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# A blueprint for measuring mathematical knowledge for teaching college algebra at community colleges

Vilma Mesa<sup>1</sup>, Inah Ko<sup>1</sup>, Claire Boeck<sup>1</sup>, Irene Duranczyk<sup>2</sup>, Bismark Akoto<sup>2</sup>, Dexter Lim<sup>2</sup>, Patrick Kimani<sup>3</sup>, Laura Watkins<sup>3</sup>, April Ström<sup>3</sup>, and Mary Beisiegel<sup>4</sup>

<sup>1</sup>University of Michigan-Ann Arbor, USA; [vmesa@umich.edu](mailto:vmesa@umich.edu)

<sup>2</sup>University of Minnesota, USA

<sup>3</sup>Maricopa Community Colleges, USA

<sup>4</sup>Oregon State University, USA

*We present the blueprint for an instrument that is designed to measure mathematical knowledge for teaching college algebra at community colleges in the United States. We begin by providing an account of efforts to describe and measure teacher knowledge in the context of practices associated with teaching mathematics, followed by a more detailed description of the approach chosen for our instrument including its composition of the hypothesized dimensions of the knowledge and sample items. We conclude with a discussion of what we have learned and possible limitations of our approach.*

*Keywords: Teacher knowledge, tasks of teaching, college algebra, postsecondary education.*

Our goal is to develop and validate the Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA), an instrument measuring the mathematical knowledge used in teaching college algebra at community colleges<sup>1</sup>. Specifically, we seek to identify multiple dimensions of this knowledge and instructor characteristics that may explain test performance on the proposed dimensions of the knowledge. Our position is that the knowledge that instructors need for teaching is fully realized through their practice as mathematics teachers (Rowland, 2020). For this reason, we focus on core tasks of teaching that generate opportunities for the hypothesized mathematical knowledge to emerge. In this paper, we begin by presenting the theoretical support for the instrument we are developing and its proposed dimensions. In the second section, we justify our choice of the research setting (community colleges and college algebra) and illustrate the differences in the hypothesized dimensions with examples that seek to target them. In the discussion, we present what we have learned from item development interviews and describe limitations that may impact our results. We are developing this instrument to measure the effectiveness of professional development programs geared to increase community college instructors' knowledge for teaching college algebra, a course that is taken by about 300,000 students in the U.S. (Blair et al., 2015, p. 162), as it is a steppingstone into a major that requires a calculus sequence.

## Efforts to Describe and Measure Teacher Knowledge

Research that seeks to establish the components of teacher knowledge has hypothesized distinct components of that knowledge theoretically, but most of them have failed to provide empirical

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<sup>1</sup> Community colleges are tertiary-type B institutions that offer courses in the first two years of post-secondary education (OECD, 2017)

support for the boundaries between those components (e.g., Copur-Genturk et al., 2019). The research consensus is that the knowledge used by teachers in their work is multidimensional. Sustained work to corroborate that mathematical knowledge for teaching is indeed composed of these distinctive dimensions has proven very difficult (Schilling, 2007). Some of the research groups (e.g., MKT, COACTIV, TEDS-M, DTAMS) further specified content knowledge or pedagogical knowledge according to cognitive level, knowledge type, or mathematical topic (Ball et al., 2008; König et al., 2011; Krauss et al., 2008; Saderholm et al., 2010).

On the other hand, Herbst and colleagues have been interested in mathematical knowledge for teaching because of a more general interest in understanding teacher decision making in teaching that is subject-specific (Herbst & Chazan, 2012; Herbst & Ko, 2019). Their work is grounded on the notions of instructional exchanges, to understand from the perspective of the teacher, how knowledge is “transacted” in classrooms. As part of their job, teachers are expected to provide students with activities or assignments that would support the development of mathematical ideas; as students produce work in response to those activities and assignments, the teacher will need to interpret the work to ascertain whether the student has learned or not. These two activities, giving problems and understanding students’ work, require teachers to sort through their own mathematical knowledge to make decisions that are guided by the goal of the instructional situation at stake. Ko and Herbst (2020) asserted that teachers’ mathematical knowledge used in the two tasks of teaching is indeed distinguishable by establishing two distinct (albeit correlated) measures for knowledge needed to choose givens for a problem and understanding students’ work. They surmise that the main challenge previous studies had in developing multiple measures is related to their knowledge-type-based framework given that there is no clear boundary among the types (e.g., common and specialized content knowledge) and teachers need to use multiple dimensions of knowledge simultaneously in the moments of teaching.

