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Knowledge of different forms of complex numbers through quantitative reasoning: The case of teachers

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This study investigated the development of five secondary mathematics teachers' conceptualizations of different forms of complex numbers through quantitative reasoning in a PD program. We report from the pre and post-written sessions with the post-interview data upon completion of a PD. Results showed that all the participants could relate the formal definition of complex numbers with the roots of quadratic equations both algebraically and geometrically. Participants could also explain the Cartesian and polar form relationship using vectors by mentioning the roots of quadratic equations. They further explained the Euler form by pointing out that the polar form of any complex number on a circle determines a function of θ from R to C . These results suggest that quantitative reasoning might lay a foundation for connecting different forms of complex numbers.

Keywords: Complex numbers, teacher knowledge, quantitative reasoning.

Background

From the early grades towards higher education, understanding numbers is essential in learning mathematics and science concepts. Specifically, researchers stated that a complete conception of complex numbers necessitates geometrically thinking of a complex number as a point on the complex plane and a vector (Fauconnier & Turner, 2002). Also, algebraically, “a complex number should be conceptualized as one number, i.e., the expression $a+ib$ is a single entity combining a real number and an imaginary number” (Nordlander & Nordlander, 2012, p. 633) such that they are mathematical objects in a well-defined set consisting of elements of the same kind or a particular category (Sfard, 1991). Researchers also pointed out that a robust conception of complex numbers includes knowing the algebraic and geometric representations of the Cartesian, polar, and exponential forms, making sense of their connections, and the flexibility among them (Karakok et al., 2015). Similar to Turkish curricula, Hodgen et al. (2010) pointed that several OECD countries such as Singapore, Japan, and the USA cover complex numbers as a subject in the later years of high school to prepare learners for college-level mathematics courses.

However, research studies indicated that secondary teachers (Conner et al., 2007; Karakok et al., 2015) have difficulty in understanding complex numbers. Specifically, working with three secondary school mathematics teachers on their connections of Cartesian, polar, and exponential forms of complex numbers, Karakok et al. (2015) showed that one teacher had difficulty visualizing complex numbers as points on the complex plane. Similarly, once she needed to represent ‘ i ’ as a point, “..she was hesitant whether it was located one unit up from the origin...” (p. 339). Also, two teachers “...both had difficulties relatingvector representations of the Cartesian form” (p. 345). Furthermore, one teacher “did not recognize how and why $re^{i\theta}$ represents a complex number or what the symbols represent” (p. 340).

Similarly, prospective teachers did not reconnect complex numbers to the roots of quadratic equations (Conner et al., 2007). Mathematics teachers are key in the preparation of students for their careers (NCTM, 2000). Therefore, they need to have an "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Specifically, it is important for teachers to have understanding of complex numbers (Karakok et.al., 2015).

We propose that while developing teachers' conceptions of complex numbers, quantitative reasoning might provide a robust thinking process for teachers. Johnson (2016) pointed the importance of quantitative reasoning in science and mathematics-related ideas. Thompson (1990) defined quantitative reasoning as a person's "analysis of a situation into a quantitative structure" (p. 13). From Thompson's (1994) perspective, "quantities are conceptual entities" such that "a person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability" (p. 184). We regard complex numbers as a union of two quantities as directed distances obtained from the analysis of a situation (a mathematical object such as quadratic functions) into a network of quantities, which are the roots and the abscissa of the vertex as distances to the origin and quantitative relationships. By quantitative relationships, we mean how the roots and abscissa of the vertex position on the number line as a point and as a distance and relate to each other. Based on the aforementioned studies and the theory of quantitative reasoning, Saraç and Karagoz Akar (2017) studied a prospective teacher's development of the Cartesian form of complex numbers. Results showed that the prospective teacher could conceptualize complex numbers as a single entity, an element of a well-defined set. Notably, she conceptualized complex numbers as the set of the roots of any quadratic equation with real coefficients.

Building on and extending previous research on complex numbers through quantitative reasoning, quadratic equations might be a starting to build up complex numbers. Further, this might help develop teachers' connections among different forms (Cartesian, polar and Euler forms) of complex numbers and also different representations (algebraic and geometric) of these forms (See Figure 1) by reasoning quantitatively.

