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# Implementing a teaching model for enhancing students' formal written mathematical communication

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*In a teacher-researcher collaboration, a teaching model designed for developing students' competence in producing formal written mathematical communications of solutions to problems has been piloted. The sequence was implemented in a mathematics course in the International Baccalaureate program. By taking advantage of a standardized thesis writing task in the course, with standardized assessment, we examined if the trained communication competence in the problem-solving context transferred to the thesis writing context. By using a control group design, we concluded that the work with formal mathematical communication during certain mathematics lessons has had transfer effects on the general handling of students' written mathematical communication.*

*Keywords: Mathematics, written communication, implementation.*

## Introduction

The present paper presents results from the implementation of a research-based teaching sequence for teaching formal written mathematical communication in one upper secondary class within the International Baccalaureate (IB) program. The IB program provides an especially interesting arena for implementation research because it is a program that is highly similar across many countries and applies highly formalized principles for assessment. Implementation research on formal written mathematical communication within the IB mathematics curriculum may have wider generalizability than research conducted within smaller or national curricula, and the use of an outcome measure that stakeholders care about, such as assessment leading to grades, can increase the relevance and applicability of the research findings and be complementary to researchers self-developed instruments. In a teacher-researcher collaboration project, the first author of this paper serves as a pilot teacher, testing and co-developing the ideas, principles, and teaching models with her classes at the IB program. The two other authors are researchers and developers in the project. The project has a plan for scaling the innovation, but here we are looking at the piloting phase of the implementation effort. As pointed out by Century and Cassata (2016) it can already in the pilot phase be important to gather information on how the intended mechanisms of intervention work practically.

The teaching model we are implementing, and the research background for it, is described in Teledahl et al. (2023) and we will describe the most important principles behind it in the background section. The teaching model aims to develop students' formal written mathematical communication competence. Communication competence can be viewed on a continuum between two poles: the practical context-specific ability to produce good formal written communications and the complex cognitive capability underlying such an ability, that can be “analytically divided into several cognitive and affective-motivational traits (or resources)” (Blömeke et al., 2015, p. 3).

In the case we report here we will examine if our teaching model has the intended effect on students' competence in producing formal written mathematical reasoning by looking at a transfer situation, which is a way of tapping into the cognitive aspect of communication competence. Our teaching design involves students producing written communications of solutions to problems, but to gauge the transfer effect, we will examine the effect on the mathematics students produce when they write a thesis with mathematical content instead of assessing task solutions with the dedicated instrument we are developing (Teledahl et al. 2023). As pointed out by Greeno and colleagues the issue of transfer concerns "how learning to participate in an activity in one situation can influence (positively or negatively) one's ability to participate in another activity in a different situation" (Greeno et al., 1993, p. 100). Marton (2006) instead focuses on that transfer can occur if one can identify enough sameness between situations. Solving problems chosen by the teacher and communicating the solution is certainly different from communicating mathematical reasoning that has been produced in a thesis writing context.

We are interested in the question: Are students' written communication of mathematics when writing a thesis affected by a teaching model focusing on writing solutions to mathematical problems? We will examine this question by using the IB program's standardized assessment rubric for mathematical communication in the thesis writing task which is also standard in the IB program and compare the class of 2023, which has been subjected to our teaching design, with the class of 2022, which act as a control group.

## **Background**

Teachers have forever tried to get their students to write clear solutions to problems they solve. In our preliminary research, we have found that there is no progress made in terms of the quality of written communications of problem solutions from when students are around 11 years old to when they are around 16 years old (Teledahl et al., 2022). Despite this depressing state, we believe that the change logic or theory of change (Weiss, 1995) underlying our teaching model can contribute greatly to students' development of formal written communication competence. The theory of change for our program builds on three main principles as explained in detail in Teledahl (2023).

The first principle is that by separating the work of solving a problem from the work of producing a formal written communication of the solution we allow students that might have difficulties solving the problem to still take part in quality work related to converting the reasoning associated to a particular solution into a formal written presentation. This is done by first administering a standard problem-solving session in groups, with a follow-up whole class discussion, which typically results in some roughly sketched solutions on the whiteboard. This part of the teaching can be said to follow the principles described by Stein et al. (2008). The objective of the teacher in the discussion is to make sure that all students, or at least all groups, understand at least one way of solving the problem. The groups then get the task to choose one solution and produce a formal written communication of that solution. Note that students do not need to continue working on the solution they previously worked on. Typically, the teaching phases described above take one lesson.

