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Preservice teachers' feedback to students' solutions on linking linear function equation and graph: a Polish and Slovak case

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Mathematics Teachers' Specialised Knowledge model (Carrillo et al., 2018) was used as a tool for analysing preservice secondary school teachers' knowledge with respect to functions particularly to linking function graph and equation of linear function. Our aim was to explore their knowledge of the topic in this area and their ability to identify the students' misconceptions that have appeared in provided students' solutions as well as their ability to formulate a valuable feedback to the written solutions. 37 participants from Poland and Slovakia took part in the study. The ability to provide feedback and to identify misconceptions about learning functions was demonstrated in a number of feedbacks and explanations. However, the misconception about scaling was not identified by most of them; moreover, there is an evidence that they shared this misconception. This gap should be addressed in the future teacher training.

Keywords: Understanding functions, preservice teachers, function representations, linear function, axis scales.

Introduction

The concept of function is one of the most important and fundamental concepts both in mathematics and its education (e.g., Leinhardt et al., 1990; Thompson and Carlson, 2017). In light of the emphasis on functions in school mathematics, it is important to pay attention to *preservice secondary school mathematics teachers'* (PSTs) training on developing functional thinking in students and on functions in general from the educational perspective.

This contribution is part of a larger study in which we developed a tool to explore the specialized knowledge of PSTs in particular with respect to linking representations of functions and identifying invariant properties of a function across its different representations. Our aim is also to explore the ability of PSTs to identify students' misconceptions in the area of functions and graphs that have appeared in provided students' solutions to the tasks.

Theoretical framework

Linking representations

Linking representations is an important area in understanding function (e.g. Ronda, 2015). According to Ronda (2015), five growth points in linking representations of a function were distinguished. These growth points were developed from a study of secondary school students and distinguish the extent to which students use local properties of functions (e.g. calculating intersections), global properties (determining the slope, pattern, trend on the graph) and invariant properties, which show the conceptualization of the function as a mathematical object of a given function when solving the task. Working with a gridless graph puts more emphasis on identifying the invariant properties of the explored function.

Students' misconceptions on function graphs

Several theoretical and research approaches describe students' misconceptions in the area of functions and graphs, e.g. Leinhardt et al., (1990) and Hadjidemetriou and Williams (2002). One of these misconceptions is closely connected to the task we are focused in this paper and used in our research (see Figure 1) in which the graph of the function is without the scale (gridless graph). Researchers (e.g. Leinhardt et al., 1990; Kerslake, 1981; Ostermann, 2018) point out that when interpreting graphs, problems with the slope and shape of the graph arise, which depend largely on the choice of coordinate system. Students should understand that some properties of the graph are inherent of the graph itself (e.g. monotonicity) and some are dependent on the choice of scale on the coordinate axes in the coordinate system in which the graph is constructed (e.g. the slope of the graph).

Why gridless or different scales graphs are important?

Awareness of the fact that the choice of scale on the axes of the coordinate system changes the look of the graph is an important knowledge not only in mathematics. Leinhardt et al. (1990, p.17) mentioned that:

“It is interesting that scale is an issue when using graphs for scientific data analysis and in computer-based instruction, but usually is not an issue when introducing graphing in mathematics classes. It may be that, because the scale is often assumed or given in mathematics instruction (normally the scale is the same on each axis), it then becomes difficult to use or access later in science classes.” Leinhardt et al. (1990, p. 17)

The ability of using and interpreting graphs with different scaling and understanding invariant function properties is a skill needed not only at mathematics lessons on functions and graphs but is therefore needed in mathematical modelling activity (within mathematics and outside mathematics) as well as in everyday life to prevent young people from being confused or manipulated by the different types of data presenting.

Polish and Slovak curriculum concerning functions

There are many similarities concerning both curriculum regarding functions as well as general way of mathematics teaching in Poland and Slovakia, as a result of historical and geographical circumstances. In both countries the notion of function is introduced only in secondary school, linear function is introduced at grade 9 (together with antiproportional relationships), general properties of functions in grade 9 (Poland) and 10 (Slovakia), and in grades 9-13 in the curriculum there are: formal changes of representations, basic graphs transformations, emphasis is put on detailed learning of the same classes of functions (power, root, quadratic, polynomial, rational, exponential, logarithmic, trigonometric) and sequences. The main difference between the two countries is that since 1990 in Slovakia there is no compulsory final exam in mathematics at the end of secondary school.

