if its graph is a closed convex cone, which amounts to requiring that*

$$(1 - \lambda)\mathcal{F}(x_1) + \lambda\mathcal{F}(x_2) \subset \mathcal{F}((1 - \lambda)x_1 + \lambda x_2),$$

for each $x_1, x_2 \in \operatorname{dom}(\mathcal{F})$ and all $\lambda \in [0, 1]$. Similarly, a closed process $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ with convex images is called a fan provided that

$$\mathcal{F}((1 - \lambda)x_1 + \lambda x_2) \subset (1 - \lambda)\mathcal{F}(x_1) + \lambda\mathcal{F}(x_2)$$

for each $x_1, x_2 \in \operatorname{dom}(\mathcal{F})$ and all $\lambda \in [0, 1]$.

While the results of this paper rely only partially on these notions, we believe that mentioning them explicitly may pave the way for further developments in Koopman theory for control systems, owing to the large corpus of results concerned with these objects in the nonsmooth analysis literature.

Example 2.15 (Collections of linear operators as closed processes). *To see why these objects do generalise bounded linear operators, suppose that $\mathcal{F}(x) := \{Ax\}$ for each $x \in X$ and some bounded linear map $A : \mathcal{X} \rightarrow \mathcal{Y}$. Then, one may easily check that*

$$\text{Graph}(\mathcal{F}) = \left\{ (x, Ax) \text{ s.t. } x \in \mathcal{X} \right\} = \text{Graph}(A)$$

is an additive closed cone, which implies in particular that $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is both a convex process and a fan. In the case in which $A : \text{dom}(A) \rightarrow \mathcal{Y}$ is a linear operator defined over a dense domain $\text{dom}(A) \subset \mathcal{X}$, the mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is also convex process. Likewise, if \mathcal{A} is a convex collection of linear operators from \mathcal{X} to \mathcal{Y} , then the sets $\mathcal{F}(x) := \{Ax \text{ s.t. } A \in \mathcal{A}\} \subset \mathcal{X}$ define a fan.

2.3 Controlled dynamical systems

In what follows, we recollect elementary facts pertaining to controlled Cauchy problems of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(\tau) = x, \end{cases} \quad (2.6)$$

defined over some finite time interval $I := [0, T]$ with horizon $T > 0$, and taking a prescribed value $x \in \mathbb{R}^d$ at some time $\tau \in [0, T]$. Therein and in what follows, the set of admissible controls is given by

$$\mathcal{U} := \left\{ u : [0, T] \rightarrow U \text{ s.t. } u(\cdot) \text{ is } \mathcal{L}^1\text{-measurable} \right\}$$

for some compact metric space $(U, d_U(\cdot, \cdot))$, and the controlled vector field $f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ complies with the following set of assumptions.

Hypotheses (H).

(i) *The map $f : (x, u) \in \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ is continuous, and there exists a constant $m > 0$ such that*

$$|f(x, u)| \leq m(1 + |x|)$$

for all $(x, u) \in \mathbb{R}^d \times U$.

(ii) *For each compact set $K \subset \mathbb{R}^d$, there exists a constant $\ell_K > 0$ such that*

$$|f(x, u) - f(y, u)| \leq \ell_K |x - y|$$

for all $x, y \in K$ and each $u \in U$.

Remark 2.16 (Concerning Hypotheses (H)). *Let it be noted that since we restrict our attention to compact control sets, our assumptions encompass standard controlled linear systems of the form*

$$f(x, u) := Ax + Bu$$

where $U \subset \mathbb{R}^n$ is compact and $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times n}$ is a pair of matrices. We also stress that all our results could be extended to unbounded time intervals by adapting the relevant working assumptions.

Throughout our developments, we will often use the shorthand notation $f_u \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ for the controlled vector field $x \in \mathbb{R}^d \mapsto f(x, u) \in \mathbb{R}^d$ associated with some control $u \in U$, and denote by

$$\mathcal{F} := \left\{ f_u \in C^0(\mathbb{R}^d, \mathbb{R}^d) \text{ s.t. } u \in U \right\}$$

the set of all admissible fields of the control system. In the following proposition, we show that the latter is compact under our working assumptions, a fact that will prove very useful in the sequel.

Proposition 2.17 (Compactness of the velocity set). *Under Hypotheses **(H)**, the set $\mathcal{F} \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ is compact for the topology of local uniform convergence.*

Proof. Let $(f_n) \subset \mathcal{F}$ be a sequence of admissible velocities. By construction, there exists a corresponding sequence of controls $(u_n) \subset U$ for which $f_n = f_{u_n}$ for each $n \geq 1$. Since $(U, d_U(\cdot, \cdot))$ is a compact metric space by assumption, there exists an element $u \in U$ such that

$$d_U(u_n, u) \xrightarrow{n \rightarrow +\infty} 0$$

along a subsequence $(u_n) \subset U$ that we do not relabel. Thus, it follows from Hypothesis **(H)**-(i) that

$$|f_u(x) - f_{u_n}(x)| \xrightarrow{n \rightarrow +\infty} 0$$

for all $x \in \mathbb{R}^d$. Besides, it follows from Hypothesis **(H)**-(ii) that the maps $f_{u_n} : K \rightarrow \mathbb{R}^d$ are uniformly equicontinuous over each compact set $K \subset \mathbb{R}^d$, so that the convergence must be uniform, i.e.

$$\sup_{x \in K} |f_u(x) - f_{u_n}(x)| \xrightarrow{n \rightarrow +\infty} 0.$$

This latter identity being valid for each compact set, what we have shown is tantamount to

$$d_{cc}(f_{u_n}, f) \xrightarrow{n \rightarrow +\infty} 0,$$

which concludes the proof of our claim. \square

In the following theorem, we recall a standard Cauchy-Lipschitz well-posedness result for the controlled dynamics (2.6), along with elementary estimates satisfied by the underlying flow maps. We point the interested reader to [15, Chapter 2] for detailed proofs thereof.

Theorem 2.18 (Well-posedness, stability and representation of solutions to (2.6)). *Let $(\tau, x) \in [0, T] \times \mathbb{R}^d$ be given and suppose that Hypotheses **(H)** hold. Then for each $u(\cdot) \in \mathcal{U}$, the controlled dynamics (2.6) admits a unique solution $x_u(\cdot) \in \text{Lip}([0, T], \mathbb{R}^d)$ which can be represented explicitly as*

$$x_u(t) = \Phi_{(\tau, t)}^u(x) \tag{2.7}$$

for all times $t \in [0, T]$. Therein, the family of maps $(\Phi_{(\tau, t)}^u)_{\tau, t \in [0, T]} \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ are the flows of homeomorphisms defined as the unique solution of the Cauchy problem

$$\Phi_{(\tau, t)}^u(x) = x + \int_{\tau}^t f(\Phi_{(\tau, s)}^u(x), u(s)) ds \tag{2.8}$$

for all times $\tau, t \in [0, T]$ and each every $x \in \mathbb{R}^d$. In addition, for each $R > 0$, the flows satisfy the following stability bound

$$|\Phi_{(\tau, t)}^u(x)| \leq M_R \tag{2.9}$$

for all times $\tau, t \in [0, T]$ and every $x \in B(0, R)$, as well as the uniform Lipschitz estimate

$$|\Phi_{(\tau_1, t_1)}^u(x) - \Phi_{(\tau_2, t_2)}^u(y)| \leq L_R (|\tau_1 - \tau_2| + |t_1 - t_2| + |x - y|), \tag{2.10}$$

for all times $\tau_1, \tau_2, t_1, t_2 \in [0, T]$ and every $x, y \in B(0, R)$, where $M_R, L_R > 0$ are constants which only depend on the magnitudes of m, T and R .

We close this section by showing that flow solutions of (2.8) depend continuously on their control input for the strong topology of $L^1([0, T], U)$.

Corollary 2.19 (Continuous dependence in the control). *Under Hypotheses **(H)**, the functional*

$$u(\cdot) \in \mathcal{U} \mapsto \Phi_{(\cdot, \cdot)}^u(\cdot) \in C^0([0, T] \times [0, T] \times \mathbb{R}^d, \mathbb{R}^d)$$

which to a control signal associates the corresponding flow map is strongly continuous.

Proof. Being somewhat standard and technical, the proof of this result is deferred to Appendix A. \square

3 Set-valued Koopman operators

In this section, we define the *set-valued Koopman operators* associated with control systems, and discuss some of their fundamental properties. In what follows, we will carry out our analysis over the separable normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}) := (C_c^0(\mathbb{R}^d, \mathbb{C}), \|\cdot\|_{C^0(\mathbb{R}^d, \mathbb{C})})$ of continuous and complex-valued observables with compact support.

Definition 3.1 (Set-valued Koopman operators). *We define the set-valued Koopman operators associated with the control system (2.6) as*

$$\mathcal{K}_{(\tau,t)} : \varphi \in \mathcal{X} \rightrightarrows \left\{ \varphi \circ \Phi_{(\tau,t)}^u \text{ s.t. } u(\cdot) \in \mathcal{U} \right\} \subset \mathcal{X} \quad (3.1)$$

for all times $\tau, t \in [0, T]$.

As highlighted in the introduction, the action of the set-valued Koopman operator on an observable $\varphi \in \mathcal{X}$ can be thought of as the reachable set of observables starting from that observable, generated by all the admissible control trajectories. Before investigating the mathematical properties of these objects, a few remarks are in order. First, notice that when the dynamics is uncontrolled, e.g. when there is a unique flow $\{(\Phi_t)_{t \in [0, T]}\}$, then the mapping introduced in Definition 3.1 coincides with the classical Koopman operator, namely

$$\mathcal{K}_{(\tau,t)}(\varphi) = \{\varphi \circ \Phi_{t-\tau}\}.$$

Besides, it follows from (3.1) together with the fact that flows of nonautonomous dynamical systems form a semigroup for the composition operation that these latter satisfy the identity

$$\mathcal{K}_{(\tau,t)}(\varphi) = \mathcal{K}_{(s,t)} \circ \mathcal{K}_{(\tau,s)}(\varphi)$$

for all times $\tau, s, t \in [0, T]$, where we implicitly used the natural notation overloading

$$\mathcal{K}_{(\tau,t)}(\Phi) := \bigcup_{\varphi \in \Phi} \mathcal{K}_{(\tau,t)}(\varphi)$$

for some set $\Phi \subset \mathcal{X}$. Another direct observation that one can make is that $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ is not a convex process, but that it is almost a fan in the sense of Definition 2.14. To see why, observe that

$$(1 - \lambda)\mathcal{K}_{(\tau,t)}(\varphi_1) + \lambda\mathcal{K}_{(\tau,t)}(\varphi_2) = \left\{ (1 - \lambda)\varphi_1 \circ \Phi_{(\tau,t)}^{u_1} + \lambda\varphi_2 \circ \Phi_{(\tau,t)}^{u_2} \text{ s.t. } u_1(\cdot), u_2(\cdot) \in \mathcal{U} \right\}$$

is in general strictly larger than the set

$$\mathcal{K}_{(\tau,t)}((1 - \lambda)\varphi_1 + \lambda\varphi_2) = \left\{ ((1 - \lambda)\varphi_1 + \lambda\varphi_2) \circ \Phi_{(\tau,t)}^u \text{ s.t. } u(\cdot) \in \mathcal{U} \right\}$$

for each $\varphi_1, \varphi_2 \in \mathcal{X}$ and all $\lambda \in [0, 1]$. We shall see nonetheless that the set-valued Koopman operators enjoy very nice characterisations for their adjoints, under suitable assumptions on the underlying control system, and that they support a meaningful variant of the spectral mapping theorem.