Following Herbst and colleagues’ approach, we structured our instrument by tasks of teaching. As the work of teaching is composed of core tasks teachers do to help students learn (Ball & Forzani, 2009), it is reasonable to structure the knowledge used in the work of teaching mathematics by core tasks of teaching. When we talk about tasks of teaching, we are referring to what the literature identifies as separable components related to the work of teaching. The core tasks of teaching used in this study are *choosing problems* and *understanding student work*. We define *choosing problems* as a task of teaching that requires instructors to select activities or assignments that represent mathematical ideas accurately and give students opportunity to work on a mathematical idea at stake and define *understanding student work* as a task of teaching that requires teachers to make sense of the ideas that students produce, either orally or in writing, and evaluate their mathematical correctness. Each task demands knowledge specific to teaching mathematics. For *Understanding student work*, a teacher needs to have knowledge about interpreting students’ work, typical errors students make, various strategies for solving a problem and how they could be represented, and the mathematically correct answer. *Choosing problems* requires knowledge about mathematical ideas presented in the problem, and about how to assess if ideas in the problem align with an instructional goal.

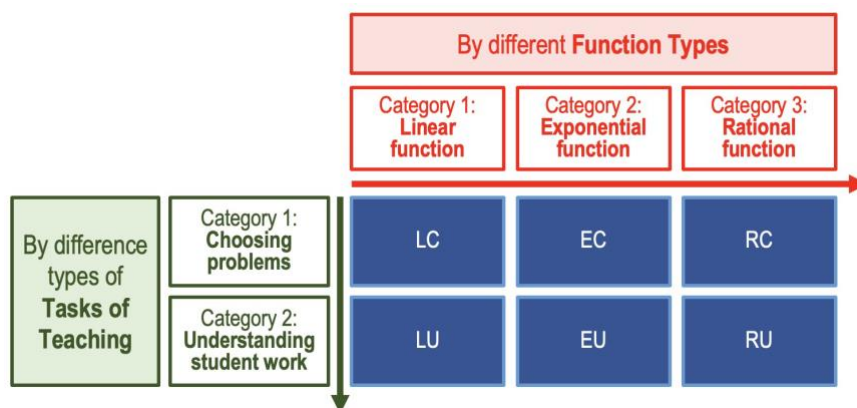
## Mathematical Knowledge for Teaching Community College Algebra

We hypothesize the knowledge that instructors need to *choose problems* for teaching college algebra at community colleges can be distinguished from the knowledge that instructors need to *understand community college students' work* in college algebra. These two tasks of teaching cut across several of Ball and colleagues' (2008) hypothesized dimensions. For example, in our conceptualization of the knowledge needed for *choosing problems*, instructors need to have both Specialized mathematical content knowledge and Knowledge of the students they are teaching to make such a decision; therefore, it is not limited to Ball's knowledge of content and students.

Our instrument is situated in the context of community college algebra courses. Community colleges specifically represent a meaningful setting for the development of the instrument structured by the two tasks of teaching, *understanding student work* and *choosing problems*. This is because community colleges in the United States have open admission policies that tend to serve students from their local communities with various degrees of preparation and understanding of the material. We presume that within any community college classroom, the diversity in students' educational backgrounds will result in great variation of student work that will require mathematical knowledge that enables instructors in making sense of their students' thinking and identifying errors in students' work that might not be typical (*understanding student work*). Similarly, instructors will need to have mathematical knowledge that enables them to choose problems that can support both individual growth and engagement of the whole class. The students' varied prior preparations might create more opportunities for instructors to engage in creating or modifying problems for their students (*choosing problems*). Likewise, individuals who teach at community colleges, may have pursued diverse career paths to become instructors of a college algebra course. In addition to full-time faculty, some instructors might be high school teachers, others are professionals in engineering or business, and others graduate students at a local university. This wide range in teacher educational and teaching backgrounds may be associated with differences in their mathematical knowledge. The heterogeneity of the level and characteristics of knowledge possessed by our instructor population would enable us to better identify differences in the dimensions of the knowledge measured by our instrument. More importantly, community college instruction has been characterized as interactive lectures (Blair et al., 2015) in which instructors bring problem examples to anchor the presentation of the material and solve them collaboratively with their students, and thus, these instructors will be engaging in both tasks of teaching (Burn & Mesa, 2017; Mesa & Herbst, 2011).