The second principle stipulates that by spending qualitative time discussing communication quality of the student-produced communication, the students will develop ways of talking about and hence

also notice different qualities of the written communications. By participating in such sessions, we believe that students will hence develop a deeper understanding of different elements that go into written communication and what makes written communication good, which they later can incorporate in their production of formal written communications. This principle is implemented through the teacher (in the pilot phase of the project, teacher and researchers) choosing a few particularly interesting written communications from lesson one and then leading a student discussion on the characteristics and qualities of these communications in the next lesson.

The third principle concerns the idea of who is the receiver of written communication and relates tightly to the notion of what is good communication in the previous principle. We posit an absent peer as the receiver. That is, the students are not writing to the teacher, or someone else already familiar with the problem and its possible solutions. Instead, they should write so that it becomes as clear as possible as to why the presented reasoning is in fact a correct solution to the problem in a way that is understandable to and convincing for a peer that has never seen the problem before. Similarly, to how Stylianides (2007) discusses what the proof concept might mean in the early grades, we believe that this anchoring of the communication in the local context can help teachers strike a balance between institutionalized ways of communicating mathematics and what is meaningful in the particular class.

We have operationalized these principles into four related but slightly different teaching sequences, each typically covering two lessons and involving 5-7 distinct teaching phases, see Nordqvist et al (2023) for an example. Among the handful of teachers, we collaborated with in the pilot phase, there has also been some local adaptation of the models. So instead of presenting our basic models formally, we will here describe the implementation with one specific teacher and class of students.

### **The local implementation of the teaching design**

On three occasions with a group of 25 students, 16-18 years of age, who, for the most part, enjoy and are ambitious about mathematics, the first author and teacher – Julia - has used the suggested teaching model in the suggested standard form. She first asked students to solve a problem, then asked students to show a few different solutions on the whiteboard, and then asked all students to choose one solution and rewrite it in a clear way “so that a student who is absent today would understand and be convinced of the validity of the solution.” Students then handed in the paper to Julia, and after some consideration, together with Ola and Linda, she showed a selection of these solutions to the class for evaluation of the quality of communication. Also, when there was an apparent need after the very first occasion, Julia explicitly taught the students about the meanings of some common mathematical symbols such as implication arrows. Initially, she asked the students to present a solution that someone in the year group below would understand. Students then produced overly detailed solutions, presenting every minutia of algebraic manipulation. It seems that it is easier for students to understand what is needed to communicate to a student in their own year, which provides some indication of the soundness of principle three described above.

The main changes Julia has seen in student communication are that they immediately started using arrows and equal signs more appropriately, and to a larger extent have both a right-hand side and a left-hand side in their equations. Over the months, students also improved the detail they include in

the presentation of their solutions so that they no longer included details that would be obvious to the intended audience. Another improvement, but slower, has been the layout of the steps of the solution on the page when writing by hand. Students tend to want to save paper space and will be creative in using all parts of the paper rather than turning the page or using another sheet. Something Julia has learned along the way and improved over the three occasions is that she must provide students with paper, or else some will tear off a corner in their notebook and write out the solution in miniature on that little piece of paper.

## Method

We used students' submitted first drafts of their thesis to gauge their competence in communicating their mathematical reasoning in text form. We compared the drafts from the class of 2022 (N=11), acting as a control group, and the class of 2023 (N=22) which had been submitted to three rounds of our teaching design and hence acted as the experimental group. By using the first drafts instead of the finished thesis, we get access to the text the students produced uninfluenced by text feedback from the teacher. Since the teacher probably changed her views of formal written mathematical communication during the intervention, we think using the drafts minimizes this effect.

The drafts were assessed by the teacher in the same way in the two years, using the IB program international standard rubrics for assessing this thesis. There are five rubrics concerning presentation (A, 1-4 points), mathematical communication (B, 1-4), engagement (C, 1-3), reflection (D) and mathematics (E, 1-6). The presentation rubric could be seen as related to our issue, but rather concerns the structure of the thesis and its content. The criteria for B comprise that the student has:

- used appropriate mathematical language (notation, symbols, terminology).
- defined key terms and variables, where required
- used multiple forms of mathematical representation, such as formulae, diagrams, tables, charts, graphs, and models, where appropriate
- used a deductive method and set out proofs logically where appropriate

Rubric B is close to the type of communication competence the project aims to develop, which includes Efficiency, Clarity, and Appropriateness of communication regarding the problem description, calculations, justifications, and conclusion. This is fully explained in Teledahl (2023).