Remark 3.2 (Set-valued Koopman operators for differential inclusions). *We would like to stress that while we define set-valued Koopman operators for controlled systems, most of our results remain valid for dynamical systems modelled by more general differential inclusions, that is*

$$\dot{x}(t) \in F(x(t)) \quad (3.2)$$

with $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ being a Lipschitz continuous set-valued mapping with nonempty compact images. In this case, one can simply define the set-valued Koopman operators as

$$(\mathcal{K}_{(\tau,t)}(\varphi))(x) := \left\{ \varphi(x(t)) \text{ s.t. } x(\cdot) \text{ solves (3.2) with } x(\tau) = x \in \mathbb{R}^d \right\}$$

for all times $t \in [0, T]$. In the situation where $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ has convex images – an assumption that is quite standard in control theory, see e.g. [7, 20, 54], and covered by Hypothesis (C) below –, the Lipschitz parametrisation theorem of [7, Theorem 9.7.1] yields the existence of a compact set $U \subset \mathbb{R}^d$ along with a Lipschitz map $f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ such that $F(x) = \{f(x, u) \text{ s.t. } u \in U\}$ for all $x \in \mathbb{R}^d$.

Remark 3.3 (Concerning time-varying systems). *We also mention that the majority of our developments would still hold – up to some technical adaptations – for time-varying vector fields $(t, x, u) \in [0, T] \times \mathbb{R}^d \times U \mapsto f(t, x, u) \in \mathbb{R}^d$ which are \mathcal{L}^1 -measurable in $t \in [0, T]$. That being said, we chose to stick to the simpler framework of time-invariant control systems for the sake of readability, and point the interested reader to [27] for some insight on how our results could be rephrased in this context.*

In the following proposition, we establish basic regularity results for the set-valued Koopman operators, akin to those already known in the context of classical dynamical systems.

Proposition 3.4 (Regularity properties of set-valued Koopman operators). *Under Hypotheses (H), the set-valued Koopman operator $\varphi \in \mathcal{X} \rightrightarrows \mathcal{K}_{(\tau,t)}(\varphi)$ is 1-Lipschitz for all times $\tau, t \in [0, T]$, and the set-valued map $(\tau, t) \in [0, T] \times [0, T] \rightrightarrows \mathcal{K}_{(\tau,t)}(\varphi)$ is strongly continuous for each $\varphi \in \mathcal{X}$.*

Proof. We start by studying the regularity of $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ for some given $\tau, t \in [0, T]$. Fix an arbitrary pair $\varphi_1, \varphi_2 \in \mathcal{X}$, and note that for every element $\psi_{(\tau,t)}^1 \in \mathcal{K}_{(\tau,t)}(\varphi_1)$, there exists some $u_1(\cdot) \in \mathcal{U}$ such that $\psi_{(\tau,t)}^1 = \varphi_1 \circ \Phi_{(\tau,t)}^{u_1}$. Then, recalling that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a vector space, one has that

$$\begin{aligned} \psi_{(\tau,t)}^1 &= \varphi_1 \circ \Phi_{(\tau,t)}^{u_1} \\ &= \varphi_2 \circ \Phi_{(\tau,t)}^{u_1} + (\varphi_1 - \varphi_2) \circ \Phi_{(\tau,t)}^{u_1} \\ &\in \mathcal{K}_{(\tau,t)}(\varphi_2) + \mathbb{B}_{\mathcal{X}}(\|\varphi_1 - \varphi_2\|_{\mathcal{X}}), \end{aligned}$$

which implies that $\mathcal{K}_{(\tau,t)}(\varphi_1) \subset \mathcal{K}_{(\tau,t)}(\varphi_2) + \mathbb{B}_{\mathcal{X}}(\|\varphi_1 - \varphi_2\|_{\mathcal{X}})$ and is tantamount to the 1-Lipschitzianity of $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ for all times $\tau, t \in [0, T]$.

Consider now an element $\varphi \in \mathcal{X}$, and note that owing to the uniform bound displayed in (2.9) of Theorem 2.18, the set

$$K_{\varphi} := \overline{\bigcup_{u(\cdot) \in \mathcal{U}} \bigcup_{\tau, t \in [0, T]} \Phi_{(t, \tau)}^u(\text{supp}(\varphi))} \subset \mathbb{R}^d$$

is well-defined and compact. Let $\tau_1, t_1 \in [0, T]$ be arbitrary, fix some $\psi_{(\tau_1, t_1)} \in \mathcal{K}_{(\tau_1, t_1)}(\varphi)$, and note that there exists a control $u_1(\cdot) \in \mathcal{U}$ such that $\psi_{(\tau_1, t_1)} = \varphi \circ \Phi_{(\tau_1, t_1)}^{u_1}$. It then follows from the flow regularity estimate (2.10) that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\sup_{x \in K_{\varphi}} |\Phi_{(\tau_1, t_1)}^{u_1}(x) - \Phi_{(\tau, t)}^u(x)| \leq \varepsilon$$

whenever $\tau, t \in [0, T]$ are such that $(|\tau_1 - \tau| + |t_1 - t|) \leq \delta$. Observing that $\varphi \in \mathcal{X}$ is uniformly continuous over its support while letting $\psi_{(\tau, t)} := \varphi \circ \Phi_{(\tau, t)}^{u_1} \in \mathcal{K}_{(\tau, t)}(\varphi)$, it then follows – up to potentially choosing a smaller $\delta > 0$ – that

$$\|\psi_{(\tau_1, t_1)} - \psi_{(\tau, t)}\|_{\mathcal{X}} \leq \varepsilon.$$

Thence, the set-valued map $(\tau, t) \in [0, T] \times [0, T] \rightrightarrows \mathcal{K}_{(\tau,t)}(\varphi) \subset \mathcal{X}$ is lower-semicontinuous at $(\tau_1, t_1) \in [0, T] \times [0, T]$ for every such pair, and thus over the whole time interval. It can be shown via very similar arguments that it is also upper-semicontinuous, which ends the proof of our claim. \square

In what follows, we build on the previous result to prove a simple, yet enlightening representation result for time-dependent Koopman observables.

Proposition 3.5 (Representation formula for curves of Koopman observables). *Given some observable $\varphi \in \mathcal{X}$, a function $(\tau, t) \in [0, T] \times [0, T] \rightarrow \psi_{(\tau,t)} \in \mathcal{X}$ is a measurable selection in the set-valued map $(\tau, t) \in [0, T] \times [0, T] \rightrightarrows \mathcal{K}_{(\tau,t)}(\varphi)$ if and only if there exists a jointly Lebesgue measurable collection of controls $\{u_{\tau,t}(\cdot)\}_{\tau,t \in [0, T]} \subset \mathcal{U}$ such that*

$$\psi_{(\tau,t)} = \varphi \circ \Phi_{(\tau,t)}^{u_{\tau,t}}$$

for \mathcal{L}^1 -almost every $\tau, t \in [0, T]$.

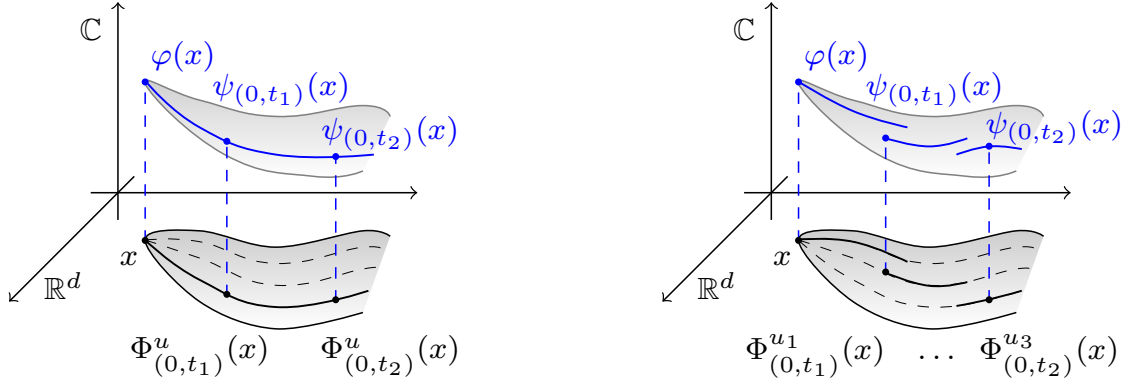


Figure 3.1: Representation of a particular curve of Koopman observables corresponding to measurements evaluated along a single controlled flow (left) and of a general curve of observables that may a priori jump in a measurable way between different controlled flows (right).

Proof of Proposition 3.5. Clearly, if $\psi_{(\tau,t)} = \varphi \circ \Phi_{(\tau,t)}^{u_{\tau,t}}$ for some jointly Lebesgue measurable family $\{u_{\tau,t}(\cdot)\}_{\tau,t \in [0,T]} \subset \mathcal{U}$, then $\psi_{(\tau,t)} \in \mathcal{K}_{(\tau,t)}(\varphi)$ for \mathcal{L}^1 -almost every $\tau, t \in [0, T]$, and the map $(\tau, t) \in [0, T] \times [0, T] \mapsto \psi_{(\tau,t)} \in \mathcal{X}$ is jointly Lebesgue measurable by the strong continuity of the mapping

$$(\tau, t, u(\cdot)) \in [0, T] \times [0, T] \times \mathcal{U} \mapsto \Phi_{(\tau,t)}^u \in C^0(\mathbb{R}^d, \mathbb{R}^d)$$

which stems from Theorem 2.18 and Corollary 2.19. Suppose now that $(\tau, t) \in [0, T] \times [0, T] \mapsto \psi_{(\tau,t)} \in \mathcal{K}_{(\tau,t)}(\varphi)$ is an arbitrary Lebesgue measurable selection in the set-valued Koopman operator, and notice that the latter can be expressed as

$$\mathcal{K}_{(\tau,t)}(\varphi) = \left\{ \Psi(\tau, t, u(\cdot)) \text{ s.t. } u(\cdot) \in \mathcal{U} \right\}$$

where $\Psi : (\tau, t, u(\cdot)) \in [0, T] \times [0, T] \times \mathcal{U} \mapsto \varphi \circ \Phi_{(\tau,t)}^u \in \mathcal{X}$ is jointly continuous, again as a consequence of Theorem 2.18 and Corollary 2.19. Because the completion of $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a separable Banach space, we may apply Filippov's measurable selection theorem twice to recover the existence of a jointly Lebesgue measurable map $(\tau, t) \in [0, T] \times [0, T] \mapsto u_{\tau,t}(\cdot) \in \mathcal{U}$ such that

$$\psi_{(\tau,t)} = \Psi(\tau, t, u_{\tau,t}(\cdot)) = \varphi \circ \Phi_{(\tau,t)}^{u_{\tau,t}}$$

for \mathcal{L}^1 -almost every $\tau, t \in [0, T]$, which closes the proof. \square

Remark 3.6 (On the meaning of the representation formula). *In essence, the previous result shows that while for a fixed pair $\tau, t \in [0, T]$, every element in $\mathcal{K}_{(\tau,t)}(\varphi)$ can be expressed using of a single controlled flow, a time-dependent family $(\tau, t) \in [0, T] \times [0, T] \mapsto \psi_{(\tau,t)} \in \mathcal{K}_{(\tau,t)}(\varphi)$ may jump arbitrarily between the realisations of different controlled dynamics from one instant of time to another, as illustrated by Figure 3.*

In what follows, many of our results will rely crucially on the assumption that the set of controlled velocities be convex, in a way which faithfully captures the fact that Koopman operators act globally – instead of pointwisely – on the state space. Quite surprisingly, this condition happens to be the same as that which appears to study mean-field control problems, see e.g. [11, 12, 13, 14] by the first author. This unexpected link shall become clearer in light of the duality results of Section 4.2.

Hypothesis (C). *The set of controlled vector fields $\mathcal{F} := \{f_u \in C^0(\mathbb{R}^d, \mathbb{R}^d) \text{ s.t. } u \in U\}$ is convex.*

A prototypical example of controlled velocity set satisfying Hypotheses (H) and (C) above is given by nonlinear control-affine systems of the form

$$f_u(x) := f_0(x) + \sum_{k=1}^n u_k f_k(x),$$

where $U \subset \mathbb{R}^n$ is convex and $f_0, \dots, f_n \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ are locally Lipschitz and sublinear vector fields. It should be stressed however that Hypothesis **(C)** is in general much stronger than the usual condition from control theory requiring that $\{f(x, u) \text{ s.t. } u \in U\} \subset \mathbb{R}^d$ be a convex set for all fixed $x \in \mathbb{R}^d$.