Representation	Form	Cartesian ($a + bi$)	Polar $r(\cos \theta + i \sin \theta)$	Exponential $re^{i\theta}$
	Purely imaginary i			
Algebraic	$i, \sqrt{-1}, (0, 1)$	$a + bi, (a, b), z$	$r(\cos \theta + i \sin \theta), z$	$re^{i\theta}$
Geometric	A point on the complex plane, a rotation operator	A point on the complex plane, a vector with a magnitude of $\sqrt{a^2 + b^2}$ and an angle of $\tan^{-1}(\frac{b}{a})$ with the positive real axis, a rotation and dilation operator	A point on the circle centered at the origin with radius r , a vector with magnitude of r and an angle of θ with the positive real axis, a rotation and dilation operator	A vector with magnitude of r and an angle of θ with the positive real axis, a point on the circle centered at the origin with radius r , a rotation and dilation operator

Figure 1: Different representations of the forms (Karakok et al., 2015, p.329)

Thus, in this study, we investigated secondary school mathematics teachers' conceptualization of complex numbers by focusing on the following research question: How do secondary school mathematics teachers conceptualize the connections among Cartesian, polar, and Euler forms of complex numbers through quantitative reasoning?

Method

This study was part of a design-based research, a Teacher Development Experiment study (TDE), which mainly focused on teachers' professional development (PD) for the progression

of the content knowledge of complex numbers. TDEs consist of both classroom teaching experiments and (multi) case studies (Simon, 2000). In this paper, we reported from the multi-case study focusing on teachers' existing knowledge base upon completion of the PD. The PD involved four teaching sessions between 120 and 150 minutes. The first two sessions were about the Cartesian form of complex numbers in relation to quadratic equations through quantitative reasoning (Saraç & Karagoz Akar, 2017), the third one was on polar form, and the last session was on Euler form. Each form was introduced by linking it with the forms that were discussed in the previous sessions. As the nature of DBR studies, the PD dwelled on the theory of quantitative reasoning by considering complex numbers in a quantitative structure as a union of two quantities, which are directed distances and the roots of any quadratic equation with real coefficients.

Participants were five mathematics teachers who were graduates of a secondary school mathematics education program with two to ten years of teaching experience. Before the PD, ten teachers completed a pre-written session on complex numbers. They were asked to algebraically and geometrically define quadratic functions and equations, different forms of complex numbers, and vectors. Based on the preliminary analysis of the pre-written session, eight participants were purposively selected based on the following criteria: They knew quadratic functions and defined and expressed vectors, which were considered background knowledge. However, they did or did not state complex numbers in the cartesian form, polar, and Euler forms and did not explain the relationship between different forms, which were considered as the pedagogical goals (Simon, 2000). Nonetheless, five participants declared their availability to attend the study. After the completion of PD, a post-written session was conducted. Then, video-recorded semi-structured interviews were carried out by the researcher who also implemented the PD to collect data about how the participants conceptualized the connections among different forms of complex numbers. Interviews lasted about 30 min. to 45 min. The written artifacts from participants were also collected.

The constant comparative method (Clement, 2000) was carried out for data analysis. To characterize how teachers conceptualize the links between different forms of complex numbers, the research team collectively read the participants' written responses and transcripts of all the interviews, watching the video recordings when necessary. The unit of analysis was the participants' statements in the transcripts ranging from a sentence to a whole paragraph on each interview question. We did the analysis using the constructs of quantitative reasoning. The analysis focused on how the teachers were making sense of complex numbers quantitatively and explaining the connections between different forms. We analyzed the data first for each participant and then for each question from different participants. Comparing and extracting the commonalities and differences among the thinking processes of different participants, we wrote narratives about how they conceptualized different forms of complex numbers through quantitative reasoning.

Results

In this section, we report on teachers' definitions of complex numbers, their explanations of the relationships between Cartesian and polar forms, and the relationships between the polar and Euler forms. In Figure 2, we share the codes and corresponding sample data that emerged from

participants' written artifacts. In the following sub-sections, we also depict particular participants' reasoning from interview data to provide detailed explanations.

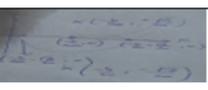
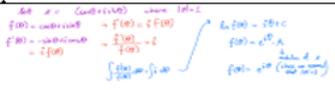
Defining complex numbers		
Codes	Teachers	Sample Data
Formal definition	All	T2: The numbers which are defined as $a+bi$ where a and b real numbers are called complex numbers.
Definition using the roots of quadratic equations	All	T3: 
Algebraic meaning of a and b using the roots of quadratic equations	All	T4: 
Geometric meaning of a and b using the roots of quadratic equations	All	T1: The roots are equidistant from the vertex. 
Relationship between Cartesian Form and Polar Form		
Codes	Teachers	Sample Data
Using vectors	T1, T2, T4, T5	Cartesian form of a complex number is $z=x+iy$ where x and y are real numbers and $i = \sqrt{-1}$. Polar form is written by using trigonometric ratios in right triangle formed by the vector of the complex number and real axis. x component is represented by $x = \sqrt{x^2 + y^2} \cdot \cos \theta$ and y component represented by $y = \sqrt{x^2 + y^2} \cdot \sin \theta$. If we plug in these representations in cartesian form, then we'll get the polar form of the same complex number.
Using formal definition (limited)	T3	T3:  $r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$
Relationship between Polar Form and Euler Form		
Codes	Teachers	Sample Data
Through function ideas	T1, T2, T4, T5	T2: 
Through the commonalities in both forms	T3	T3: In the relationship with polar form, in both of them actually we use θ the angle and modulus. 