## Results

We will present statistical results and follow up with some concrete examples. As a first indication, we present results per criterion B level, mathematical communication.

**Table 1: Relative frequencies of scores for standard rubric B, mathematical communication**

Level	1 point	2 points	3 points	4 points
2022	0.27	0.55	0.18	0
2023	0.046	0.50	0.46	0

No students in either class reached the highest level in this first draft, but the 2023 distribution shows a clear shift toward higher levels. This is confirmed by looking at the mean values which are 1.91 for the class of 2022 and 2.41 for the class of 2023 which represents a significant difference at the  $p < 0.05$  level, based on a Mann-Whitney U test. The difference between the classes when summing the scores for all five rubrics is 9.55 versus 9.91, which is not significant at the 0.05 level. From this, we conclude that it does not seem like the class of 2023 generally performs better, but they do perform better in written mathematical communication.

We will continue with some examples for illustration purposes and first look at cases where students use equations.

### A., 2022, Level 1

$$0.2181x + 0.3102y \geq 0.27 \times 100,000 = 21.81x + 32.02y \geq 27,000 \text{ (27\% of 100,000 is 27,000)}$$

### B, 2023, Level 1

$$y - \textit{intercept} = r_2 = 9.75\textit{cm}, r_1 = 6.75\textit{cm} \text{ and } h_1 = 14.3\textit{cm}.$$

### C, 2023, Level 3

At a given time and point on a road, Traffic flow ( $Q$ ) is equal to the traffic density ( $K$ ) multiplied by the traffic velocity ( $v$ ) at which the traffic is moving.

$$Q = Kv \tag{2}$$

**Figure 1: Examples of equation usage at levels 1 and 3**

The two, level 1, examples A and B in Figure 1 are both characterized by using words inside the equation. Example C has a nicely formatted equation and also in a clear way introduces the meaning of the variables in the preceding text. This is a relatively typical improvement after implementing our teaching design.

In Figure 2, example C misuses the implication arrow, and example D makes an error with indices. It should be mentioned that many students in the class of 2023 started to use the mathematical typesetting program Latex which generally makes mathematical formatting look better. However, as indicated by examples B and D, errors and bad mathematical communication practices are not eliminated by Latex.

### D, 2022, Level 2

$$\Rightarrow r^2 = \left(\sqrt{x^2 + y^2}\right)^2 + z^2 \quad (3)$$

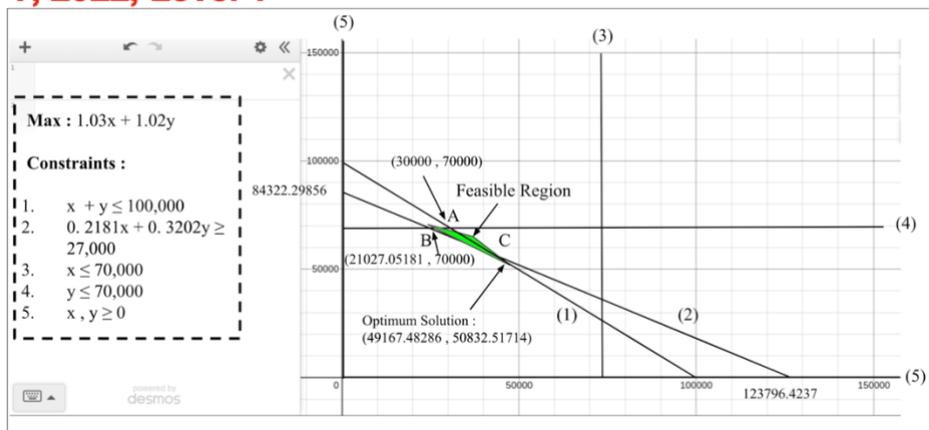
### E, 2023, Level 2

$$\mathbf{a}_m = N \frac{\mathbf{r}_{t,p} - \mathbf{r}_m}{|\mathbf{r}_{t,p} - \mathbf{r}_m|^2} \mathbf{v}_r \quad (1)$$

where  $\mathbf{a}_m$  is the acceleration of the missile,  $\mathbf{r}_m$  is the position of the missile,  $\mathbf{v}_r$  is the relative velocity between the missile and the target,  $\mathbf{r}_{t,p}$  is the predicted position of the target at the time of intercept, and  $N$  is a Pro-Nav gain constant. The comma between  $\mathbf{r}_{t,p}$  is used to indicate that these two quantities are related and should be treated as a single entity.