In the ensuing proposition, we establish a compactness result on the images of the Koopman operators, which stems from the (functional) convexity of the set of admissible velocities.

Proposition 3.7 (Topological properties of Koopman operators). *Suppose that Hypotheses **(H)** and **(C)** hold. Then, the sets $\mathcal{K}_{(\tau,t)}(\varphi) \subset \mathcal{X}$ are compact for all times $\tau, t \in [0, T]$ and each $\varphi \in \mathcal{X}$, and the operators $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ are closed processes.*

Proof. To show that the sets $\mathcal{K}_{(\tau,t)}(\varphi)$ are compact for a fixed $t \in [0, T]$ and $\varphi \in \mathcal{X}$, let $(\psi_n) \subset \mathcal{K}_{(\tau,t)}(\varphi)$ be given and $(u_n(\cdot)) \subset \mathcal{U}$ be the sequence of signals such that $\psi_n = \varphi \circ \Phi_{(\tau,t)}^{u_n}$ for each $n \geq 1$. We recall that by Theorem 2.18, there exist for each $R > 0$ a pair of positive constants $M_R, L_R > 0$ such that

$$|\Phi_{(\tau,t)}^{u_n}(x)| \leq M_R \quad \text{and} \quad |\Phi_{(\tau,t)}^{u_n}(x) - \Phi_{(\tau,t)}^{u_n}(y)| \leq L_R|x - y|, \quad (3.3)$$

for $x, y \in B(0, R)$ and each $n \geq 1$. Therefore, the sequence of maps $(\Phi_{(\tau,t)}^{u_n}) \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ is locally uniformly valued in a compact set and locally equi-Lipschitz, which by the Ascoli-Arzelà theorem (see e.g. [33, Chapter 7 Theorem 18]) yields the existence of an element $\Phi_{(\tau,t)} \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\|\Phi_{(\tau,t)} - \Phi_{(\tau,t)}^{u_{n_k}}\|_{C^0(K, \mathbb{R}^d)} \xrightarrow{k \rightarrow +\infty} 0 \quad (3.4)$$

for each compact set $K \subset \mathbb{R}^d$, along a subsequence $(u_{n_k}(\cdot)) \subset \mathcal{U}$. Observing now that the set

$$K_\varphi := \overline{\bigcup_{n \geq 1} \bigcup_{\tau, t \in [0, T]} \Phi_{(t,\tau)}^{u_n}(\text{supp}(\varphi))}$$

is well-defined and compact as a consequence of (3.3), it necessarily follows that

$$\|\varphi \circ \Phi_{(\tau,t)} - \psi_{n_k}\|_{\mathcal{X}} \xrightarrow{k \rightarrow +\infty} 0$$

along that same subsequence. Hence, to close the argument, there remains to show that

$$\Phi_{(\tau,t)} = \Phi_{(\tau,t)}^u \quad (3.5)$$

for some admissible control $u(\cdot) \in \mathcal{U}$. To this end, recall first that by Proposition 2.17, the set of admissible velocities $\mathcal{F} \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ is convex and compact for the topology of local uniform convergence. Thus, for each compact set $K \subset \mathbb{R}^d$, the collection of restricted velocities

$$\mathcal{F}_K := \left\{ f|_K \in C^0(K, \mathbb{R}^d) \text{ s.t. } f \in \mathcal{F} \right\}$$

is a compact and convex subset of the Banach space $C^0(K, \mathbb{R}^d)$. Therefore, using Hypothesis **(H)**-(i), we can inductively apply the compactness criterion of Proposition 2.3 on a countable and increasing family of balls covering \mathbb{R}^d together with a diagonal argument, to obtain the existence of an \mathcal{L}^1 -measurable map $t \in [0, T] \mapsto f(t) \in \mathcal{F}$ satisfying

$$\int_0^T \int_{\mathbb{R}^d} \zeta \cdot (f(t, x) - f_{u_{n_k}(t)}(x)) d\mu(t)(x) dt \xrightarrow{k \rightarrow +\infty} 0 \quad (3.6)$$

for every $\mu(\cdot) \in L^\infty([0, T], \mathcal{M}(K, \mathbb{R}))$, each $\zeta \in \mathbb{R}^d$ and every compact set $K \subset \mathbb{R}^d$. Therein, the convergence holds along a subsequence that may depend on the set $K \subset \mathbb{R}^d$ itself, and that we do not relabel. At this stage, we consider for any $(t, x) \in [0, T] \times \mathbb{R}^d$ the set defined by

$$K_x^t := \overline{\bigcup_{n \geq 1} \bigcup_{s \in [0, T]} \Phi_{(\tau,s)}^{u_n}(x)},$$

and note that it is compact by (3.3). Choosing now the particular measure-valued map defined by

$$\mu(s) := \mathbb{1}_{[0,t]}(s) \delta_{\Phi_{(\tau,s)}(x)} \in \mathcal{M}(K_x^t, \mathbb{R})$$

for \mathcal{L}^1 -almost every $s \in [0, T]$, it follows from (3.6) that

$$\int_0^t \zeta \cdot \left(f(s, \Phi_{(\tau,s)}(x)) - f_{u_{n_k}}(s)(\Phi_{(\tau,s)}(x)) \right) ds \xrightarrow[k \rightarrow +\infty]{} 0$$

for every $\zeta \in \mathbb{R}^d$, along a subsequence that may depend on $(t, x) \in [0, T] \times \mathbb{R}^d$. Besides, observing that

$$\int_{\tau}^t \left\| f_{u_{n_k}}(s)(\Phi_{(\tau,s)}) - f_{u_{n_k}}(s)(\Phi_{(\tau,s)}^{u_{n_k}}) \right\|_{C^0(K_x^t, \mathbb{R}^d)} ds \xrightarrow[k \rightarrow +\infty]{} 0$$

as a byproduct of (3.4) and Hypothesis (H)-(ii), we can conclude by merging the previous two equations that

$$\int_0^t f_{u_{n_k}}(s)(\Phi_{(\tau,s)}^{u_{n_k}}(x)) ds \xrightarrow[k \rightarrow +\infty]{} \int_0^t f(s, \Phi_{(\tau,s)}(x)) ds \quad (3.7)$$

along a subsequence that may depend on $(t, x) \in [0, T] \times \mathbb{R}^d$. Since the latter are arbitrary, we can deduce from (3.4) and (3.7) that

$$\Phi_{(\tau,t)}(x) = x + \int_0^t f(s, \Phi_{(\tau,s)}(x)) ds \quad (3.8)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Finally, noting that

$$f(t) \in \mathcal{F} := \left\{ f_u \in C^0(\mathbb{R}^d, \mathbb{R}^d) \text{ s.t. } u \in \mathcal{U} \right\}$$

for \mathcal{L}^1 -almost every $t \in [0, T]$ owing to what precedes, while observing that the map $u \in \mathcal{U} \mapsto f_u \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ is continuous under Hypotheses (H), one may apply Theorem 2.10 to infer the existence of some $u(\cdot) \in \mathcal{U}$ such that

$$f(t) = f_{u(t)}$$

for \mathcal{L}^1 -almost every $t \in [0, T]$. This, together with (3.8), is tantamount to (3.5) and thus concludes the proof of our first claim.

We now prove that $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ is a closed process for all times $\tau, t \in [0, T]$, namely that its graph is a closed cone. The latter of these properties is straightforward, since for each $\alpha > 0$ and every $(\varphi, \psi) \in \text{Graph}(\mathcal{K}_{(\tau,t)})$, one has that

$$\alpha\psi = \alpha\varphi \circ \Phi_{(\tau,t)}^u \in \mathcal{K}_{(\tau,t)}(\alpha\varphi)$$

where $u(\cdot) \in \mathcal{U}$. In order to prove that $\text{Graph}(\mathcal{K}_{(\tau,t)})$ is closed, we consider a sequence $((\varphi_n, \psi_n)) \subset \text{Graph}(\mathcal{K}_{(\tau,t)})$ such that

$$\|\varphi - \varphi_n\|_{\mathcal{X}} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \text{and} \quad \|\psi - \psi_n\|_{\mathcal{X}} \xrightarrow[n \rightarrow +\infty]{} 0, \quad (3.9)$$

and recall that by definition, there exists $(u_n(\cdot)) \subset \mathcal{U}$ so that

$$\psi_n = \varphi_n \circ \Phi_{(\tau,t)}^{u_n}$$

for each $n \geq 1$. Then, upon repeating the compactness argument detailed hereinabove combined with (3.9), it follows that $\psi \in \mathcal{K}_{(\tau,t)}(\varphi)$, which concludes the proof. \square

4 Set-valued Liouville and Perron-Frobenius operators

In this section, we define the set-valued counterparts of the Liouville and Perron-Frobenius operators, for which we provide several structure results and explicit characterisations. Throughout what follows, we denote by $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) := (C_c^1(\mathbb{R}^d, \mathbb{C}), \|\cdot\|_{C^1})$ the separable normed space of continuously differentiable complex-valued functions with compact support.

4.1 Set-valued Liouville operators and Koopman dynamics

In what ensues, we define the set-valued Liouville operator as the collection of all Liouville operators associated with each individual controls, and show that the latter can be rigorously related to the infinitesimal behaviour of the set-valued Koopman semigroup $(\mathcal{K}_{(\tau,t)})_{\tau,t \in [0,T]}$ at time $t = \tau$. These results provide theoretical underpinning to the works defining the Koopman and Liouville operators for control systems as the collections of operators associated to each control element, see e.g. [45, 46].

Definition 4.1 (Set-valued Liouville operators). *We define the set-valued Liouville operator $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ associated with (2.6) as*

$$\mathcal{L}(\varphi) := \left\{ \nabla_x \varphi \cdot f_u \text{ s.t. } u \in U \right\} \subset \mathcal{X} \quad (4.1)$$

for each $\varphi \in \mathcal{D}$.

We start by establishing some simple topological properties on these Liouville operator. These results will prove insightful when studying its point spectrum further down in Section 4.3.

Proposition 4.2 (Topological properties of the Liouville operator). *Suppose that Hypotheses (H) and (C) hold. Then, the set-valued Liouville operator $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ is a fan with compact images.*

Proof. The fact that $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ has convex and compact images is a direct consequence of Hypothesis (C) on the one hand, and of Hypotheses (H) together with Proposition 2.17 on the other hand. Besides, it follows from the very definition of these set-valued mappings that

$$\begin{aligned} \mathcal{L}((1-\lambda)\varphi_1 + \lambda\varphi_2) &= \left\{ ((1-\lambda)\nabla_x \varphi_1 + \lambda\nabla_x \varphi_2) \cdot f_u \text{ s.t. } u \in U \right\} \\ &\subset \left\{ (1-\lambda)\nabla_x \varphi_1 \cdot f_{u_1} + \lambda\nabla_x \varphi_2 \cdot f_{u_2} \text{ s.t. } u_1, u_2 \in U \right\} = (1-\lambda)\mathcal{L}(\varphi_1) + \lambda\mathcal{L}(\varphi_2) \end{aligned}$$

for each $\varphi_1, \varphi_2 \in \mathcal{D}$ and all $\lambda \in [0, 1]$. Hence, there simply remains to show that $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ is a closed process, namely that its graph is a closed cone. The latter of these properties is straightforward, as for each $\alpha > 0$ and every $(\varphi, \psi) \in \text{Graph}(\mathcal{L})$, it clearly holds that

$$\alpha\psi = \alpha\nabla_x \varphi \cdot f_u \in \mathcal{L}(\alpha\varphi)$$

where $u \in U$ is some fixed control value. Suppose now that we are given a sequence $((\varphi_n, \psi_n)) \subset \text{Graph}(\mathcal{L})$ such that

$$\|\varphi - \varphi_n\|_{\mathcal{D}} \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \|\psi - \psi_n\|_{\mathcal{X}} \xrightarrow{n \rightarrow +\infty} 0, \quad (4.2)$$

for some $(\varphi, \psi) \in \mathcal{X} \times \mathcal{D}$. This fact together with the characterisation (4.1) of the Liouville operator yield the existence of a sequence $(u_n) \subset U$ for which

$$\psi_n = \nabla_x \varphi_n \cdot f_{u_n}$$

for all $n \geq 1$. The conclusion simply follows then from the compactness of the set of admissible velocities established in Proposition 2.17. \square

In the following theorem, we establish a result inspired by [28, Section 2] which shows a connection between the Liouville operator and set-valued derivatives of the Koopman semigroup at time $t = \tau$.