Mathematics Teachers' Specialised Knowledge model

In our research, we use *Mathematics Teachers' Specialised Knowledge* model (MTSK, Carrillo et al., 2018) as a tool for analysing the PSTs' knowledge of linking function graph and equation of linear function. Our focus here will be on three of the six subdomains of the MTSK model.

- **Knowledge of Topics (KoT)** – knowledge on functions and their representations, especially being aware and understanding of functions in different representations, being able to link different representations and to identify invariant properties of a function across its different representations as well as being able to use all the mathematical methods and knowledge to solve tasks, and to include those aspects of the function that are related to real contexts (scale issue).
- **Knowledge of features of learning mathematics (KFLM)** – in particular, knowledge that can be expressed by providing valuable explanation to student's solution. According to Carrillo et al., (2018) if the teacher provides valuable explanation to the student's solution, including an explanation of possible student's difficulties, problems, and misconceptions, this subdomain is concerned.
- **Knowledge of mathematics teaching (KMT)** – specifically in this study we focus on teachers' feedback to students' solution that could help students scaffold in the development of his/her functional thinking and understanding of functions. Such knowledge belongs to KMT subdomain (Carrillo et al., 2018).

Methodology

This work is a descriptive exploratory study focusing on the specialized knowledge of PSTs from Slovakia and Poland at the end of their studies (5th year of mathematical studies). The research question is: What specialized knowledge (**KoT**, **KFLM**, **KMT**), on linking linear function equation and graph is revealed in written contributions of PSTs?

Participants of our study were 37 PSTs from Poland and Slovakia. The Polish group consisted of 23 PSTs from the Pedagogical University of Krakow and the Slovak group consisted of 14 PSTs from Pavol Jozef Šafárik University in Košice. All the PSTs studied their last year of studies and had passed all the compulsory classes from pure mathematics and almost finished their teacher preparation (general pedagogy, psychology and didactics of mathematics).

The comparative approach towards results is not our priority because both countries have the similar curriculum and teaching tradition concerning functions. However, we identify some similarities and differences, which are communicated in the conclusion of the paper.

The research tool was composed from two questionnaires. The first one consisted of 8 tasks which were taken from the research of Ronda (2015). The second questionnaire consisted of three parts and each part consisted of 7 or 8 questions or tasks (22 questions or tasks in total).

In order to create the second questionnaire, we collected 105 Slovak high school students' solutions to the tasks from the first questionnaire (Šabáková, 2023). Students were 16-17 years old and had already completed the topic of functions, where they have encountered all the basic elementary functions and the basic properties of functions. From these students' solutions, we selected 3 solutions with typical argumentation (Ann's, Boris's and Daniels's – see Table 1) and 2 solutions with nonstandard argumentation to be analysed by the PSTs in the second questionnaire.

In this paper, we are going to focus only on one task from both questionnaires: Task 1 (see Figure 1) from the first questionnaire and chosen 5 students' solution of Task 1 were included in the second questionnaire.

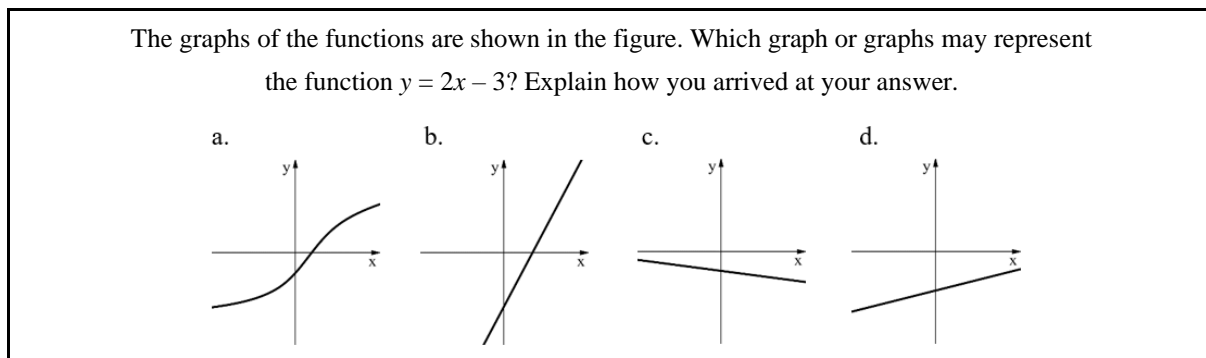


Figure 1: Task 1, first questionnaire

The task for preservice teachers was as follows:

Look at the following student's solutions to the task. Write at least one question in response to each student's solution. Explain why you chose that question.