Figure 2: Codes and corresponding sample data

Defining complex numbers

In this section, we report on teachers' definitions of complex numbers. As Figure 2 showed, all the participants formally defined complex numbers in the form of $x+iy$ where $x, y \in \mathbb{R}$. They also defined complex numbers as the roots of quadratic equations with real coefficients. More interestingly, they made the connections between the two definitions both algebraically and geometrically. Below, we provide a participant's, T5, reasoning from the interview. When asked how she would define complex numbers, T5 stated, "We can define complex numbers as the roots of quadratic equations... But of course, we state the format of numbers as the following: the numbers that can be written as $a+ib$ where a and b are real numbers."

When asked about the relationship between these definitions, she clarified her statement about the coefficients and stated:

T5: Let's say $x+iy$ to be more accurate because a and b we're talking about, these are the coefficients in $a+ib$... They are not the real coefficients of the second-degree equations (Referring to $ax^2 + bx + c = 0$). Therefore, if I say $x+iy$ it will not be confusing. So x is $-b/2a$, and y is $\sqrt{(-\Delta)/2a}$ so the format of the root of quadratic equations can be written as $-b/2a \pm \sqrt{-1} \sqrt{(-\Delta)/2a}$.

She also sketched a 3D figure and placed the roots on the figure as follows:

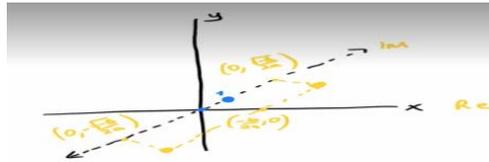


Figure 3: T5's placing $(-b/2a, 0)$ and $(0, \pm\sqrt{-\Delta}/2a)$ on the Complex plane.

These data showed that T5 comprehended the formal definition of complex numbers regarding the general form of the roots of quadratic equations. Specifically, she stated that "x" in the expression $x+yi$ refers to " $-b/2a$ " and "y" refers to " $\sqrt{(-\Delta)}/2a$ " when compared to the roots of quadratic equations. T5 also mentioned geometric meanings of "x" as the abscissa of the vertex of the parabola and "y" as the distance between one root and the abscissa of the vertex of the parabola. She also showed the x and y components as points on the imaginary and real axis in the figure.

Relationship between Cartesian Form and Polar Form

As Figure 2 depicted, the codes and the corresponding sample data that emerged from participants' explanations showed that all participants except T3 could relate these two forms using vectors. Below, we provide T4's explanations from the interview as she provided detailed reasoning in her response. T4 drew and wrote down the following (See Figure 4a,b,c) about the polar form.

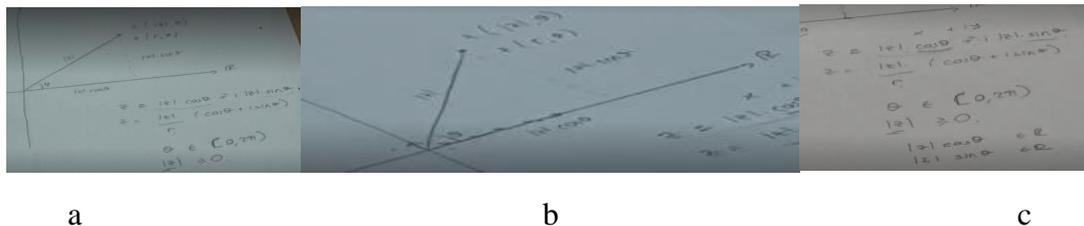


Figure 4: T4's written work for polar form

As shown in Figure 4, T4 stated the polar form of complex numbers relating the components "x" and "y" in the Cartesian form with $|z|\cos\theta$ and $|z|\sin\theta$ in the polar form, respectively, by pointing to the possible values of θ and $|z|$. She also stated that depending on the changes in θ , she could think of the maximum and minimum values of $\cos\theta$ and $\sin\theta$ as 1 and -1, respectively. Then, stating, "...but when we multiply with $|z|$, positive for instance, again we get a real number. Actually, that is why we say x and y are real numbers...", she could obtain any value of "x" and "y" as real numbers. This showed that she could explain why "x" and "y" are real numbers using the polar form. When further probed, she explained:

T4: If the angle is 0, the point is on the real axis. If I form an angle, if there exists an angle, I can always find the cosine and sine values in the right triangle. Actually, after writing the cosine and sine in the right triangle I write in this form (referring to $|z|\cos\theta$ and $i|z|\sin\theta$) but when I separated the components as vectors actually there is a direction on real axis and there is a direction in the imaginary axis. If I think like a vector, as u and v being added end-to-end as many as that (pointing to $|z|\cos\theta$ and $|z|\sin\theta$) I can write these two components. According to the angle, its adjacent part is cosine and the opposite part is sine therefore, let's call $|z|$ or r, $r(\cos\theta + i\sin\theta)$, we might think why "i", but I transit from real to imaginary axis, so I first multiply with "i". Therefore, I show $i\sin\theta$, and they become cis. If I calculate little by little,

our vector is 1 in here, “i” in here (referring to unit vectors on the axes). As far as the distance of this point (z) over there, it becomes this one such as this point’s (referring to $(|z|, \Theta)$ projection on x axis is this (referring to $|z|\cos\Theta$), on y axis is this $(|z|\sin\Theta)$... I do vector addition. Okay, I will say 1 to this (referring 1 on the real axis). I add the vector “1” as much as $|z|\cos\Theta$ and I make the x component from here, when I express it in binomial form it becomes $|z|\cos\Theta$. In here x and y parts (pointing to $x+iy$) so they correspond to $-b/2a$ and $(\sqrt{4ac - b^2})/2a$ respectively...

Importantly, data showed that T4 justified the polar form by considering vectors. Specifically, taking first the angle Θ as zero, she considered the x component as a point on the real axis having a magnitude and direction. Notably, she considered x made up of iterated additions of unit vector 1 until $|z|\cos\Theta$. Similarly, she considered y as a point on the imaginary axis having a magnitude and direction where she considered y (i.e., $|z|\sin\Theta$) as iterated additions of unit vector i, with magnitude 1. This allowed her to reason that z is a vector and further yielded her thinking that the “+” sign refers to the vector addition. Therefore, she thought of “x” and “y” components of complex number z both geometrically as points on the related axes having distances to the origin; and algebraically referring to “ $-b/2a$ ” and “ $(\sqrt{4ac - b^2})/2a$ ”, and, $|z|\cos\Theta$ and $|z|\sin\Theta$ in a related way. This suggested she conceived a complex number z in a quantitative structure, referring to simultaneously algebraic and geometric (points and vectors) representations.

Relationship between Polar Form and Euler Form

As the codes and the corresponding sample data in Figure 2 showed, all the participants except T3 could relate polar and Euler forms using function ideas. Below, we again provide data from T4’s interview, where she explained her detailed reasoning about what she wrote in the post-written session (see Figure 2). When asked, T4 first mentioned all the participants' discussions in PD. She stated that when asked whether the complex number in polar form (referring to Figure 4a) could determine a function, they first thought about x and y values in the Cartesian plane making up a circle. This led them to decide that the values would not determine a function. She further commented that they need to consider the relationship between the variables in the Complex plane. Specifically, she knew that the variables she was working with were Θ and the ordered pair $(|z|\cos\Theta, |z|\sin\Theta)$. Her consideration of the correspondence of the points from a point to plane leveraged her reasoning that the relationship between variables represents a function. This is important because her thinking suggests that she conceived the ordered pair $(|z|\cos\Theta, |z|\sin\Theta)$ as one and only one entity. She also knew that the values of Θ were real numbers. Then she continued commenting on her written work: as follows:

- R: ...In here you have a statement like this “ $z \in \mathbb{R} \rightarrow C_1$ ” can you explain this?
T4: If I determine the region accordingly, and say the regions the complex number might be located on, I create a small set. When I think of the unit circles infinitely many points might be located on and extend these circles I can construct complex number set. When I say C_1 , I get the circle according to the complex number that I choose...I determined one point for instance according to z let’s say this point. The locations that the point might be is here according to radius and Θ ...Therefore, if I define a small set I can call this one as C_1 . In this way, the values of $|z|$ are from zero to infinity. Starting from the beginning, I can actually scan an infinite number of circles like this. We can say circles or interior circles. The union of these infinitely many C ’s construct the complex number set...