Theorem 4.3 (Set-valued Liouville operators as generators of the Koopman semigroups). *Suppose that Hypotheses (H) hold and fix a time $\tau \in [0, T]$. Then, the following inclusions*

$$\mathcal{L}(\varphi) \subset \text{Liminf}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau,t)}(\varphi) - \varphi}{t - \tau} \quad (4.3)$$

and

$$\text{Limsup}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau,t)}(\varphi) - \varphi}{t - \tau} \subset \overline{\text{co}} \mathcal{L}(\varphi) \quad (4.4)$$

are satisfied for each $\varphi \in \mathcal{D}$. Furthermore, if Hypothesis **(C)** holds, then the set-valued Liouville operator $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ is the infinitesimal generator of the Koopman semigroup, in the sense that

$$\mathcal{L}(\varphi) = \text{Lim}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau,t)}(\varphi) - \varphi}{t - \tau} \quad (4.5)$$

for each $\varphi \in \mathcal{D}$.

Proof. It is quite clear from the definition of lower and upper limits of sequences of sets provided in Section 2.2 above that

$$\text{Liminf}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau,t)}(\varphi) - \varphi}{t - \tau} \subset \text{Limsup}_{t \rightarrow \tau} \frac{\mathcal{K}_{(\tau,t)}(\varphi) - \varphi}{t - \tau}$$

for each $\varphi \in \mathcal{D}$. Remarking thus that under Hypothesis **(C)**, it holds that

$$\overline{\text{co}} \mathcal{L}(\varphi) = \overline{\text{co}} \left\{ \nabla_x \varphi \cdot f \text{ s.t. } f \in \mathcal{F} \right\} = \left\{ \nabla_x \varphi \cdot f \text{ s.t. } f \in \overline{\text{co}} \mathcal{F} \right\} = \mathcal{L}(\varphi),$$

it would directly follow from (4.3) and (4.4) that the set-valued Liouville operator coincides with the derivative displayed in (4.5).

We start by establishing the liminf inclusion (4.3), which amounts to finding for each element $\bar{u} \in U$ a curve of Koopman observables $t \in [0, T] \mapsto \psi_{(\tau,t)} \in \mathcal{K}_{(\tau,t)}(\varphi)$ such that

$$\lim_{t \rightarrow \tau} \frac{\psi_{(\tau,t)} - \varphi}{t - \tau} = \nabla_x \varphi \cdot f_{\bar{u}}. \quad (4.6)$$

To this end, consider the constant control signal given by $u(t) := \bar{u}$ for all times $t \in [0, T]$, and let $\psi_{(\tau,t)} := \varphi \in \Phi_{(\tau,t)}^u$. Observe then that by construction, one has that

$$\begin{aligned} \Phi_{(\tau,t)}^u(x) &= x + \int_{\tau}^t f\left(\Phi_{(\tau,s)}^u(x), \bar{u}\right) ds \\ &= x + (t - \tau) f_{\bar{u}}(x) + \int_{\tau}^t \left(f_{\bar{u}}(\Phi_{(\tau,s)}^u(x)) - f_{\bar{u}}(x) \right) ds, \end{aligned}$$

for all $x \in \mathbb{R}^d$. By combining Hypothesis **(H)**-(ii) and the estimate of (2.10) in Theorem 2.18 within the latter identity, one further obtains that

$$\sup_{x \in K_{\varphi}} \left| \Phi_{(\tau,t)}^u(x) - x - (t - \tau) f_{\bar{u}}(x) \right| \leq C_{\varphi} |t - \tau|^2 \quad (4.7)$$

for all times $t \in [0, T]$, where $C_{\varphi} > 0$ is a constant and $K_{\varphi} \subset \mathbb{R}^d$ is the compact set defined by

$$K_{\varphi} := \overline{\bigcup_{u(\cdot) \in \mathcal{U}} \bigcup_{t \in [0, T]} \Phi_{(t,\tau)}^u(\text{supp}(\varphi))}, \quad (4.8)$$

whose existence is guaranteed by (2.9). Thence, recalling that $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{R})$, it follows from (4.7) combined with (4.8) that

$$\lim_{t \rightarrow \tau} \left\| \frac{\varphi \circ \Phi_{(\tau,t)}^u - \varphi - (t - \tau) \nabla_x \varphi \cdot f_{\bar{u}}}{t - \tau} \right\|_{\mathcal{X}} = 0,$$

which then yields (4.6) by what precedes, since $\bar{u} \in U$ is arbitrary.

To complete the proof, there remains to derive the limsup inclusion (4.4), which can be equivalently recast as the requirement that

$$\lim_{t_n \rightarrow \tau} \frac{\psi_{(\tau,t_n)} - \varphi}{t_n - \tau} \in \overline{\text{co}} \mathcal{L}(\varphi) \quad (4.9)$$

for every pair of sequences $(t_n) \subset \mathbb{R}_+$ and $(\psi_{(\tau,t_n)}) \subset \mathcal{X}$ such that $\psi_{(\tau,t_n)} \in \mathcal{K}_{t_n}(\varphi)$ for each $n \geq 1$, and along which the aforewritten limit exists. Fixing such a pair, it follows from Proposition 3.5 that

$$\psi_{(\tau,t_n)} = \varphi \circ \Phi_{(\tau,t_n)}^{u_n}$$

for each $n \geq 1$ and some sequence $u_n(\cdot) \in \mathcal{U}$. Then, by definition of the characteristic flows

$$\begin{aligned}\Phi_{(\tau, t_n)}^{u_n}(x) &= x + \int_{\tau}^{t_n} f\left(\Phi_{(\tau, s)}^{u_n}(x), u_n(s)\right) ds \\ &= x + (t_n - \tau) \left(\frac{1}{t_n - \tau} \int_{\tau}^{t_n} f(x, u_n(s)) ds \right) + \int_{\tau}^{t_n} \left(f\left(\Phi_{(\tau, s)}^{u_n}(x), u_n(s)\right) - f(x, u_n(s)) \right) ds\end{aligned}\tag{4.10}$$

for each $n \geq 1$. Denoting by $K_{\varphi} \subset \mathbb{R}^d$ the compact set associated with $\varphi \in \mathcal{X}$ via (4.8), it holds that

$$\sup_{x \in K_{\varphi}} \left| \int_{\tau}^{t_n} \left(f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) - f(x, u_n(s)) \right) ds \right| \leq C_{\varphi} |t_n - \tau|^2,\tag{4.11}$$

for some constant $C_{\varphi} > 0$ and each $n \geq 1$. Besides, regarding for any given $n \geq 1$ the map $t \in [0, T] \mapsto f_{u_n(t)} \in C^0(K_{\varphi}, \mathbb{R}^d)$ as an elements of the space of Bochner integrable maps $L^1([0, T], C^0(K_{\varphi}, \mathbb{R}^d))$, it follows from Proposition 2.2 that

$$\frac{1}{t_n - \tau} \int_{\tau}^{t_n} f_{u_n(s)} ds \in \overline{\text{co}} \mathcal{F}$$

for each $n \geq 1$. Under Hypotheses **(H)**, it can be checked that the mappings

$$F_n : x \in K_{\varphi} \mapsto \frac{1}{t_n - \tau} \int_{\tau}^{t_n} f_{u_n(s)}(x) ds \in \mathbb{R}^d$$

form a uniformly bounded and equi-Lipschitz family in $C^0(K_{\varphi}, \mathbb{R}^d)$, which by the standard Ascoli-Arzelà theorem (see e.g. [49, Theorem 11.28]) combined with the fact that $\overline{\text{co}} \mathcal{F} \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ is closed as a consequence of Proposition 2.17, implies that there exists an element $F \in \overline{\text{co}} \mathcal{F}$ such that

$$\lim_{k \rightarrow +\infty} \|F - F_{n_k}\|_{C^0(K_{\varphi}, \mathbb{R}^d)} = 0\tag{4.12}$$

along a vanishing subsequence $t_{n_k} \rightarrow \tau$. Owing to the fact that $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{C})$, it then follows from (4.10), (4.11) and (4.12) that

$$\psi_{(\tau, t_{n_k})} = \varphi \circ \Phi_{(\tau, t_{n_k})}^{u_{n_k}} = \varphi + (t_{n_k} - \tau) \nabla_x \varphi \cdot F + o(|t_{n_k} - \tau|)$$

as $t_{n_k} \rightarrow \tau$. Since we assumed that the limit in (4.9) exists, this last identity ultimately implies that

$$\lim_{k \rightarrow +\infty} \frac{\psi_{(\tau, t_{n_k})} - \varphi}{t_{n_k} - \tau} = \nabla_x \varphi \cdot F \in \overline{\text{co}} \mathcal{L}(\varphi)$$

which closes the proof of Theorem 4.3. \square

Remark 4.4 (On the relation between the set-valued Koopman and Liouville operators). *The previous theorem provides us with two insights on the structure of the Koopman and Liouville operators for control systems. Firstly, the identities (4.3) and (4.4) show that, even when the dynamics is not convex, the infinitesimal behaviour of the set-valued Koopman operators is captured by the natural set-valued counterpart of the Liouville operators introduced in Definition 4.1. In Theorem 4.17 below, we shall see how this particular fact allows for the derivation of a set-valued spectral mapping theorem, that can in particular be applied to finite control sets which are inherently non-convex.*

*Secondly, the identity (4.5) means in turn that, when the admissible velocities are convex in the sense of Hypothesis **(C)**, the set-valued Liouville operator is the infinitesimal generator of the Koopman operator. This transcribes the fact that even though time-dependent Koopman observables cannot be represented in general by a single controlled flow – see Proposition 3.5 and Figure 3.1 above –, the set of all possible infinitesimal evolutions at time $t = \tau$ coincides exactly with those generated by a controlled vector field $f_u \in \mathcal{F}$ with $u \in U$.*

Remark 4.5 (On set-valued derivatives and Liouville operators). *The set-theoretic limit of difference quotient appearing in (4.5) is usually called adjacent or intermediate derivative in the literature, see e.g. [7, Section 5.2]. This denomination comes from the fact that its graph is the adjacent cone (see Definition 2.12 above) to that of the set-valued mapping $t \in [0, T] \rightrightarrows \mathcal{K}_{(\tau, t)}(\varphi) \in \mathcal{X}$ at $t = \tau$. While one could alternatively define $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ by means of other set-valued derivatives, involving e.g. contingent or Clarke cones [7, Section 5.1 and 5.2], adjacent derivatives are known to naturally appear when investigating the infinitesimal behaviour of reachable sets, see for instance [26, 28].*

In the next proposition, we close this section by showing how the set-valued Liouville operator encodes the dynamics of those time-dependent Koopman observables which are generated by precisely one admissible control signal, and thus follow an admissible trajectory of the system.

Proposition 4.6 (Dynamics of time-dependent Koopman observables). *Fix some $\varphi \in \mathcal{D}$, suppose that Hypotheses (H) hold and assume that $f_u \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ for each $u \in U$. Then, a family of Koopman observables $(\tau, t) \in [0, T] \times [0, T] \mapsto \varphi_{(\tau, t)} \in \mathcal{K}_{(\tau, t)}(\varphi)$ is of the form $\varphi_{(\tau, t)} = \varphi \circ \Phi_{(\tau, t)}^u$ for some $u(\cdot) \in \mathcal{U}$ if and only if it is a strong solution of the differential inclusion*

$$\begin{cases} \partial_\tau \varphi_{(\tau, t)} \in -\mathcal{L}(\varphi_{(\tau, t)}), \\ \varphi_{(t, t)} = \varphi, \end{cases} \quad (4.13)$$

in the space of observables $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$.

