



**HAL**  
open science

## Calculus at the school to university transition: early stages of a structuralist perspective in real analysis

Ludwig Laukert, Thomas Hausberger, Reinhard Hochmuth

### ► To cite this version:

Ludwig Laukert, Thomas Hausberger, Reinhard Hochmuth. Calculus at the school to university transition: early stages of a structuralist perspective in real analysis. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04406746

**HAL Id: hal-04406746**

**<https://hal.science/hal-04406746v1>**

Submitted on 19 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Calculus at the school to university transition: early stages of a structuralist perspective in real analysis

Ludwig Laukert<sup>1,2</sup>, Thomas Hausberger<sup>1</sup> and Reinhard Hochmuth<sup>2</sup>

<sup>1</sup>University of Montpellier, France

<sup>2</sup>Leibniz University Hannover, Germany; [laukert@idmp.uni-hannover.de](mailto:laukert@idmp.uni-hannover.de)

*Building on our prior work which described the notion of structuralist praxeology for the theory of teaching and learning of Abstract Algebra, we focus in this article on the case of Real Analysis. The theoretical perspective of the Anthropological Theory of the Didactic by Chevallard helps us to analyse the data which is drawn from an introductory course in Real Analysis at university level. We identify structuralist aspects and show how they develop into a structuralist praxeology later in the curriculum. Further, we introduce the notion of pre-structuralist praxeology as a characteristic of the School to University transition.*

*Keywords: Real analysis, university mathematics education, school-university transition, anthropological theory of the didactic, pre-structuralist praxeologies.*

## Introduction

The teaching and learning of Calculus at the transition between School and University has been the subject of intensive research (Bressoud et al., 2016). These studies focus on core calculus concepts such as real numbers, sequences, series, derivatives, and integrals with the concept of limit as a cornerstone. In such an epistemological perspective, the modern vision of Analysis is mainly that of the turn of rigour due to Cauchy with corresponding issue in the school-university transition the implementation of proofs around formal definitions and their consequences.

Winsløw (2009) also pointed out a shift towards abstraction since students gradually learn to relate properties of specific functions to general and abstract properties of arbitrary functions, expressed in conceptual terms (e.g., the existence of a maximum due to continuity on a compact interval). Using the Anthropological Theory of the Didactic (ATD) as theoretical framework and the notion of praxeology (Chevallard & Bosch, 2020) as key theoretical construct, he proposed a three-phase model to characterise the evolution of students' activity in the analysis path at university. In a first transition, techniques learned at school are justified and unified on a theoretical level (by means of properties and theorems) and lead to the learning of "complete" Calculus praxeologies, endowed with a consistent mathematical logos (in the ATD sense). Secondly, theoretical tasks that relate to the latter logos are introduced: in this second transition, students are required to engage with generic functions and their abstract properties to produce modest theoretical developments.

The rationale of the praxeologies resulting from the second transition is not further elucidated by Winsløw. In our view, it is related to another striking feature of modern analysis that is not due to Cauchy: the rise of abstract analysis, in the avenue opened up by abstract algebra, under the influence of mathematical structuralism as a new epistemology of mathematics. This trend, led by mathematicians such as the Bourbaki group in France, resulted in a refoundation of mathematical domains around key mathematical structures. As a consequence, Real Analysis evolved into a body

of knowledge called Abstract Analysis and organized around such notions as topological spaces, metric spaces or Hilbert and Banach spaces (Hausberger & Hochmuth, 2023).

The didactical consequences of mathematical structuralism are being investigated in a research program opened up by Hausberger (2018) in the paradigmatic context of Abstract Algebra. Theoretical ideas, expressed in the language of ATD (in particular, the key notion of structuralist praxeology), nevertheless apply to any context where structures are involved. The initiating paper (Hausberger & Hochmuth, 2023) aims to show the relevance of the notion of structuralist praxeology to account for the knowledge taught in Analysis courses at the transition from the Bachelor to graduate studies where Abstract Analysis lies in the horizon.

In this paper, we rather focus on the transition from school to university with a view to identifying how structuralist thinking influences the teaching and learning of Real Analysis at early stages of the journey to Abstract Analysis. The research question is: What are the structuralist aspects in the teaching and learning of Real Analysis in the first year of Bachelor courses? We begin by a description of our theoretical ideas and methodology, then present a praxeological analysis of excerpts from first-year Analysis course material. We then discuss the identified structuralist aspects and their significance with respect to the didactic issues of the school-university transition.

