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# Mathem-Ethics for teenagers: An experiment about “loci”

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*The aim of this paper is to show an experiment of giving to mathematics' topics (on geometric loci) ethical meanings, so that students can own the mathematical topic, seen as connected with their own lives, and can use mathematical reasoning to reflect on their own states of mind and feelings. The theoretical frameworks to design the activity are the Teaching for Robust Understanding Framework and the Mathem-Ethics methodology. Investigated students are teenagers. Analyses of students' productions are briefly described at the end of the paper.*

*Keywords: Mathematics education, ethics, personal change, teaching for robust understanding.*

## Introduction

Too many students exhibit alienation from mathematics (Nardi & Steward, 2003), too often reduced to a ready set of rules to memorize and blindly apply (OFSTED, 2012; Foster, 2013). Mathematics is seen as far from students' lives and at most useful for technical reasons. “Discussions about the connections between mathematics and democracy amongst the general populace have not been explicitly well rehearsed, other than to either assume that mathematics has nothing to do with anything political, being neutral in form and practice, so it has nothing to do with democracy, which is something political; or that it is implicitly democratic.” (Swanson & Appelbaum, 2012, p 1). But nowadays research in mathematics education and teachers' effort aim more and more towards the development of skills useful for students seen as future citizens in democratic societies (Jablonka, 2003; Skovsmose, 2007), (Aguilar & Zavaleta, 2012). And teachers, and mathematics' teachers, have to take into account their ethical obligations (Ernest, 2019), and rethink mathematics' teaching in view of a social dimension. But still, if mathematics can be useful for a future role of citizens, students do not think of mathematics as connected to their own present lives. Mathematical interventions aiming to a life change have been designed by one of the authors in prison, with adults inmates (Ferrarello et al., 2021), (Battaglia & Ferrarello, 2021), (Ferrarello & Mammana, 2022), starting to use a methodology called Mathem-Ethics. Starting from one of the topics described by Ferrarello et al. (2021), a mathem-ethical path was experimented in a regular class of teenagers in the south of Italy, for the master degree thesis of one of the authors. The mathematical content is on *loci* (conics, to be precise), and it is used to make students think of their states of mind. We used a wordplay because of the double meaning of the Italian word “*luogo*”, used both for “place” and for “locus” (in mathematical sense). In this paper we will use the Latin word “*locus*”, both in a mathematical sense (a conic) and in a metaphorical sense (a state of mind). From the “little prince worlds”, used to explore students feelings on their “undesired places”, we talked especially about ellipse and hyperbola. All the conics were constructed by means of mathematical machines (Bartolini Bussi & Maschietto, 2006). The machines were introduced in a virtual version, on GeoGebra, and presented by means of GeoGebra classrooms. Paper worksheets were used too, to collect students' thoughts and feelings.

The activity was based on the Teaching for Robust Understanding (TRU) framework together with the Mathem-Ethic methodology. Results are analysed by considering the following research question: “Is it possible to reach a robust understanding of a rich content, with an ownership of the topic, by means of a mathem-ethical path and to make teenager students reflect on their own lives by using mathematics?”

### Theoretical framework

We chose to design the activity and review the results by using the Teaching for Robust Understanding (TRU) framework proposed by Schoenfeld (2016) and the Mathem-Ethics methodology (Ferrarello & Mammana, 2022). The TRU framework consists of five dimensions (D1, D2, D3, D4 ad D5) for powerful classrooms, described in Table 1.

D1: The Content	D2: Cognitive Demand	D3: Equitable Access to Content	D4: Agency, Ownership, and Identity	D5: Formative Assessment
<p>The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.</p>	<p>The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called a “productive struggle”.</p>	<p>The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which a small number of students get most of the “air time” are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.</p>	<p>The extent to which students are provided opportunities to “walk the walk and talk the talk”, to contribute to conversations about disciplinary ideas, to build on others’ ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.</p>	<p>The extent to which classroom activities elicit student thinking and the subsequent interactions in response to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction “meets students where they are” and gives them opportunities to deepen their understanding.</p>