Proof. We start by assuming that $\varphi_{(\tau, t)} = \varphi \circ \Phi_{(\tau, t)}^u \in \mathcal{D}$ for some $u(\cdot) \in \mathcal{U}$, and note that by classical results from nonautonomous dynamical systems theory (see e.g. [15, Theorem 2.3.3]), the strong limit

$$\partial_\tau \varphi_{(\tau, t)} := \lim_{h \rightarrow 0} \frac{\varphi_{(\tau+h, t)} - \varphi_{(\tau, t)}}{h} \quad (4.14)$$

is well-defined in \mathcal{X} for every $(\tau, t) \in [0, T] \times \mathbb{R}^d$. Besides, one can easily check that for all $(t, y) \in \mathbb{R}^d$, the map $\tau \in [0, T] \mapsto \varphi_{(\tau, t)} \circ \Phi_{(t, \tau)}^u(y)$ is constant, so that

$$\begin{aligned} \frac{d}{d\tau} \varphi_{(\tau, t)} \circ \Phi_{(t, \tau)}^u(y) &= \partial_\tau \varphi_{(\tau, t)} \circ \Phi_{(t, \tau)}^u(y) + \nabla_x \varphi_{(\tau, t)} \circ \Phi_{(t, \tau)}^u \cdot \partial_\tau \Phi_{(t, \tau)}^u(y) \\ &= \left(\partial_\tau \varphi_{(\tau, t)} + \nabla_x \varphi_{(\tau, t)} \cdot f_{u(\tau)} \right) \circ \Phi_{(t, \tau)}^u(y) \\ &= 0 \end{aligned}$$

for \mathcal{L}^1 -almost every $\tau \in [0, T]$. Noting that in the previous identity, one may choose the particular point $y = \Phi_{(\tau, t)}^u(x)$ for some arbitrary $x \in \mathbb{R}^d$, it then follows that $(\tau, x) \in [0, T] \times \mathbb{R}^d \mapsto \varphi_{(\tau, t)}(x) \in \mathbb{R}^d$ is the unique classical solution of the Cauchy problem

$$\begin{cases} \partial_\tau \varphi_{(\tau, t)} + \nabla_x \varphi_{(\tau, t)} \cdot f_{u(\tau)} = 0, \\ \varphi_{(t, t)} = \varphi, \end{cases} \quad (4.15)$$

in $[0, T] \times \mathbb{R}^d$. At this stage, there remains to notice that the map

$$(\tau, x) \in [0, T] \times \mathbb{R}^d \mapsto \nabla_x \varphi_{(\tau, t)}(x) \cdot f_{u(\tau)}(x) \in \mathbb{R}^d$$

is \mathcal{L}^1 -measurable in $\tau \in [0, T]$ as well as continuous in $x \in \mathbb{R}^d$, which implies in particular that its functional lift $\tau \in [0, T] \mapsto \nabla_x \varphi_{(\tau, t)} \cdot f_{u(\tau)} \in \mathcal{X}$ is \mathcal{L}^1 -measurable, see e.g. [44, Page 511]. This latter fact together with (4.14) and (4.15) allows us to conclude that $\tau \in [0, T] \mapsto \varphi_{(\tau, t)} \in \mathcal{D}$ solves (4.13).

To prove the converse implication, observe that one may rewrite the evaluation of the Liouville operator along the curve $\tau \in [0, T] \mapsto \varphi_{(\tau, t)} \in \mathcal{D}$ as

$$\mathcal{L}(\varphi_{(\tau, t)}) = \left\{ \Psi_t(\tau, u) \text{ s.t. } u \in U \right\}$$

wherein $\Psi_t : (\tau, u) \in [0, T] \times U \mapsto \nabla_x \varphi_{(\tau, t)} \cdot f_u \in \mathcal{X}$. It is clear from Hypothesis (H) that the function $u \in U \mapsto \Psi_t(\tau, u) \in \mathcal{X}$ is continuous for \mathcal{L}^1 -almost every $\tau \in [0, T]$, and we just showed

$\tau \in [0, T] \mapsto \Psi_t(\tau, u) \in \mathcal{X}$ is \mathcal{L}^1 -measurable. Thus, because $\tau \in [0, T] \mapsto \partial_\tau \varphi_{(\tau, t)} \in \mathcal{X}$ is an \mathcal{L}^1 -measurable map and since the completion of $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a separable Banach space, it follows from the measurable selection principle of Theorem 2.10 that

$$-\partial_\tau \varphi_{(\tau, t)} = \Psi_t(u_\varphi(\tau)) = \nabla_x \varphi_{(\tau, t)} \cdot f_{u_\varphi(\tau)} \quad (4.16)$$

for \mathcal{L}^1 -almost every $\tau \in [0, T]$ and some admissible control signal $u_\varphi(\cdot) \in \mathcal{U}$. Owing to the regularity assumptions that we posited, it follows from classical well-posedness results for transport equations (see e.g. [1, Proposition 2.3]) that

$$\varphi_{(\tau, t)}(x) = \varphi \circ \Phi_{(\tau, t)}^{u_\varphi}(x)$$

for all $(\tau, x) \in [0, T] \times \mathbb{R}^d$, which concludes the proof of our claim. \square

Remark 4.7 (Comparing Proposition 4.6 with its counterpart in classical Koopman theory). *In the familiar situation in which $(\mathcal{K}_t(\varphi))_{t \in [0, T]}$ is the usual Koopman semigroup generated by a single family of flows $(\Phi_t)_{t \in [0, T]} \subset C^0(\mathbb{R}^d, \mathbb{R}^d)$ corresponding to a well-posedness differential equation whose right-hand side is a continuously differentiable and sublinear vector field $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, namely*

$$\Phi_t(x) = x + \int_0^t f(\Phi_{(\tau, s)}(x)) ds \quad (4.17)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$, it is well-known (see e.g. [39, Section 7.6]) that the curve of observables $t \in [0, T] \mapsto \varphi_t := \mathcal{K}_t(\varphi) \in \mathcal{X}$ is the unique strong solution of the Koopman dynamics

$$\begin{cases} \partial_t \varphi_t = \nabla_x \varphi_t \cdot f, \\ \varphi_0 = \varphi. \end{cases} \quad (4.18)$$

In the previous expression, the right-hand side coincides with the evaluation of the classical Liouville operator $\mathcal{L} : \varphi \in \mathcal{D} \mapsto \nabla_x \varphi \cdot f \in \mathcal{X}$ along $(\varphi_t)_{t \in [0, T]}$.

It is worth noting that this result is essentially contained in Proposition 4.6. Indeed, in the autonomous case, the Koopman observables take the simpler form $\varphi_{(\tau, t)} = \varphi \circ \Phi_{t-\tau}$, and in particular

$$\partial_\tau \varphi_{(\tau, t)} = -\partial_t \varphi_{(\tau, t)} \in -\mathcal{L}(\varphi_{(\tau, t)}) = \{-\nabla_x \varphi_{(\tau, t)} \cdot f\}.$$

Thence, up to potentially redefining the time variable, the set-valued dynamics presented in (4.13) becomes equivalent to that of (4.18).

Remark 4.8 (Pointwise versus functional Koopman dynamics). *In Proposition 4.6, we proved that when the controlled vector field is sufficiently smooth, the set-valued Liouville operators encode the dynamics of Koopman observables in a functional sense, as they generate solution curves to a differential inclusion in the space of observables. However, much like what is exposed in [39, Remark 7.6.1], these operators could also be used to describe the pointwise evolution of Koopman observables. Indeed, under Hypotheses (H), it follows from the usual characteristic formula along with Rademacher's theorem (see e.g. [2, Theorem 2.14]) that $(\tau, x) \in [0, T] \times \mathbb{R}^d \mapsto \varphi_{(\tau, t)}(x) := \varphi \circ \Phi_{(\tau, t)}^u(x) \in \mathbb{R}^d$ satisfies*

$$\partial_\tau \varphi_{(\tau, t)}(x) + \nabla_x \varphi_{(\tau, t)}(x) \cdot f_{u(\tau)}(x) = 0$$

for \mathcal{L}^1 -almost every $\tau, t \in [0, T]$ and \mathcal{L}^d -almost every $x \in \mathbb{R}^d$. This can informally be thought of as a pointwise inclusion of the form $\partial_\tau \varphi_{(\tau, t)}(x) \in -\mathcal{L}(\varphi_{(\tau, t)})(x)$ for \mathcal{L}^d -almost every $x \in \mathbb{R}^d$, and should in all likelihood correspond to a weak version of the functional dynamics (4.13).

4.2 Set-valued Perron-Frobenius operators and adjoint Koopman dynamics

In this section, we propose a formal definition for the set-valued Perron-Frobenius operator, and precisely discuss its relation with the Koopman operators as well as that of its infinitesimal generator with the Liouville operator.

Definition 4.9 (Set-valued Perron-Frobenius operators). *We define the set-valued Perron-Frobenius operators $\mathcal{P}_{(\tau,t)} : \mathcal{X}^* \rightrightarrows \mathcal{X}^*$ by*

$$\mathcal{P}_{(\tau,t)}(\mu) = \left\{ \Phi_{(\tau,t)\sharp}^u \mu \text{ s.t. } u(\cdot) \in \mathcal{U} \right\} \subset \mathcal{X}^* \quad (4.19)$$

for all times $\tau, t \in [0, T]$ and each $\mu \in \mathcal{X}^*$.

In what follows, we provide mathematical grounding for this definition by showing that the set-valued Perron-Frobenius operators are the adjoints of the Koopman operators in a suitable sense. As already highlighted in Section 3 above, the Koopman and Liouville operators are not convex processes, and thus not amenable to the application of the classical notion of duality put forth by Rockafellar and presented e.g. in [6, Chapter 2] and [48, Chapter 9]. Nevertheless, we are able to establish meaningful duality characterisations for these processes by leveraging the following definition, the formulation of which was inspired by the notion of adjoint dynamics for differential inclusions investigated in [26], see also the surveys [6, Chapter 7] and [54, Chapter 7]. It should also be noted that our notion of duality for non-convex processes coincides with that which was introduced to study fans, see e.g. [32], as soon as the latter makes sense.

Definition 4.10 (Local and full adjoints of closed processes). *Given two complete separable Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, we define the local adjoint of a closed process $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ at some point $(x, y) \in \text{Graph}(\mathcal{F})$ by*

$$\mathcal{F}^*(x, y)(\mu) := \left\{ \nu \in \mathcal{X}^* \text{ s.t. } (\nu, -\mu) \in N_{\text{Graph}(\mathcal{F})}^b(x, y) \right\}, \quad (4.20)$$

for all $\mu \in \mathcal{Y}^*$, and subsequently define its full full adjoint by

$$\mathcal{F}^*(\mu) := \left\{ \nu \in \mathcal{X}^* \text{ s.t. for all } x \in \text{dom}(\mathcal{F}) \text{ there exists } y \in \mathcal{F}(x) \text{ for which } \nu \in \mathcal{F}^*(x, y)(\mu) \right\}$$

for all $\mu \in \mathcal{Y}^*$.