Table 1 describes the provided 5 students' solutions to be analysed.

Table 1: Task 2.1, second questionnaire

Student	Answer	Solution (translation of the original solution into English)	What could be observed from the student's solution
Ann	b)	"The function is linear, has a linear term and an absolute term. The graph is a straight line. A $2x$ means the inclination of the line that it is closer to the y -axis, and -3 means that the graph is shifted down by 3."	Ann linked the graph and the equation based on the invariant property - it is a linear function whose graph is a straight line, and the local property - the coordinates of the intersection with the y -axis. She used informal language - "the inclination of the line is closer to the y -axis". Possible problems of the student: understanding the slope, scales on the x and the y axes needs to be symmetrical.
Boris	b)	"Solution using intersections: $P_x = \frac{3}{2}; P_y = -3$ "	Boris linked the graph to the equation based on the individual values. He noticed only the local properties of the function. Possible problems of the student: misconception concerning linearity - the graph of the function that pass through two given points is a straight line, scales on the x and the y axes needs to be symmetrical. The notation of the intercepts in the solution does not fit the convention.
Cecilia	b)	"The function $y = x$ passes through the point $[0; 0]$ and is a straight line (therefore a) drops out), $y = x - 3$ is a shift on the x -axis to the right by 3, therefore $f: y = 2x - 3$ is b ."	Cecilia linked the graph and the equation based on the global property - the graph of the function $y = 2x - 3$ is obtained by transforming the function $y = x$. She doesn't transform the graph in the correct way. Possible problem with understanding function as object. She doesn't consider different scales on x -axis and y -axis.
Daniel	b), d)	"It must be a rising line and start below zero. Graph d) has a different ratio of coordinates, but this is allowed."	Daniel noticed invariant properties that do not depend on the scaling on the x and y axes. He does not explain why the function is increasing and its intersection is in the negative part of y -axis. He used informal language - using the word rising instead of increasing function and writing about the start below the zero instead of the intersection with the y -axis.

Ed	b), d)	“b) d) the formula of the function tells us that $y = 2x - 3$, so $P_x: 0 = 2x - 3 \mid +3$ $3 = 2x \mid :2$ $1.5 = x \rightarrow$ positive number since P_x is positive, it's increasing f and there is no scale at b) d), theoretically both can be.”	Ed used the local property of the function - calculating the intersection with the x-axis. There is the misconception - he identifies the slope of the function with the intersection with the x-axis. Possible problem of the student: Misconception - linearity - the graph of the function that pass through two given points is straight line.
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The questionnaires were completed during the training classroom in each country. PSTs had 60 ± 10 minutes for completing the first questionnaire and 100 ± 10 minutes for the second one. Once the data for each country had been collected, we implemented qualitative content analysis (Mayring, 2015) using both deductive as well as an inductive approach to create a proper code scheme. When developing it, we have taken into account both theoretical framework together with the research questions and the results from Slovakia and Poland. We coded a separately posed question (q) asked of the student related to the solution (or a reaction given by the teacher to the student's solution) to each of the five students' solutions to explore KMT. And separately we coded an explanation (e) of the purpose of this question to explore KFLM. The coding scheme is provided in Table 2.