**Table 1: The five dimensions of the TRU model**

The framework identifies these five dimensions, useful for a truly effective teaching/learning context and fostering a deep understanding, achieving a robust teaching/learning. Briefly, “if the content is rich, the students get to engage, they get powerful ideas, they build on each other’s ideas, they can build positive identities with the teacher adjusting the level of instruction so that it is right for the students to engage productively” (Schoenfeld, video in <https://truframework.org/>). A peculiarity of the TRU framework is that it does not tell the teacher how to teach. There are many ways to reach a robust understanding. For instance, a laboratorial methodology (Anichini et al., 2004), together with a mathematical discussion (Bartolini Bussi et al., 1995), could be a good way to engage all the students (D3) and make them think, letting them the possibility to be active thinkers (D1) and produce their own ideas (D4), without any fear of being judged (D5), even if the content is rich and not trivial (D2). Our project was focused especially on Dimension 4 (Agency, Ownership, and Identity) and Dimension 3 (Equitable Access to Content), with the aid of Mathem-Ethics, as a methodology. The Mathem-Ethics is a methodology to teach/learn mathematics, by giving precise ethical meanings to mathematical topics. The muse of the Mathem-Ethics is Polya’s “How to solve it” (Polya, 1957), where several methodologies to solve mathematical problems are analysed. Reading “How to solve it” with a wider look, we think that mathematics could help to solve life problems. The idea is to link a mathematical entity (a definition, a theorem, a procedure, ...) to a personal attitude (a state of mind, a feeling, an action in life, ...), with the final target to face personal problems with a similar method used to solve the connected mathematical problem. It is possible to design mathem-ethical activities, aiming both to mathematical and social skills. Giving to mathematics meanings related to life can help students to see mathematics alive and be engaged with it, but also better understand topics with a less formal but deeper approach. Solving mathematical problems can give students a self-efficacy (Bandura, 1997): the attitude to solve mathematical problems connected with life problems get students used to think that it is possible to solve life problems and can be useful for supporting students in their lives (Ferrarello & Mammana, 2022). The method was successfully experimented by one of the authors in a prison context (Ferrarello et al., 2021), (Battaglia & Ferrarello, 2021), (Ferrarello & Mammana, 2022), and here, for the first time, out of prison, with teenager students.

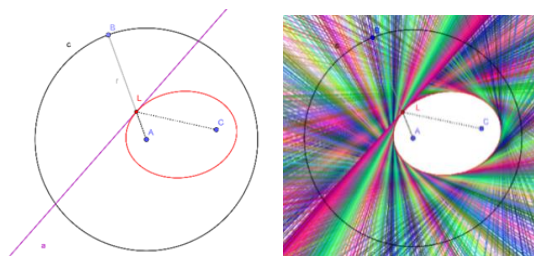
## **Description of the project and methods**

The class, 11-th grade at “Galilei-Campailla” high-school in Modica (Sicily), consisted of 24 students, the main part well disposed toward mathematics. But the aim was to propose a content that is rich, not only from a mathematical but also from an educational point of view. The researcher (one of the authors) presented the project to the class teacher. It is important, in fact, to have a close collaboration between researchers and teachers; many researchers have explored this cooperation, promoting interventions carried out directly by researchers and teachers, together (William & Lester, 2008). The whole project was structured in two meetings, for a total of 3 hours. It is possible to design short-term interventions in mathematics education that can alleviate the typical problems of classroom practice: in this way teachers can benefit from the observed methodology without disrupting their curricular activity (Stylianides & Stylianides, 2013). The activities involved the use of technological devices and the dynamic geometry software GeoGebra. GeoGebra was used to construct virtual version of mathematical machines, useful to build geometric *loci* (conics). Students were divided into groups of 4, in order to implement a collaborative learning, making students able

to learn from each other, formulate conjectures and solve problems together. The researcher presented the problems (“what does this virtual machine do?”) and guided the discussion, interacting with students, observing them working, helping if necessary. After a worksheet on the “Little Prince” story, and constructions of already known *loci* (circumference and parabola), students were guided to construct ellipses and hyperbolas, both accompanied by astronomic applications (Kepler laws for ellipse and Fly-past effect for hyperbola). The two meetings were audio-recorded. Analysed data came from the transcription of these recordings, from paper worksheets produced by students and from digital worksheets on GeoGebra classrooms. Textual data were analysed from a qualitative point of view by using the Discourse Analysis: “Analysis is based on the details of speech [...] or writing that are arguably deemed relevant in the context and that are relevant to the arguments the analysis is attempting to make.” (Gee, 2005, p. 117). The first author selected those phrases among the data related to the important issues of the research question; the second author organized the data, guided by the themes (the 5 dimensions of the TRU and the Mathem-Ethic), but both authors paid attention to any other possible result seeming to emerge from the data.