Example 4.11 (Full adjoint of families of linear operators seen as closed processes). *Let us go back to the familiar situation in which $\mathcal{F}(x) := \{Ax\}$. Then, one may verify that for each $x \in \mathcal{X}$, it holds*

$$T_{\text{Graph}(\mathcal{F})}^b(x, Ax) = \text{Graph}(A) \quad \text{and} \quad N_{\text{Graph}(\mathcal{F})}^b(x, Ax) = \text{Graph}(A)^\perp.$$

Thus, given some $\mu \in \mathcal{Y}^*$, one has that

$$\begin{aligned} \mathcal{F}^*(x, Ax)(\mu) &= \left\{ \nu \in \mathcal{X}^* \text{ s.t. } (\nu, -\mu) \in \text{Graph}(A)^\perp \right\} \\ &= \left\{ \nu \in \mathcal{X}^* \text{ s.t. } \langle \nu, x \rangle_{\mathcal{X}} - \langle \mu, Ax \rangle_{\mathcal{Y}} = 0 \text{ for all } x \in \mathcal{X} \right\} = \{A^*\mu\}, \end{aligned}$$

from whence it readily follows $\mathcal{F}^*(\mu) = \{A^*\mu\}$ for each $\mu \in \mathcal{Y}^*$. Similarly, if one posits that $\mathcal{F}(x) := \{Ax \text{ s.t. } A \in \mathcal{A}\}$ for some collection of linear operators \mathcal{A} from \mathcal{X} to \mathcal{Y} , it can be easily verified from the above definition and by repeating the same reasoning that $\mathcal{F}^*(\mu) = \{A^*\mu \text{ s.t. } A \in \mathcal{A}\} \subset \mathcal{X}^*$.

In the following theorem, we provide explicit characterisations of the local and global adjoints of the set-valued Liouville and Koopman operators.

Theorem 4.12 (Characterisation of the adjoints of the Liouville and Koopman operators). *Suppose that Hypotheses (H) hold. Then, for each $(\varphi, \psi) \in \text{Graph}(\mathcal{L})$, one has that*

$$\mathcal{L}^*(\varphi, \psi)(\mu) = \left\{ -\text{div}_x(f_u \mu) \text{ s.t. } \psi = \nabla_x \varphi \cdot f_u \text{ with } u \in \mathcal{U} \right\} \quad (4.21)$$

for every $\mu \in \mathcal{X}^*$. Similarly, for all times $t \in [0, T]$ and each $(\varphi, \psi_{(\tau,t)}) \in \text{Graph}(\mathcal{K}_{(\tau,t)})$, it holds that

$$\mathcal{K}_{(\tau,t)}^*(\varphi, \psi_{(\tau,t)})(\mu) = \left\{ \Phi_{(0,t)\sharp}^u \mu \text{ s.t. } \psi_{(\tau,t)} = \varphi \circ \Phi_{(\tau,t)}^u \text{ with } u(\cdot) \in \mathcal{U} \right\} \quad (4.22)$$

for all $\mu \in \mathcal{X}^*$. In particular, the global adjoints of the set-valued Koopman and Liouville operators are given explicitly by

$$\mathcal{L}^*(\mu) = \left\{ -\operatorname{div}_x(f_u \mu) \text{ s.t. } u \in U \right\} \quad \text{and} \quad \mathcal{K}_{(\tau,t)}^*(\mu) = \left\{ \Phi_{(\tau,t)\sharp}^u \mu \text{ s.t. } u(\cdot) \in \mathcal{U} \right\}$$

for all times $\tau, t \in [0, T]$ and each $\mu \in \mathcal{X}^*$. In particular, the set-valued Perron-Frobenius operators are the adjoints of the set-valued Koopman operators.

Proof. In order to derive the expression (4.21) for the local adjoint of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ at some $(\varphi, \psi) \in \operatorname{Graph}(\mathcal{L})$, we start by computing the adjacent cone to its graph in the sense of Definition 2.12. Following e.g. [7, Section 4.1.3 p.128], the latter can be characterised in terms of sequences as

$$T_{\operatorname{Graph}(\mathcal{L})}^b(\varphi, \psi) = \left\{ (\zeta, \xi) \in \mathcal{D} \times \mathcal{X} \text{ s.t. for each } h_n \rightarrow 0^+ \text{ there exists } (\zeta_n, \xi_n) \rightarrow (\zeta, \xi) \text{ in } \mathcal{D} \times \mathcal{X} \right. \\ \left. \text{such that } (\varphi + h_n \zeta_n, \psi + h_n \xi_n) \in \operatorname{Graph}(\mathcal{L}) \text{ for each } n \geq 1 \right\}.$$

Given some $h_n \rightarrow 0^+$ and a sequence $(\zeta_n, \xi_n) \subset \mathcal{D} \times \mathcal{X}$ satisfying these conditions, it follows from the definition of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ that

$$\xi_n = \nabla_x \zeta_n \cdot f_{u_n} + \frac{1}{h_n} (\nabla_x \varphi \cdot f_{u_n} - \psi) \quad (4.23)$$

for some sequence $(u_n) \subset U$ given in such a way that $\psi + h_n \xi_n = \nabla_x(\varphi + h_n \zeta_n) \cdot f_{u_n}$ for each $n \geq 1$. In particular, up to considering a subsequence $(u_n) \subset U$ that we do not relabel, it follows from Proposition 2.17 that there exists some $u \in U$ for which

$$d_{cc}(f_{u_n}, f_u) \xrightarrow{n \rightarrow +\infty} 0,$$

wherein the limit vector field satisfies $\psi = \nabla_x \varphi \cdot f_u$ as a consequence of (4.23). Moreover, since the sequences $(\xi_n) \subset \mathcal{X}$ and $(\nabla_x \zeta_n) \subset \mathcal{X}$ are also convergent, there must exist some $w_\psi \in \mathcal{X}$ such that

$$\left\| \frac{1}{h} (\nabla_x \varphi \cdot f_{u_n} - \psi) - w_\psi \right\|_{\mathcal{X}} \xrightarrow{n \rightarrow +\infty} 0. \quad (4.24)$$

Therefore, upon letting $h_n \rightarrow 0^+$ in (4.23) along that same subsequence, one readily gets

$$\xi = \nabla_x \zeta \cdot f_u + w_\psi,$$

where $w_\psi \in \mathcal{X}$ is some arbitrary function. It then follows that $(\nu, -\mu) \in N_{\operatorname{Graph}(\mathcal{L})}^b(\varphi, \psi)$ if and only if

$$\langle \nu, \zeta \rangle_{\mathcal{D}} - \langle \mu, \nabla_x \zeta \cdot f_u + w_\psi \rangle_{\mathcal{X}} \leq 0,$$

which by the definition (2.1) of the divergence distribution may be equivalently rewritten as

$$\langle \nu + \operatorname{div}_x(f_u \mu), \zeta \rangle_{\mathcal{D}} \leq \langle \mu, w_\psi \rangle_{\mathcal{X}}$$

for some $w_\psi \in \mathcal{X}$ and all $\zeta \in \mathcal{D}$. Observing finally that the elements $\zeta \in \mathcal{D}$ span the whole space, the latter identity can only hold provided that

$$\nu = -\operatorname{div}_x(f_u \mu),$$

which together with the fact that $\psi = \nabla_x \varphi \cdot f_u$ allows us to obtain (4.21).

We now compute the local adjoint of $\mathcal{K}_{(\tau,t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ at some $(\varphi, \psi_{(\tau,t)}) \in \operatorname{Graph}(\mathcal{K}_{(\tau,t)})$. Resorting again to the characterisation of adjacent cones in terms of sequences, there exists for each $(\zeta, \xi_{(\tau,t)}) \in T_{\operatorname{Graph}(\mathcal{K}_{(\tau,t)})}^b(\varphi, \psi_{(\tau,t)})$ and any $h_n \rightarrow 0^+$ a sequence $(\zeta^n, \xi_{(\tau,t)}^n) \rightarrow (\zeta, \xi_{(\tau,t)})$ in $\mathcal{X} \times \mathcal{X}$ such that

$$\xi_{(\tau,t)}^n = \zeta^n \circ \Phi_{(\tau,t)}^{u_n} + \frac{1}{h_n} (\varphi \circ \Phi_{(\tau,t)}^{u_n} - \psi_{(\tau,t)})$$

for each $h > 0$ and some $(u_n(\cdot)) \in \mathcal{U}$. By using the compactness result of Proposition 3.7, one further obtains by letting $h_n \rightarrow 0^+$ along an adequate subsequence that

$$\xi_{(\tau,t)} = \zeta \circ \Phi_{(\tau,t)}^u + \varpi_t,$$

where $u(\cdot) \in \mathcal{U}$ is such that $\psi_{(\tau,t)} = \varphi \circ \Phi_{(\tau,t)}^u$ and $\varpi_t \in \mathcal{X}$ is arbitrary. Therefore, one may verify that $(\mu_{(\tau,t)}, -\mu) \in N_{\text{Graph}(\mathcal{K}_{(\tau,t)})}^b(\varphi, \psi_{(\tau,t)})$ if and only if

$$\langle \mu_{(\tau,t)}, \zeta \rangle_{\mathcal{X}} - \langle \mu, \zeta \circ \Phi_{(\tau,t)}^u + \varpi_t \rangle_{\mathcal{X}} \leq 0,$$

which can be equivalently recast using the change of variable formula (2.2) for image measures as

$$\langle \mu_{(\tau,t)} - \Phi_{(\tau,t)\#}^u \mu, \zeta \rangle_{\mathcal{X}} \leq \langle \mu, \varpi_t \rangle$$

for some $\varpi_t \in \mathcal{X}$ and every possible $\zeta \in \mathcal{X}$. Noting again that $\zeta \in \mathcal{X}$ spans the whole space, the latter identity amounts to requiring that $\nu_t = \Phi_{(\tau,t)\#}^u \mu$ and concludes the proof of (4.22).

From the characterisations of the local adjoint for the set-valued Koopman and Liouville operators derived above, we can easily infer the expressions of their full adjoint following Definition 4.10 as

$$\mathcal{L}^*(\mu) = \left\{ -\text{div}_x(f_u \mu) \text{ s.t. } u \in U \right\}$$

and

$$\mathcal{K}_{(\tau,t)}^*(\mu) = \left\{ \Phi_{(\tau,t)\#}^u \mu \text{ s.t. } u(\cdot) \in \mathcal{U} \right\}$$

respectively, for all times $\tau, t \in [0, T]$ and each $\mu \in \mathcal{X}^*$. One may then verify that the Perron-Frobenius operators introduced in Definition 4.9 are indeed the adjoints of the Koopman operators. \square

We end this section by showing that under our working assumptions, the infinitesimal generator of the Perron-Frobenius semigroup is the adjoint of the Liouville operator.

Theorem 4.13 (Infinitesimal generator of the Perron-Frobenius semigroup). *Suppose that Hypotheses (H) hold. Then, the following inclusions*

$$\mathcal{L}^*(\mu) \subset \text{Liminf}_{t \rightarrow \tau} \frac{\mathcal{P}_{(\tau,t)}(\mu) - \mu}{t - \tau} \quad (4.25)$$

and

$$\text{Limsup}_{t \rightarrow \tau} \frac{\mathcal{P}_{(\tau,t)}(\mu) - \mu}{t} \subset \overline{\text{co}} \mathcal{L}^*(\mu) \quad (4.26)$$

are satisfied for each $\mu \in \mathcal{X}^*$. Furthermore, if Hypothesis (C) holds, then the adjoint of the set-valued Liouville operator is the infinitesimal generator of the Perron-Frobenius semigroup, in the sense that

$$\text{Lim}_{t \rightarrow \tau} \frac{\mathcal{P}_{(\tau,t)}(\mu) - \mu}{t - \tau} = \mathcal{L}^*(\mu) \quad (4.27)$$

with the limit being taken in the weak-* topology for all $\mu \in \mathcal{X}^*$.

Proof. As in the proof of Theorem 4.3, we start by noting that if Hypothesis (C) holds, then necessarily

$$\overline{\text{co}} \mathcal{L}^*(\mu) = \overline{\text{co}} \left\{ -\text{div}_x(f \mu) \text{ s.t. } f \in \mathcal{F} \right\} = \left\{ -\text{div}_x(f \mu) \text{ s.t. } f \in \overline{\text{co}} \mathcal{F} \right\} = \mathcal{L}^*(\mu),$$

and it directly stems from the inclusions (4.25) and (4.26) that (4.27) holds, namely that the adjoint of the Liouville operator is the infinitesimal generator of the Perron-Frobenius operator.