Figure 2: Examples of equation usage at level 2

### F, 2022, Level 1



### G, 2023, Level 3

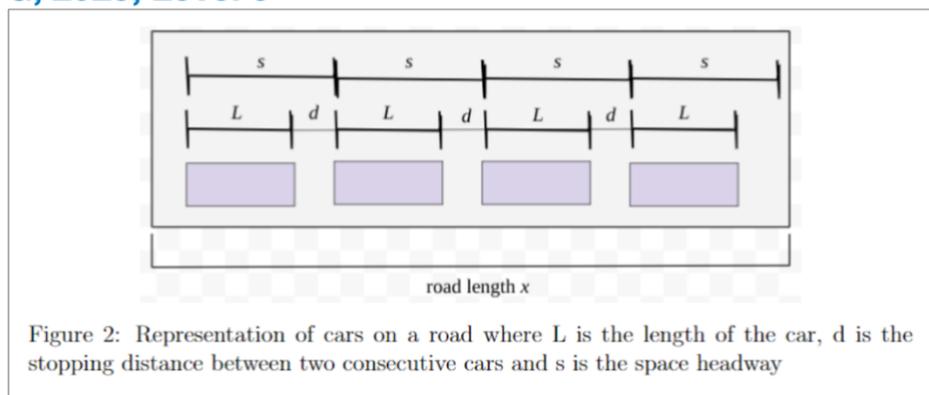


Figure 2: Representation of cars on a road where  $L$  is the length of the car,  $d$  is the stopping distance between two consecutive cars and  $s$  is the space headway

Figure 3: Illustrations at levels 1 and 3

Also, Figure 3 illustrates an improvement from the class of 2022 to the class of 2023. Example F includes all necessary information but crams it up in a tight space with no thought-through formatting. At the same time, example G uses a precise figure with variables explained in the text.

## **Discussion**

We have presented a single case implementation of a teaching designed aiming to develop students' ability to produce formal written communications of their solutions to mathematical problems. We examined if the trained ability to produce such written communications produced transfer effects to a different situation, where students are instead writing mathematical theses. In the latter situation, all mathematics they put in writing is chosen by themselves as opposed to being a reaction to a problem posed by a teacher. We claim the two situations are different enough for the thesis writing to represent a transfer relative to producing written communication of problem solutions. We claim that it is interesting from an implementation perspective to even in this early stage examine if the competence trained in a specific situation, transfers to other situations.

Obviously, the results should be treated with some care since it is derived from only comparing two classes and only involves one teacher. A difference between the years was also that the class of 2023 used the typesetting program Latex, which might have contributed to their improved results. As illustrated through, also with Latex, 2023 students make certain communicational errors. The IB program in which we carried out the experiment can also be considered to be of a high standard and the students are relatively old relative to the age span our project targets. It is possible that these factors positively influence the transfer effect we have examined here. Nevertheless, it can be considered relatively impressive to get measurable results after only three iterations of our teaching model, in particular when considering transfer rather than trained performance in the problem-solving context. We consider the transfer effect as an indication that students' development in producing written communication of solutions to problems is not just a locally trained skill, but reflects a shift in cognitive disposition and attitude towards formal mathematical writing.

The pilot case presented here is one of a handful of cases where our teaching model has been implemented. We used it with teachers for children around 11 years old, and in well organized as well as not so well organized schools, with great social problems. So far, it has been surprisingly easy to implement the teaching model and get visible student development. However, as described above, the researcher-teacher collaboration has been quite tight, with meetings between lessons 1 and 2 in a teaching cycle where we evaluate the written communications as preparation for lesson 2.

We are presently scaling this up, by training facilitators, whom themselves start by implementing our teaching model with support from us, at the same time that we are planning an upscale to around 50-100 teachers who will be advised by the facilitators rather than us directly. Taking this step is the next implementation challenge for the project.

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