## Theoretical framework

Hochmuth et al. (2021) highlight the following main features of ATD that justify its frequent use in research on transitions: the consideration of knowledge as living within *institutions*, the institutionalisation of knowledge seen as the result of complex *didactic transposition* processes, and finally the central 4T-model of *praxeologies* (task, technique, technology, theory) that allows researchers to build *reference models* of the knowledge to be taught for application to teaching-learning phenomena. As these theoretical constructs (presented in a concise manner in the mathematics education encyclopaedia; Chevallard & Bosch, 2020) are increasingly used in university mathematics education research, we will focus now on the additional tools, specific to the point of view of mathematical structuralism, that have been developed.

The starting point is the consideration of mathematical structuralism as a methodology, which consists of reasoning in terms of classes of objects, relations between these classes and stability properties for operations on structures (Hausberger, 2018). The general view of structures thus allows particular properties of objects to be demonstrated by making them appear as consequences of more general facts (theorems about structures). Dually, generalisations are put to the test of objects, hence a *dialectical relationship between objects and structures*. In praxeological terms, the structuralist method consists in the passage from a praxeology  $\mathbf{P} = [T/?/?/\Theta_{\text{particular}}]$  where it is unclear which technique to apply, to a structuralist praxeology  $\mathbf{P}_s = [T^g/\tau/\theta/\Theta_{\text{structure}}]$  where, modulo generalisation of the type of tasks ( $T^g$ ), the theory of a given type of structure guides the mathematician in solving the problem.

As a paradigmatic illustration in the context of Abstract Algebra, Hausberger (2018) analysed a thread from an online forum. In order to prove that the ring of decimal numbers is a principal ideal domain PID (T: show that a given ring is a PID), a forum participant proposes to prove that any subring of  $\mathbf{Q}$  is a PID ( $T^g$ : show that any subring of a given PID is a PID), with the (erroneous) structuralist

technique  $\tau$ : show that any subring of a PID is again a PID. The underlying structuralist technology is the idea that striking features of rings are preserved by taking subrings, which unfortunately is not true for being a PID ( $\mathbf{Z}[X] \subset \mathbf{Q}[X]$  is a counterexample). The objects-structures dialectic thus gives rise to abstract tasks  $t \dashv \tau$  that are meant to provide new powerful technologies to solve tasks of type T (here, showing that the given ring is included in a PID). In other words, the objects-structures dialectic subsumes both a *dialectic of the particular and the general*, and of *the concrete and the abstract*, which are interrelated.

Furthermore, Hausberger (2018) distinguishes several *levels of structuralist praxeologies*: at level 1, structures act as a vocabulary and appear mainly through definitions (e.g., a task of type T is solved by showing, by hand, that the definition is satisfied); at level 2, the technique mobilises general results about structures (in our example, one shows the existence of an Euclidean algorithm, which invokes in the logos of the praxeology the structuralist theorem that any Euclidean ring is a PID).

Hausberger (2018) explained how analyses of structuralist praxeologies may shed light on transition issues in the development of structuralist thinking among learners. This program was carried out in a study of course materials in Abstract Algebra throughout the Bachelor in France and Switzerland (Jovignot & Hausberger, 2022). The present study aims at investigating similar issues in the context of Real Analysis in Germany.

## Methods

To identify structuralist aspects in mathematical analysis we chose an Analysis course at a medium-sized German university that was offered during the winter semester 2021/2022. This introductory course was designed for the first year of the bachelor's program of mathematics, of pre-service teachers of mathematics, of physics and of meteorology. In this context, the students had to turn in homework assignments on a weekly basis. The problems assigned for homework were discussed in a tutorial that took place after the homework assignment was turned in by the students. In these tutorials, a document presenting the standard solution of the problems was made available to the students and to us. In the tutorials, the students were presented an additional set of problems, whose contents correspond to the problems assigned for homework, and to the lecture. This additional material as well as the textbook used for the lecture, were made available to us.