We now aim at establishing the liminf inclusion (4.25), which amounts here to showing that given some $\nu = -\text{div}_x(f_{\bar{u}} \mu) \in \mathcal{L}^*(\mu)$, there exists a control signal $u(\cdot) \in \mathcal{U}$ for which

$$\lim_{t \rightarrow \tau} \frac{\Phi_{(\tau,t)\#}^u \mu - \mu}{t - \tau} = -\text{div}_x(f_{\bar{u}} \mu)$$

in the weak*-topology. To this end, we simply consider the constant control signal given by $u(t) := \bar{u}$ for all times $t \in [0, T]$, and observe that in this case, it holds for any $\zeta \in \mathcal{D}$ that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \langle \frac{1}{t} (\Phi_{(0,t)\sharp}^{\bar{u}} \mu - \mu), \zeta \rangle_{\mathcal{X}} &= \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} \frac{1}{t} (\zeta \circ \Phi_{(\tau,t)}^{\bar{u}}(x) - \zeta(x)) d\mu(x) \\ &= \int_{\mathbb{R}^d} \nabla_x \zeta(x) \cdot f_{\bar{u}}(x) d\mu(x) \\ &= \langle -\operatorname{div}_x(f_{\bar{u}} \mu), \zeta \rangle_{\mathcal{D}} \end{aligned}$$

by a simple application of Lebesgue's dominated convergence theorem, which is licit since $\zeta \in C_c^1(\mathbb{R}^d, \mathbb{C})$ and yields the liminf inclusion. To prove the limsup inclusion (4.26), one needs to show

$$\lim_{t_n \rightarrow \tau} \frac{\Phi_{(\tau,t_n)\sharp}^{u_n} \mu - \mu}{t_n - \tau} \in \mathcal{L}^*(\mu)$$

for every $(u_n(\cdot)) \subset \mathcal{U}$ and any vanishing sequence $(t_n) \subset \mathbb{R}_+$ along which this limit exists in the weak*-topology. Given some $\zeta \in \mathcal{D}$, we consider the compact set

$$K_\zeta := \overline{\bigcup_{u(\cdot) \in \mathcal{U}} \bigcup_{\tau, t \in [0, T]} \Phi_{(t,\tau)}^u(\operatorname{supp}(\zeta))}$$

and note by reproducing the reasoning detailed in the proof of Theorem 4.3 that there exists some $\bar{u} \in U$ such that

$$\sup_{x \in K_\zeta} |\Phi_{(\tau,t_n)}^{u_n}(x) - x - (t_n - \tau) f_{\bar{u}}(x)| \leq o(|t_n - \tau|)$$

along a subsequence $t_n \rightarrow \tau$ that we do not relabel. This latter identity further implies that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \langle \frac{1}{t_n} (\Phi_{(\tau,t_n)\sharp}^{u_n} \mu - \mu), \zeta \rangle_{\mathcal{X}} &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \frac{1}{t_n} (\zeta \circ \Phi_{(0,t_n)}^{u_n}(x) - \zeta(x)) d\mu(x) \\ &= \int_{\mathbb{R}^d} \nabla_x \zeta(x) \cdot f_{\bar{u}}(x) d\mu(x) \\ &= \langle -\operatorname{div}_x(f_{\bar{u}} \mu), \zeta \rangle_{\mathcal{D}} \end{aligned}$$

by Lebesgue's dominated convergence theorem along with the definition (2.1) of the divergence distribution and the fact that $\zeta \in C_c^1(\mathbb{R}^d, \mathbb{C})$, which concludes the proof of our claim. \square

As a consequence of Theorem 4.13, it can be shown¹ by adapting the arguments of Proposition 4.6 that a curve $t \in [0, T] \mapsto \mu_{(\tau,t)} \in \mathcal{P}_{(\tau,t)}(\mu)$ is of the form $\mu_{(\tau,t)} = \Phi_{(\tau,t)\sharp}^u \mu$ for some fixed control signal $u(\cdot) \in \mathcal{U}$ if and only if it is a weak*-solution of the differential inclusion

$$\begin{cases} \partial_t \mu_{(\tau,t)} \in \mathcal{L}^*(\mu_{(\tau,t)}), \\ \mu_{(\tau,\tau)} = \mu. \end{cases}$$

Interestingly, it follows from the characterisation derived in Theorem 4.12 that the adjoint Koopman dynamics is exactly a differential inclusion in the space of measures as introduced by the first author in [11, 13] to investigate the mathematical properties of mean-field control problems.

Remark 4.14 (On the definition of the full adjoint for closed processes). *As mentioned at the beginning of this section, we can give a practical justification for departing from the usual definition of adjoint mapping for closed processes proposed e.g. in [6, Chapter 3, Definition 12]. Indeed, considering for instance the Liouville operator, the latter definition would amount to requiring that $(\nu, -\mu) \in \mathcal{L}^*(\varphi, \psi)(\mu)$ for all (instead of some) $\psi \in \mathcal{L}(\varphi)$. Therefore, such a notion did not seem relevant since*

$$\left\{ \nu \in \mathcal{D}^* \text{ s.t. } \nu = -\operatorname{div}_x(f_u \mu) \text{ for all } u \in U \right\}$$

¹The precise argument involves abstract measurable selection theorems borrowed from [18] to handle weak*-topologies – which are not metrisable but Souslin, see e.g. [51, Theorem 7 page 112] –, and falls beyond the scope of this article.

defines an empty set whenever there exist two control parameters $u_1, u_2 \in U$ for which $\operatorname{div}_x(f_{u_1}\mu) \neq \operatorname{div}_x(f_{u_2}\mu)$, i.e. as soon as the controlled dynamics is not trivial.

However, we have already explained above that our notion of duality coincides with that introduced to study fans, when the latter makes sense, and shown in Example 4.11 that it reduces to the familiar notion of adjoint mapping when the underlying operators are single-valued. In particular, if $(\mathcal{K}_t(\varphi))_{t \in [0, T]}$ is the usual Koopman operator associated with some flow $(\Phi_t)_{t \in [0, T]}$ as in (4.17) of Remark 4.7, then the set-valued Perron-Frobenius operators and its infinitesimal generators coincide with the linear adjoints of the Koopman and Liouville operators.

4.3 Point spectra of the set-valued Koopman and Liouville operators

In this section, we study the interplay between the point spectrum of the set-valued Koopman and Liouville operators, understood in the following sense.

Definition 4.15 (Eigenvalues and eigenvectors of closed processes). *Given two Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, we say that a complex number $\lambda \in \mathbb{C}$ is an eigenvalue of a closed process $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ if there exists an eigenvector $x_\lambda \in \mathcal{X}$ such that*

$$\lambda x_\lambda \in \mathcal{F}(x_\lambda).$$

The set of all such eigenvalues is called the point spectrum of \mathcal{F} , and is denoted by $\sigma_p(\mathcal{F}) \subset \mathbb{C}$.

Example 4.16 (Link with the usual notion of point spectrum). *Again, one may easily check that the definition introduced above reduces to the usual notion of point spectrum of an operator when $\mathcal{F}(x) = \{Ax\}$ is single-valued, since then*

$$\lambda x_\lambda \in \mathcal{F}(x_\lambda) = \{Ax_\lambda\} \quad \text{if and only if} \quad Ax_\lambda = \lambda x_\lambda$$

for each $\lambda \in \sigma_p(\mathcal{F})$.

In the following theorem, we prove a set-valued version of the classical spectral mapping theorem (see e.g. [24, Chapter IV - Theorem 3.7]), which precisely relates the point spectra of the Liouville and Koopman operators.

Theorem 4.17 (Relationship between the eigenvalues of the Koopman and Liouville operators). *Suppose that Hypotheses (H) hold and fix some $\tau \in [0, T]$. Then for all times $t \in [0, T]$, it holds that*

$$e^{\sigma_p(\mathcal{L})(t-\tau)} \subset \sigma_p(\mathcal{K}_{(\tau, t)}), \tag{4.28}$$

namely if an observable $\varphi_\lambda \in \mathcal{D}$ is an eigenfunction of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ with eigenvalue $\lambda \in \sigma_p(\mathcal{L})$, then it is an eigenfunction of $\mathcal{K}_{(\tau, t)} : \mathcal{X} \rightrightarrows \mathcal{X}$ with eigenvalue $e^{\lambda(t-\tau)} \in \sigma_p(\mathcal{K}_{(\tau, t)})$. Furthermore, if Hypothesis (C) holds, then for each measurable curve $t \in [0, T] \mapsto \lambda_{(\tau, t)} \in \mathbb{C}$ satisfying

$$\lambda_{(\tau, t)} \varphi_\lambda \in \mathcal{K}_{(\tau, t)}(\varphi_\lambda)$$

for a given and time-independent $\varphi_\lambda \in \mathcal{D}$, there exists a sequence $t_n \rightarrow \tau$ for which

$$\lim_{n \rightarrow +\infty} \lambda_{(\tau, t_n)} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{\lambda_{(\tau, t_n)} - 1}{t_n - \tau} \in \sigma_p(\mathcal{L}). \tag{4.29}$$

In particular, the converse spectral inclusion

$$\left\{ \lambda_{(\tau, t)} \in \sigma_p(\mathcal{K}_{(\tau, t)}) \setminus \{0\} \text{ s.t. } \lambda_{(\tau, t)} = e^{\lambda(t-\tau)} \text{ for some } \lambda \in \mathbb{C} \right\} \subset e^{\sigma_p(\mathcal{L})(t-\tau)} \tag{4.30}$$

is satisfied for all times $\tau, t \in [0, T]$.

Proof. We start by proving the first inclusion (4.28). By Theorem 4.3, the fact that $\lambda\varphi_\lambda \in \mathcal{L}(\varphi_\lambda)$ for some $\lambda \in \sigma_p(\mathcal{L})$ and $\varphi_\lambda \in \mathcal{D}$ is tantamount to the existence of some control value $\bar{u}_\lambda \in U$ for which

$$\lambda\varphi_\lambda = \nabla_x \varphi_\lambda \cdot f_{\bar{u}_\lambda}. \quad (4.31)$$

Hence, considering the constant control signal defined by $\bar{u}_\lambda(t) := \bar{u}_\lambda$ for all times $t \in [0, T]$, one has

$$\frac{d}{dt}\varphi \circ \Phi_{(\tau,t)}^{u_\lambda}(x) = \langle \nabla_x \varphi_\lambda, f_{\bar{u}_\lambda} \rangle \circ \Phi_{(\tau,t)}^{\bar{u}_\lambda}(x) = \lambda\varphi_\lambda \circ \Phi_{(\tau,t)}^{\bar{u}_\lambda}(x)$$

for each $x \in \mathbb{R}^d$, and it follows from a standard Cauchy-Lipschitz uniqueness argument that

$$\varphi \circ \Phi_{(\tau,t)}^{u_\lambda}(x) = e^{\lambda(t-\tau)}\varphi_\lambda(x)$$

for all $(\tau, t, x) \in [0, T] \times [0, T] \times \mathbb{R}^d$. The thesis follows then from the observation that $\varphi \circ \Phi_{(0,t)}^{\bar{u}_\lambda} \in \mathcal{K}_{(\tau,t)}(\varphi_\lambda)$, by the very definition of the set-valued Koopman operators.