### The mathematics’ loci

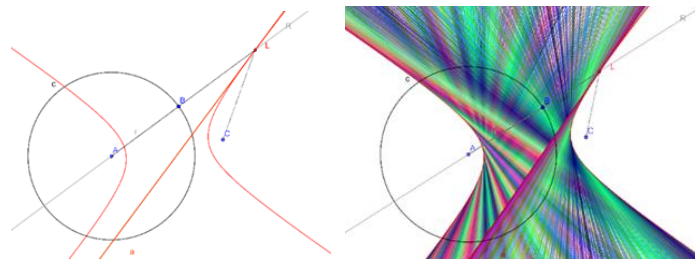
As mentioned, we dealt with conics, as mathematics’ *loci*, all constructed by means of virtual mathematical machines. In this section, we only enter into the details of the mathematical construction of an ellipse and a hyperbola by using the so-called director circle. We start with the construction of the ellipse (Figure 1): we set a circle  $c$ , called director circle, with centre in a fixed point  $A$  and radius  $r$ . We also set a point  $B$  free to move on the boundary of  $c$  and a fixed point  $C$  inside  $c$ . We construct the line  $a$ , that is the axis of the (not explicitly constructed) line segment  $BC$ . Finally, we construct the point  $L$ , intersection point between  $r$  and  $a$ . The geometric locus described by  $L$ , whenever the point  $B$  moves on  $c$  is the ellipse whose *foci* are  $A$  and  $C$ . In fact, since  $a$  is the axis of  $BC$ ,  $\overline{LB} = \overline{LC}$ . So, the sum  $\overline{AL} + \overline{LC} = \overline{AL} + \overline{LB} = r$ . And  $r$ , that is the radius of  $c$ , is a fixed distance. We can visualize the ellipse in GeoGebra also by moving the point  $B$  and considering all the positions taken by the line  $a$  (all with different colours in Figure 1). The ellipse is get by an envelope of tangent lines. This is useful to visualize the ellipse as a “closed” *locus*, being the white part, the only not covered by coloured lines, limited in space. (GeoGebra applet: <https://www.geogebra.org/m/c5a3ykvjg>).



**Figure 1: Ellipse with director circle on GeoGebra**

To construct a hyperbola by means of the director circle, we can use the previous construction, with just two differences: point  $C$  has to be fixed outside  $c$ , and point  $L$  is the intersection of axis  $a$  with the line  $R$  that is the line containing segment  $r$  (Figure 2). The geometric *locus* described by  $L$ , whenever the point  $B$  moves on  $c$  is the hyperbola whose *foci* are  $A$  and  $C$ . In fact, since  $a$  is the axis of  $BC$ ,  $\overline{LB} = \overline{LC}$ . So, the difference  $|\overline{AL} - \overline{LC}| = |\overline{AL} - \overline{LB}| = r$ . And  $r$ , that is the radius of  $c$ , is a

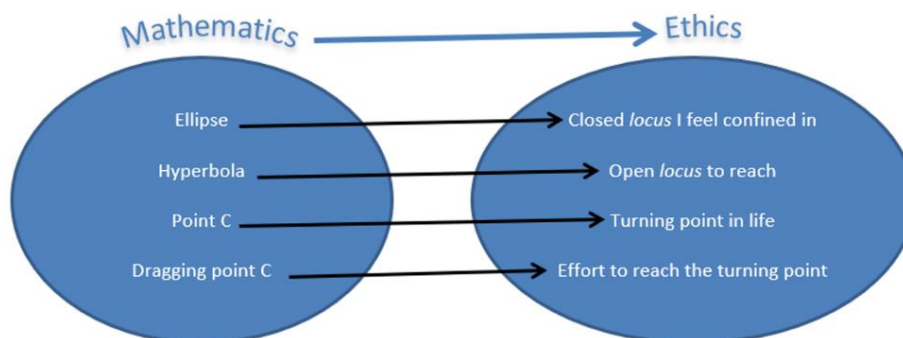
fixed distance. We can visualize the hyperbola in GeoGebra also by moving the point B and considering all the positions taken by the line  $a$  (all with different colours in Figure 2). The hyperbola is get by an envelope of tangent lines. This is useful to visualize the hyperbola as an “open” *locus*, being the white part, the only not covered by coloured lines, not limited in space (GeoGebra applet: <https://www.geogebra.org/m/myagrzqv>).



**Figure 2: Hyperbola with director circle on GeoGebra**

### The Mathem-Ethics’ loci

The Mathem-Ethic meaning of the Mathematics’ content have two steps: “**Loci**”, aimed at the construction of the ellipse, as a “closed” *locus*; and “**Changes**”, aimed at the passage from ellipse to hyperbola, by dragging a point outside the director circle. The step “*Loci*” is the consideration that an ellipse is a finite geometric *locus*, in the Riemann’s sense, i.e. it has infinite points, but it is limited in space. The ethical meaning is related to the feeling to dwell *loci* that we do not want to belong to, and to feel confined into these *loci*, without the possibility to escape, such as the characters of the “Little prince” story. Students were invited to think about those *loci* and to express their feelings. The step “Changes” is the consideration that an escape from those closed *loci* that we do not like is possible, such as the transformation of an ellipse into a hyperbola, when a point ( $C$  in figure 2), is outside the director circle. But it is not always easy. It requires an effort: the dragging of  $C$ . Because if you want to change the *locus*, you have to change conditions to be in the *locus* (no more the sum of the distance from two fixed points, as in the ellipse, but the difference, as in the hyperbola): you should leave your comfort zone, the closed *locus*, to reach the new open *locus*.