R: Okay, can you tell us again how this constructs a function?
T4: According to the Θ value, Θ value in here it goes to $|z|\cos\Theta$ or $|z|\sin\Theta$. Right now, according to the value of Θ , I determine two values, “x” and “y”. Therefore, in here the variable for me is Θ . Then, I can define the function that sends Θ to $(\cos\Theta, \sin\Theta)$.

Here, she further thought about the range of the function in the Complex Plane such that she knew that the points in the range form a particular set, C_1 , depending on the radius of the circle. She also indicated that as the radius changes, increasing from 0 to infinity, the circles get bigger. This resulted in her thinking that these points on infinitely many circles form the Complex Plane. This suggested that the circles referred as C_1 also changes, which further indicated that she thought of the Complex Plane as a union of circles.

Conclusion and Discussion

Results showed teachers conceptualized complex numbers as a quantitative structure, allowing them to make sense of each form both algebraically and geometrically and conceptualize relationships among them. Notably, teachers conceptualized the formal definition of complex numbers in terms of the roots of quadratic equations (Saraç & Karagoz Akar, 2017), both algebraically and geometrically, which allowed them to comprehend complex numbers as the elements of a well-defined set (Sfard, 1991). Also, teachers conceived a complex number z as a union of two directed distances. This enabled them to think of the Cartesian form of complex numbers as vectors, leveraging their thinking about the Cartesian and polar forms concurrently and relatedly. Previous research reported that teachers did not link quadratic equations with complex numbers (Conner et.al., 2007) and had difficulty thinking about the Cartesian form's vector representations (Karakok et.al., 2015). In this study, teachers' reasoning quantitatively to relate quadratic equations and vectors, except T3, to make sense of different forms of complex numbers was important as teachers' awareness of how mathematical ideas are related within the horizon of mathematics is necessary (Ball et al., 2008, p. 403). T3's difficulty might stem from her lack of knowledge on vectors. Also, Karakok et al. (2015) proposed that function ideas are essential in learning complex numbers. Supporting their claim, results showed that teachers made sense of the Euler form of complex numbers by thinking about the polar form as a function of the angle a complex number makes with the positive real axis. We conclude that quantitative reasoning is propitious for further productive meanings on complex numbers. We propose to research to determine learning trajectories on each form of complex numbers and other concepts, such as constructing a complex plane through quantitative reasoning.

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References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of teacher education*, 59(5), 389–407.
<https://doi.org/10.1177/0022487108324554>

- Clement, J. (2000) Analysis of clinical interviews: Foundations and model viability. In R. Lesh, and A. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 341–385). Hillsdale, NJ: Lawrence Erlbaum.
- Conner, E., Rasmussen, C., Zandieh, M., & Smith, M. (2007, February 22–25). *Student understanding of complex numbers* [Conference presentation]. the 10th Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education, San Diego, CA, United States. <http://sigmaa.maa.org/rume/crume2007/papers/conner-rasmussen-zandieh-smith.pdf>
- Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. Basic Books.
- Hodgen, J., Pepper, D., Sturman, L., & Ruddock, G. (2010). *Is the UK an outlier? An international comparison of upper secondary mathematics education* (Report). (n/a ed.) The Nuffield Foundation.
- Johnson, H. L. (2016). Quantitative reasoning in mathematics education: Directions in research and practice. In R. A. Duschl, & A. S. Bismack (Eds.), *Reconceptualizing STEM education: the central role of practices* (pp. 117–146). Routledge.
- Karakok, G., Soto-Johnson, H., & Dyben, S. A. (2015). Secondary teachers' conception of various forms of complex numbers. *Journal of Mathematics Teacher Education*, 18(4), 327–351. <https://doi.org/10.1007/s10857-014-9288-1>
- National Council of Teachers of Mathematics-NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Nordlander, M.C., & Nordlander, E. (2012). On the concept image of complex numbers. *International Journal of Mathematical Education in Science and Technology*, 43(5), 627–641. <https://doi.org/10.1080/0020739X.2011.633629>
- Saraç, M., & Karagoz Akar, G. (2017). A Prospective Secondary Mathematics Teacher's Development of the Meaning of Complex Numbers Through Quantitative Reasoning. In E. Galindo, & J. Newton, (Eds.), *Proceedings of the 39th annual meeting of the North American chapter of the international group for the psychology of mathematics education*, (pp. 267–271). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educator.
- Simon, M. (2000). Research on mathematics teacher development: The teacher development experiment. In A. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 335–359). Lawrence Erlbaum Associates Publishers.
- Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic*. Center for Research in Mathematics & Science Education: San Diego State University.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). SUNY Press.