Conversely, let us assume that there exist $t \mapsto \lambda_{(\tau,t)} \in \mathbb{C}$ and $\varphi_\lambda \in \mathcal{D}$ such that $\lambda_{(\tau,t)}\varphi_\lambda \in \mathcal{K}_{(\tau,t)}(\varphi_\lambda)$ for \mathcal{L}^1 -almost every $t \in [0, T]$, and suppose that Hypothesis (C) holds. By Proposition 3.5, there exists a measurable family of control signals $\{u_t^\lambda(\cdot)\}_{t \in [0, T]} \in \mathcal{U}$ such that

$$\lambda_{(\tau,t)}\varphi_\lambda = \varphi_\lambda \circ \Phi_{(\tau,t)}^{u_t^\lambda} \in \mathcal{K}_{(\tau,t)}(\varphi_\lambda) \quad (4.32)$$

for \mathcal{L}^1 -almost every $t \in [0, T]$. By repeating the arguments yielding the limsup inclusion of Theorem 4.3, one may infer the existence of an element $\bar{u}_\lambda \in U$ along with a sequence $t_n \rightarrow \tau$ such that

$$\varphi_\lambda \circ \Phi_{(\tau,t_n)}^{u_{t_n}^\lambda} = \varphi_\lambda + (t_n - \tau)\nabla_x \varphi_\lambda \cdot f_{\bar{u}_\lambda} + o(|t_n - \tau|).$$

This along with (4.32) implies that $\lim_{n \rightarrow +\infty} \lambda_{(\tau,t_n)} = 1$, as well as

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\lambda_{(\tau,t_n)} - 1}{t_n - \tau} \varphi_\lambda &= \lim_{n \rightarrow +\infty} \frac{\varphi_\lambda \circ \Phi_{(0,t_n)}^{u_{t_n}^\lambda} - \varphi_\lambda}{t_n - \tau} \\ &= \nabla_x \varphi_\lambda \cdot f_{\bar{u}_\lambda}, \end{aligned}$$

which belongs to $\mathcal{L}(\varphi_\lambda)$ as a consequence of Theorem 4.3. Thus, we have shown the limit inclusion (4.29), from which the spectral inclusion (4.30) easily follows. \square

Example 4.18 (The spectral mapping theorem for linear feedback controls). *To illustrate the results of the previous theorem, let us focus on the simple example of a linear time-invariant system in with feedback controls*

$$\dot{x}(t) = (A + BK)x(t)$$

wherein $(A, B) \in \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times m}$ and $K \in \mathcal{K}_{\text{adm}}$ with $\mathcal{K}_{\text{adm}} \subset \mathbb{R}^{m \times d}$ being a compact set. In practice, one may consider for instance that $\mathcal{K}_{\text{adm}} := \{K_1, \dots, K_n\}$ is comprised of finitely many feedback matrices, or maybe fix $\mathcal{K}_{\text{adm}} := \{K \in \mathbb{R}^{m \times d} \text{ s.t. } \|K\|_{\text{F}} \leq 1\}$ as the closed unit ball for the Frobenius norm. In this context, the set-valued Koopman and Liouville operators can be computed explicitly as

$$\mathcal{K}_{(0,t)} : \varphi \in \mathcal{X} \Rightarrow \left\{ x \in \mathbb{R}^d \mapsto \varphi\left(\exp(t(A + BK))x\right) \in \mathbb{C} \text{ s.t. } K \in \mathcal{K}_{\text{adm}} \right\} \subset \mathcal{X}$$

and

$$\mathcal{L} : \varphi \in \mathcal{D} \Rightarrow \left\{ x \in \mathbb{R}^d \mapsto \nabla_x \varphi(x) \cdot \left((A + BK)x\right) \in \mathbb{C} \text{ s.t. } K \in \mathcal{K}_{\text{adm}} \right\} \subset \mathcal{X}.$$

In this context, a map $\varphi_\lambda \in \mathcal{D}$ is an eigenvalue of the Liouville operator if there exists some $\lambda \in \mathbb{C}$ along with a $K_\lambda \in \mathcal{K}_{\text{adm}}$ such that

$$\lambda\varphi_\lambda(x) = \nabla_x \varphi_\lambda(x) \cdot (A + BK_\lambda)x$$

for all $x \in \mathbb{R}^d$. Following [41], one may for instance consider candidates linear eigenfunctions of the form $\varphi_\lambda(x) := e_\lambda \cdot x$, with $\lambda \in \mathbb{C}$ and $e_\lambda \in \mathbb{C}^d$ being respectively an eigenvalue and eigenvector of $(A + BK_\lambda)^* \in \mathbb{C}^{d \times d}$. One may then check via elementary computations that $e^{\lambda t} \in \sigma_p(\mathcal{K}_{(0,t)})$ for each $\lambda \in \sigma_p(\mathcal{L})$, and vice versa.

We end this section by a simple proposition which shows how one may produce new eigenfunctions and eigenvalues of the set-valued Liouville operator by combining already known ones.

Proposition 4.19 (On the structure of the point spectrum). *Let $\varphi_{\lambda_1}, \varphi_{\lambda_2} \in \mathcal{D}$ be two eigenfunctions of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ associated with the same control value $u_\lambda \in U$. Then for every $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\varphi_{\lambda_1}^{\alpha_1} \varphi_{\lambda_2}^{\alpha_2} \in \mathcal{D}$, the latter is an eigenfunction of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ with the eigenvalue $\alpha_1 \lambda_1 + \alpha_2 \lambda_2 \in \sigma_p(\mathcal{L})$.*

Proof. By definition, there exists an element $u_\lambda \in U$ for which the eigenfunctions $\varphi_{\lambda_1}, \varphi_{\lambda_2} \in \mathcal{D}$ satisfy

$$\lambda_1 \varphi_{\lambda_1} = \nabla_x \varphi_{\lambda_1} \cdot f_{u_\lambda} \quad \text{and} \quad \lambda_2 \varphi_{\lambda_2} = \nabla_x \varphi_{\lambda_2} \cdot f_{u_\lambda}.$$

Thus, given a pair of real numbers $\alpha_1, \alpha_2 \in \mathbb{R}$ for which $\varphi_{\lambda_1}^{\alpha_1} \varphi_{\lambda_2}^{\alpha_2} \in \mathcal{D}$, one easily gets that

$$\begin{aligned} \nabla_x(\varphi_{\lambda_1}^{\alpha_1} \varphi_{\lambda_2}^{\alpha_2}) \cdot f_{u_\lambda} &= \alpha_1 \varphi_{\lambda_1}^{\alpha_1-1} \varphi_{\lambda_2}^{\alpha_2} \nabla_x \varphi_{\lambda_1} \cdot f_{u_\lambda} + \alpha_2 \varphi_{\lambda_1}^{\alpha_1} \varphi_{\lambda_2}^{\alpha_2-1} \nabla_x \varphi_{\lambda_2} \cdot f_{u_\lambda} \\ &= (\alpha_1 \lambda_1 + \alpha_2 \lambda_2) \varphi_{\lambda_1}^{\alpha_1} \varphi_{\lambda_2}^{\alpha_2}, \end{aligned}$$

which proves the statement. \square

The main subtlety in the previous proposition – which again results from the lack of convexity of the graph of $\mathcal{L} : \mathcal{D} \rightrightarrows \mathcal{X}$ – is that one cannot combine eigenvalues and eigenfunctions corresponding to different control values in order to produce new eigenelements.

Appendices

A Proof of Corollary 2.19

In this appendix section, we detail the proof of Corollary 2.19 for the sake of self-containedness.

Proof of Corollary 2.19. Consider an elements $u(\cdot) \in \mathcal{U}$ as well as a sequence $(u_n(\cdot)) \subset \mathcal{U}$ such that

$$\int_0^T d_U(u_n(t), u(t)) dt \xrightarrow{n \rightarrow +\infty} 0.$$

It then follows from classical results in measure theory (see e.g. [2, Remark 1.18]) that $(u_n(\cdot))$ converges to $u(\cdot)$ in measures, that is if we consider the set $\mathcal{A}_n^\delta := \{t \in [0, T] \text{ s.t. } d_U(u_n(t), u(t)) \geq \delta\}$, it holds

$$\mathcal{L}^1(\mathcal{A}_n^\delta) \xrightarrow{n \rightarrow +\infty} 0 \tag{A.1}$$

for each $\delta > 0$. In turn, given a compact set $K \subset \mathbb{R}^d$, it follows from (2.9) in Theorem 2.18 that

$$\sup_{n \geq 1} |\Phi_{(\tau, t)}^{u_n}(x)| \leq R_K$$

for all $(\tau, t, x) \in [0, T] \times K$ and some constant $R_K > 0$ which only depends on the magnitudes of m, T and $\sup_{x \in K} |x|$. Then, introducing the notation $K' := B(0, R_K)$, one can use Hypothesis **(H)**-(ii) to estimate the discrepancy between the flow maps generated respectively by $u_n(\cdot)$ and $u(\cdot)$ as

$$\begin{aligned} |\Phi_{(\tau, t)}^u(x) - \Phi_{(\tau, t)}^{u_n}(x)| &\leq \int_\tau^t \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^{u_n}(x), u_n(s)\right) \right| ds \\ &\leq \int_\tau^t \left(\ell_{K'} |\Phi_{(\tau, s)}^u(x) - \Phi_{(\tau, s)}^{u_n}(x)| + \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) \right| \right) ds. \end{aligned}$$

Hence, it directly follows from an application of Grönwall's lemma that

$$|\Phi_{(\tau, t)}^u(x) - \Phi_{(\tau, t)}^{u_n}(x)| \leq \left(\sup_{x \in K} \int_0^T \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) \right| ds \right) e^{\ell_{K'} T} \tag{A.2}$$

for all $(\tau, t, x) \in [0, T] \times [0, T] \times K$ and each $n \geq 1$.

At this stage, given some arbitrary $\varepsilon > 0$, it follows from Hypotheses **(H)** along with the fact that $[0, T] \times [0, T] \times K \times U$ is a compact metric space that there exists some $\delta > 0$ for which

$$\sup_{(\tau, t, x) \in [0, T] \times [0, T] \times K} \left| f\left(\Phi_{(\tau, t)}^u(x), u_1\right) - f\left(\Phi_{(\tau, t)}^u(x), u_2\right) \right| < \frac{\varepsilon}{2Te^{\ell_{K'}T}} \quad (\text{A.3})$$

whenever $u_1, u_2 \in U$ are such that $\mathbf{d}_U(u_1, u_2) < \delta$, and where $m > 0$ is the constant of Hypothesis **(H)**-(i). Moreover, following (A.1), there exists for that same $\delta > 0$ some integer $N_{\varepsilon, K} \geq 1$ such that

$$\mathcal{L}^1(\mathcal{A}_n^\delta) < \frac{\varepsilon}{4m(1 + R_K)e^{\ell_{K'}T}} \quad (\text{A.4})$$

for each $n \geq N_\varepsilon$. Therefore, by combining the information from (A.3) and (A.4), one may infer that

$$\begin{aligned} & \int_0^T \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) \right| ds \\ & \leq \int_{\mathcal{A}_n^\delta} \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) \right| ds \\ & \quad + \int_{[0, T] \setminus \mathcal{A}_n^\delta} \left| f\left(\Phi_{(\tau, s)}^u(x), u(s)\right) - f\left(\Phi_{(\tau, s)}^u(x), u_n(s)\right) \right| ds \\ & \leq 2m \left(1 + \sup_{(\tau, s, x) \in [0, T] \times [0, T] \times K} |\Phi_{(\tau, s)}^u(x)| \right) \mathcal{L}^1(\mathcal{A}_n^\delta) \\ & \quad + \sup_{(\tau, s, x) \in ([0, T] \setminus \mathcal{A}_n^\delta) \times ([0, T] \setminus \mathcal{A}_n^\delta) \times K} \left| f\left(\Phi_{(0, s)}^u(x), u(s)\right) - f\left(\Phi_{(0, s)}^u(x), u_n(s)\right) \right| \mathcal{L}^1([0, T] \setminus \mathcal{A}_n^\delta) \\ & \leq \frac{\varepsilon}{e^{\ell_{K'}T}} \end{aligned}$$

for all $x \in K$ and each $n \geq N_{\varepsilon, K}$, where we used Hypothesis **(H)**-(i). Plugging this last inequality in (A.2) finally yields the uniform estimate

$$\sup_{(\tau, t, x) \in [0, T] \times [0, T] \times K} \left| \Phi_{(\tau, t)}^u(x) - \Phi_{(\tau, t)}^{u_n}(x) \right| \leq \varepsilon$$

valid for each $n \geq N_{\varepsilon, K}$, which concludes the proof since both $K \subset \mathbb{R}^d$ and $\varepsilon > 0$ are arbitrary. \square

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