Table 2: Coding scheme for task 2.1, second questionnaire

Explanation of code	Abbr.	Example from PSTs' solutions
PST is not satisfied with use of Informal Language	IL	“What does it mean nearest to the y axis?” (q)
PST mentions that student have to be aware of different possibilities SCaling	SC	“The student is aware of the possibility of different scales.” (e)
PST mentions problem with Graph Transformation	GT	“The student should be aware that translating along the x-axis differs as the linear term changes.” (e)
PST mentions that student have to explain that it is Linear Function or that it is not sufficient to find intersection(s)	LF	“The student should realize that knowing two points of intersection does not tell him anything about the graph of a function.” (e)
PST wants student to Add Arguments	AA	“He doesn't explain why it is so, I need more arguments.” (e)
PST gives CounterExample to the solution	CE	“What about the item a)? Imagine the function $y = -2x + 3$ ” (q)
PST gives Positive Reaction on solution	PR	“I would like to praise the student.” (q)
PST accepts 2. Answer	2A	“Yes, the task can also be solved like that.” (q) (We also use this code if PST have two solutions in his/her solution.)
PST points out that second Answer is Missing	MA	“Why the item d) is not correct?” (q)
PST wants to have only 1 Answer	1A	“The student does not realize that the function is uniquely given and therefore cannot have two different graphs. He does not see the difference between the graphs.” (e) (We used this code only if PST explicitly wrote that student must have one solution.)
PST Lacks Calculations	LC	“The student did not undertake the calculation” (e)
PST wants more about Understanding of Slope – twice a value or inclination/tangent of angle	US	“Understanding the expression $2x$ not only as a tendency but also as twice the value.” (q)
PST identifies misconception Px and Slope	PxS	“If P_x is positive, must it be an increasing function?” (q)
PST wants student to find intersection Point with y-axis and/or x-axis	Pxy	“What about the intersection of the graph of the function f with the OY axis?” (q)

MISunderstanding the student's solution	MIS	(We used this code if PST accept wrong solution of the student or wrote that (s)he doesn't understand the solution or provided incomprehensible or incorrect feedback.)
PST pays attention on interpretation slope (Coefficient, Slope, Monotonicity)	CSM	"The student does not determine the monotonicity of functions as indicated by the coefficient <i>a</i> ." (e)
PST pays attention on another mathematical aspect (Specific Question)	SQ	"What the zero of a function means?" (q)

Both authors coded the data of the PSTs from their country after agreeing on the coding scheme and precise rules for coding. Then we had checked each other's assigned codes and agreed on the final version of the coding. In some PSTs' work, the question (q) to the student and the explanation (e) were intertwined. We take it into account when coding and distinguish what could be interpreted as feedback to the student (KMT) and what as an explanation, an attempt to understand student's solution and to identify his/her doubts or misconceptions (KFLM).

Results

With regard to the first questionnaire, the results of the PSTs were similar to those of the students (see Šabáková, 2023). PSTs from both Poland and Slovakia overwhelmingly indicated only answer (b), assuming the same scale on both axes (see Table 3), a typical misconception also in the students' solutions. In contrast to the students, none of the PSTs chose both options (a) and (c) as the correct answer. On the other hand, 2 PSTs chose option (b) only on the basis of generating values of the function (creating a table) or using other local properties (e.g. calculating intercepts).

Table 3: Results - Task 1, first questionnaire

	Only answer <i>b</i>)	Both answers <i>b</i>) & <i>d</i>)	Answer <i>b</i>) & doubts on <i>d</i>)
Slovak PSTs, n=14	10	3	1
Polish PSTs, n=23	19	2	2

We believe that these results are rather natural and were predictable. They are a consequence of habit and of the conventions of working in mathematics lessons. In both countries, the same units on both axes of the coordinate system are most often (or even always) selected during mathematics lessons, moreover, in both countries the most common formulation of questions is the multiple choice test with the only one correct answer. In addition, the PSTs rated this task as easy, which also favoured giving a quick, intuitive answer without making a longer analysis. However, it was important for us to see how the PSTs would react to the students' answers, in which Ed and Daniel indicated two correct answers.

The frequency of the codes that we have identified in PSTs reactions at Slovak (SVK) showed that the most common feedback was a counterexample (code CE, $\sum q=36$, $\sum e=6$), which they used to point out an incorrect or incomplete solution (see Figure 2, abbreviation Aq indicates PST's questions (feedback) to Ann's solution, Ae indicates PST's explanations to Ann's solution, Bq and Be are corresponding to Boris's solution, etc.). On the one hand, in some situations this can help the student to quickly see the mistake; on the other hand, sometimes it prevents the student from discovering the mistake, finding its sources and constructing the knowledge on his/her own. Very often, the use of a counterexample in the questions to the student corresponds to other code in the explanation (see codes in Figure 2) that contains the reason to construct the counterexample.