These tasks of teaching need to be realized in concrete mathematical content. Across the US, there is great variation in the content that is covered, but most college algebra courses explore various types of functions, including polynomial, rational, trigonometric, and exponential. Substantial research in K-12 and teacher preparation contexts suggest that notions of covariation, function transformation, algebra of functions, rate of change, behavior of a function over a whole domain, and exponential growth and decay are difficult for students, yet these are essential for building a conceptual understanding of ideas in a calculus sequence—a sequence required for many students interested in pursuing a degree in science, technology, engineering, or mathematics (STEM, Thompson & Carlson, 2017). This course also has a reputation as a gatekeeper for STEM fields, with reports that between 30% and 80% of students fail it every year (Gonzalez-Muñiz et al., 2012). To select topics covered

by our items across these two tasks of teaching, we considered content that is foundational for building an understanding of mathematical functions and that is typically taught in college algebra courses in the US. We chose linear, exponential, and rational functions, as related concepts (e.g., covariation, function transformation, algebra of functions, rate of change, behavior of a function over a whole domain, etc.) are fundamental for building a conceptual understanding of ideas and are needed for courses within a calculus sequence. The Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA) instrument that we developed includes 34 multiple-choice and testlet items balanced across two organizers, tasks of teaching and topics, resulting in a proposed knowledge structure with six dimensions (Figure 1).



**Figure 1: Blueprint for the development of the MKT-CCA instrument**

Figures 2 and 3 present examples of items measuring the knowledge used in *choosing a problem* and in *understanding student work* in exponential functions respectively. We will use these items to illustrate how the knowledge dimensions can be mapped onto the types of work community college instructors do. First, these two tasks are mathematical because they require instructors to use their mathematical knowledge to make sense of them. But they could be seen as distinct in nature, whereas they both require managing some kinds of exchange between an instructor and students. In choosing a problem (Figure 2), instructors must sort through various possibilities to determine how the chosen problem can help achieve an instructional goal. For example, assess whether the terms used in a problem (e.g., exponential growth patterns) align with those learned from other sources, such as textbook problems, prior lesson plans, online resources, or curriculum documents. In addition, instructors need to assess whether there are any mathematical errors in the given information, and whether all the mathematical elements presented in the problem are consistent. In this case, prior experience with mathematical ideas and how they interact with each other would be useful in choosing a problem. But in the second task (Figure 3), the claim that an instructor needs to make is of a different nature; in this task, the instructor needs to claim that the students either have or have not learned a particular piece of the mathematical knowledge at stake—in this case, interpreting the role of the parameter  $\frac{1}{5}$  in the expression of an exponential function based on the responses from the students. In this case, prior experience grading students' work, answering student questions, or even tutoring would be useful for interpreting a student solution. For these reasons, we claim that these two tasks, while relying on mathematical knowledge, are essentially different. In addition, the management of the instructional exchange is different for each task of teaching. *Understanding*

*student work* starts with student work for the problem and points to the mathematical knowledge at stake, whereas *choosing problems* starts with the mathematical knowledge at stake and goes to anticipate student work on the problem. Second, we surmise that given the context in which we are situating our work, it is more likely that instructors will have more experience understanding student work rather than choosing problems, as it is more likely for instructors to use college-approved materials in their work (which obviates the need to select or create problems for teaching). Thus, we conjecture that items related to this type of knowledge will be more difficult to answer as teachers have less experience choosing problems.

Mr. Trevena is creating a lesson about applications of exponential functions. Which of the following news headlines would be the most likely to provoke a discussion about exponential growth patterns? **Choose one option only.**

- A. Census data show that the population of the city tripled from 1970 to 1980 then doubled from 1980 to 2000.
- B. Prices have increased 7% over the past 5 years, while usage of public transportation has decreased 5% over the same period.
- C. Twice as many single people ages 20-25 are living with their parents today as they were 20 years ago and economists predict this trend will continue.
- D. The water level of the local river has increased by 13% in the past decade and 35% during the past 30 years.

**Figure 2: Item addressing choosing problems in exponential functions**

Ms. Ucci's class has just finished discussing the notation of  $f(t)=a(b)^{rt}$  and the meaning of parameters and variables for exponential functions. To assess her students' understanding she gave them this task:

A radioactive substance decays according to the function  $A(t)=100(1/2)^{(1/5)t}$  where  $t$  is measured in years and  $A(t)$  denotes the amount of the unstable portion of the substance in micrograms at  $t$ . What is the meaning of  $1/5$  in the function?