**Figure 3: Mathem-ethic correspondences in the mathem-ethic path**

Students reflected about the possibility of changing conditions to change their undesired *loci*.

### Brief discussion and conclusion

Here we briefly analyze the results of the effectiveness of the TRU framework and the Mathem-Ethics methodology, in the school context in which the experimentation took place. The research question,

“Is it possible to reach a robust understanding of a rich content, with an ownership of the topic, by means of a mathem-ethical path and to make teenager students all reflect on their own lives by using mathematics?”, links back to the dimensions of the TRU (see Table 1), and we analyze Mathem-Ethics at the end of section. Students showed to be engaged and to have reached an **agency, ownership and identity** (D4 of the TRU) thanks to the reflection of their own lives:

Student: I take with myself a little life lesson, more than a mathematics’ one, [...] it has stayed with me the way we linked mathematics with things we live everyday [...] We were more involved with respect to a normal lecture, because we had fun and we learned at the same time.

Since the first activity (the “Little Prince” worksheets) students opened their own hearts:

Student: I would like to succeed in doing what I want, without depriving myself of things I like, just for fear of the others’ thinking.

Student: I am stuck in fear of the judgment of the people surrounding me [...] I wish to be myself with everyone, because I often wear a mask in order to please the others.

and many other comments identifying the *locus* they feel confined in as the fear of the others’ judgements. They also claimed to own the mathematical topics:

Student: I liked the practical application of the studied concepts, because it will not forget anything [...]; the memory, indeed, needs images to be able to work actively. I also clarified some doubts I had on this topic.

Since the first geometric locus, the circumference, was introduced, **all the students** (D3 of TRU) were involved and showed the availability to help each other:

Student: I have never used GeoGebra, how can I do?

Researcher: You can work together trying to solve the problem [...]

Student: I got it! Can I help my classmates?

Researcher: Good! You can help who is left behind.

This collaboration among students was possible also because they knew that they would not be evaluated: a **formative assessment** (D5 of TRU) let students be free to help each other and free to make mistakes, because every mistake is not judged, but rather used to reach a better comprehension. At the beginning, in fact, some students seemed to be confused about the functioning of the virtual machine to construct *loci*, for instance parabola, as in the dialogue involving student F we report:

Researcher: What does this machine realize?

Student: A parabola! It is evident

Researcher: And why is it a parabola?

Student F: I see that it is a parabola, but I’m not able to understand the reason why this conic comes out. [...] I got it, but anyway I still don’t understand

Student F is not afraid to share his difficulty. The proposed **content** is not trivial, it is **challenging** (D2 of TRU). After some notes from a student and the researcher, the following dialogue took place:

Student F: And what is the focus here? The vertex upside in the rhombus, right?

Researcher: Exact. Very good!

Student F: Now, finally, it is clear!

Similar discussions took place for the construction of ellipse. Students did not listen to a ready lesson, but rather they constructed their own knowledge, starting from the exploration of the virtual machine and showing to be real **thinkers** (D1 of TRU):



Student: This non-traditional method has made everything easier [...] I find it useful [...] because we are the ones who directly experience and work.

More, as we can read from their worksheets, they reached the definition of hyperbola by themselves, through the dragging of point *C* (Figure 2). And here the Mathem-Ethic part comes out. If in the first meeting students expressed their thoughts about the feeling to be confined in a closed locus (as the others' judgement), in the second meeting they dragged the point outside the circle and realized that a change is possible, and they could deal with the difficulty, free themselves in an open *locus*:

Student: I'm taking home a little life lesson [...] thanks to this meeting I am more willing to change "locus" in my life

Student: I think that a moment of crisis allows you to question ourselves and to have a broader conception of ourselves.

Of course, a change can be hard to reach, as in the fly-past effect, and it requires a meeting with an important situation (as a big planet) or a huge use of fuel, to change the orbit:

Student: From this experience I understood that change costs and that when you want to change, you must be willing to travel new and unknown paths [...]. This method is fruitful because it allows for reflection and deepening of the examined objects [...]. I think this experience is positive and a source of enrichment of mathematical knowledge which, before participating, I did not think it was connected so much to our lives

Two more aspects arose from data, not directly connected to the research question. Firstly, students claimed to have clarified doubts on topics already known (which was not a target for us), suggesting that the Mathem-Ethic methodology could be used to strengthen mathematical concepts. Secondly, they put in evidence the importance of a collaborative way of learning.

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