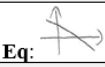
Aq: Is the function $\frac{x^2+2x+1}{x+1}$ linear? Ae: The student should realise that just because a function has a linear term and an absolute term does not mean that it is a linear function.

Eq: P_x is also positive. Is this function increasing? Ee: To see that the conclusion is wrong.
Cq: Does the argumentation justify the rejection of answer d? Ce: The graph of d is also a straight line and can arise as a shift to the right.

Figure 2: Examples of PSTs' work that mention counterexample

The most common feedback in Polish case concerned objections to the use of informal language by students (code IL, $\sum q=21$, $\sum e=20$), most often to solutions of Ann ($\sum q=10$, $\sum e=11$) and Daniel ($\sum q=9$, $\sum e=8$). This type of feedback was also common among Slovak PSTs ($\sum q=15$, $\sum e=10$) (see Figure 3). On the one hand, it is certainly encouraging that PSTs are trying to take care of the purity of mathematical language in students, but on the other hand, the focus on the language aspect overlooks the more important aspects of students' understanding of the topic.

Aq: What does it mean that the inclination of the line is closer to the y-axis? Ae: The student is not expressing himself correctly.
Dq: What does the expression 'there must be a line' mean? De: All graphs are drawn with a line (perhaps a straight line was meant).
Bq: What does it mean to solve by intersections? Be: I would like to check that the student understands what they are doing.

Figure 3: Examples of PSTs' work that mention informal language

Scale was mentioned quite often in the questions to the students and in the explanation of the students' solutions in both cases (code SC, $\sum q=29$, $\sum e=28$). It is interesting that in some PSTs' reactions it was the reason why solution d) is wrong and in some it served as an argument why solution d) is correct. Examples of such reactions to Edo's solution can be found in Figure 4.

Eq: Would answer d be correct if we keep the same scale on both axes? Ee: I would like to clarify if the student can see that in point d we need to select different scales for both axes and not just different scales than in b.
Eq: If there is the same scale on the x-axis and the y-axis in the graphs in the task which possibility will be correct one? Very good point. It is also possible to solve the problem in this way. Ee: We must say, that the task has more than one solution. The teacher has to add scale to it (to get only one correct solution).
Eq: What is the ratio of the numbers $3/2$ and 3 ? Is it preserved in d? Try to measure it with your finger. Ee: Graph d would suggest that $ -3 < 3/2 $
Ee: (...) but if "there is no scale" then perhaps "2x" actually tells us something - leading us to eliminate answer d.

Figure 4: Examples of PSTs' work that mention scaling

In the Polish case, despite Ed's and Daniel's correct answers, the majority of PSTs did not accept the second answer (on 25 occasions we could notice the strong belief that only answer b could be the correct answer – code 1S). In the Slovak case, the explicit exclusion is in 8 occasions but 4 other PSTs avoid to make any comment to second solution of Daniel and Ed although it is in contradiction to their solution of the task. Some of PSTs were so convinced that judge the students' proper arguments as methodologically incorrect. On the other hand, it is positive that some of PSTs' influenced by the students' solutions, made a reflection and changed their own answer and expected in their feedbacks the second answer in solutions of Ann, Boris & Cecilia. This demonstrates their analytical thinking and reflexivity so much needed in the mathematics teaching profession.

Conclusion

PSTs' ability to provide feedback to the student that can scaffolds his/her learning, as well as deliver valuable explanations of the students' solution, were demonstrated in a number of responses. They were able to identify common misconceptions about learning functions such as linearity, relationship between x-axis intersection and slope, understanding and interpreting the importance of coefficients in linear function equation, difficulties with graph translation, etc. However, the misconception about scaling - that scales on axes must be symmetrical - is

not identified by most of them. Moreover, there is an evidence that they have this misconception. On the other hand, the task presented in the second questionnaire has the potential to disrupt this misconception. In both countries, when studying functions at university, PSTs encounter the graph representation of a function separately from applications and sciences. Scaling issues seem irrelevant in the abstract study of functions, where an understanding of the function as an object is already expected. This gap in the preparation of PSTs in both countries, concerning their KoT - category phenomenology and application, should be addressed in the future. The gap in their KoT affected their understanding of the students' solution (KFLM) and the feedback they gave to the student (KMT).

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