Which of the following students' responses demonstrates a correct understanding for the meaning of  $1/5$ ? **Choose one option only.**

- A.  $1/5$  means that the substance needs to decrease four more times to get to half of its original amount.
- B.  $1/5$  means that  $1/5$  of the substance  $A$  is left after half of a year.
- C.  $1/5$  indicates that it will take five years for the substance  $A$  to reach its half life.
- D.  $1/5$  means that  $1/5$  of the substance  $A$  has decayed in half of a year.

**Figure 3: Item addressing understanding student work in exponential functions**

## Preliminary support for our working hypotheses

We aim at establishing distinguishable measures of mathematical knowledge for teaching college algebra at community colleges using an assessment-type instrument. Each item is designed to present the knowledge used in one of the two tasks of teaching—*understanding student work* and *choosing problems*, across the three mathematical functions—*linear*, *rational*, and *exponential functions*. We provided the theoretical background and rationale supporting our hypothesis on the distinctions among the dimensions of the knowledge. As we are still collecting data via an online survey, we share some findings that support the differences between the dimensions based on item development interviews and our own experiences as researchers. Item development interviews are considered a best practice in item development (Meadows, 2021) and were performed to assess if participants were using the anticipated knowledge to answer the items and to improve the items being written. We

conducted these interviews in May 2021 with 12 community college faculty using 36 items. The interview data suggested differences among our hypothesized knowledge dimensions with respect to the participants' cognitive process in solving the problems and the types of functions involved. For example, to answer a *choosing problems* item, teachers evaluated whether numbers used in a problem make sense in a real-world situation as misalignment between numbers used in a problem and their intended meaning can cause confusion (e.g., in the context of rational function items: "because a vertical asymptote ' $h$ ' would be a negative number and a negative height above the earth's surface might be misleading to a student", "[option D] is not a practical example because if  $x$  gets smaller and smaller, the rectangular box is, you know, basically disappearing, so I don't think that's a good example for students to consider as  $x$  approaches zero."). On the other hand, when solving *understanding student work* items, instructors tried to infer students' thinking or strategies using students' statements or work and decide whether the thinking process or answer was mathematically correct. For example, instructors read a student's statements or written work, determined whether the inferred process was correct, and tried to identify the error of the student's logic (e.g., "student B's answer would seem to imply that the  $y$ -intercept would be zero, which is not correct, so student B incorrectly set up the linear function."). Likewise, when asked about the mathematical topics the items were addressing, participants identified the three types of functions in the items (e.g., "I'd typically use similar examples like this to ask students to translate an application into a linear function", "I think you can infer that this has to do with rational functions", "this is probably a topic on exponential functions").

### **Limitations and next steps**

One possible limitation we are anticipating in distinguishing knowledge along the tasks of teaching dimension is that our task, choosing problems is much less specific than Ko and Herbst's (2020) as they focused exclusively on choosing *givens* (set of givens that has enough information for students to solve the problem) for problems in geometry. Their specificity might have contributed to make a clear distinction possible. Given that in our study, *choosing problems* is a task of teaching that requires instructors to select activities or assignments that include more than specific givens, the broader definition of the task may cause multidimensionality within the choosing problems dimension. A second limitation is related to the format in which the test is given, online, with a request for faculty not to consult other sources. However, our interest in reaching faculty with diverse experiences has made this format the most appropriate for our work: the variation can help us in identifying differences along the hypothesized dimensions. Our data collection, which requires 600 community college faculty will conclude in April 2023, and at that time we will start the psychometric modeling of the data to assess dimensionality, item difficulty, and discrimination; we are also recruiting a sample of 100 undergraduate students without teaching experience to take the instrument, to investigate whether teaching experience is associated with the knowledge measured by our instrument. We hypothesize a difference in the average item difficulty between the two tasks of teaching: we surmise that choosing problem items can be more difficult for less experienced participants. This is because when they choose problems for their students, novice instructors often overlook whether there is something mathematically important for their students to learn when solving the problem (Fraser et al., 2019). The task of choosing problems, however, requires teachers'

mathematical knowledge to check and validate the correctness of the mathematics problems and their fit with their instructional goal, given the content already covered; in other words, teachers cannot assume that the problems given in the curriculum (or found on the internet) are mathematically correct or appropriate for their instructional goals.

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