This course is usually attended by students in their first year of studies. Thus, it is situated in the transition from School to University. This study is part of a research project of the first author, that aims to provide course material to support students in the transition process. This course material is going to refer to problem sets that students encounter throughout their first year of studies of Analysis. Hence, we focus on the homework assignments and the problem sets from the tutorials as well as on the solutions that are suggested by the lecturer, to conduct our praxeological analysis.

## Data Analysis and discussion

The problems that we analysed reveal different structuralist aspects. There are a few problems that allowed us to conclude that their purpose is to develop a structuralist praxeology. Among them there are some problems whose potential to develop a structuralist praxeology is not used in this course. We chose two examples to illustrate how the potential to develop a structuralist praxeology is used

successfully in one case, and how it is not in the other. In addition, these two examples contain some justification for structuralist aspects (e.g., particular/general, and concrete/abstract dialectics) which we could identify in many problems of the set. For each problem, we first present the task and its solution as it was suggested by the lecturer. Then the discussion will follow.

### Analysis and discussion of Problem 11

Two main tasks are assigned in this problem, labelled (a) and (b).

Let  $(a_n), (c_n)$  be two convergent sequences so that  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n$ . Let  $(b_n)$  be a sequence, so that  $a_n \leq b_n \leq c_n$  for all  $n \in N$ . Show that  $(b_n)$  is convergent and  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n$ .

#### Standard solution proposed by the teacher:

From  $a_n \leq b_n \leq c_n$  follows  $0 \leq b_n - a_n \leq c_n - a_n$ . But we have

$$\lim_{n \rightarrow \infty} (c_n - a_n) = \lim_{n \rightarrow \infty} c_n - \lim_{n \rightarrow \infty} a_n = 0.$$

From tutorial problem set 3 follows  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$  and it follows, that  $b_n = (b_n - a_n) + a_n$  is a convergent sequence since  $(b_n - a_n)$  and  $(a_n)$  are convergent. Thus it also follows, that

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (b_n - a_n) + \lim_{n \rightarrow \infty} a_n = 0 + \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n.$$

#### Figure 1: Task 11 (a) and its standard solution

Task 11 (a) is about proving the squeeze theorem (figure 1). Since the squeeze theorem is used in Analysis to show the convergence of sequences, the praxeology  $\mathbf{P}_1$  that gives this piece of knowledge its raison d'être can be modelled as follows:

$\mathbf{T}_1$ : Prove the convergence of a given real sequence and determine its limit;  $\mathbf{t}_1$ : Find upper and lower bounds which converge to the same known limit;  $\mathbf{\Theta}_1$ : Squeeze theorem;  $\mathbf{\Theta}_1$ : Real Analysis.

Indeed, we could find a problem in the tutorial material that corresponds to this praxeology. It also relates to problems that could be found in German Schools (approximation of the circle by polygons, integrals as areas...). By contrast, task 11 (a) would rather not appear in Schools, due to its level of abstraction (it deals with generic sequences). In praxeological terms, it aims at justifying the technology  $\mathbf{\Theta}_1$  of the praxeology  $\mathbf{P}_1$ , in other words at completing the School praxeology. This is an example of the first transition identified by Winsløw.

The sequence is a variable of the type of tasks  $\mathbf{T}_1$ . Several other techniques may arise depending on the value taken by this variable. By contrast, the transition from an explicitly given sequence of real numbers to a generic sequence entails a switch in the technique and logos of the resulting abstract praxeology, if task 11 (a) falls within a type of tasks - which we will defend, of a different nature. Our proposed praxeological model is indeed the following:

$\mathbf{T}_2$ : Prove that any real sequence that fulfils certain conditions is convergent and determine the limit (in terms of the conditions);  $\mathbf{t}_2$ : Apply the definition of convergence; use the conditions and standard bounding techniques;  $\mathbf{\Theta}_2$ : Definition of convergence; properties of real numbers;  $\mathbf{\Theta}_2$ : Real Analysis.

The formal definition of convergence is now a key element of the technology of the abstract praxeology  $\mathbf{P}_2$ . In the point of view of structuralist praxeologies, we may note  $\mathbf{T}_2 = \mathbf{T}_1^g$  (a

generalisation) and call  $\mathbf{P}_2$  a structuralist praxeology of level 1 since it invokes a conceptual definition. In fact, we prefer to call it *pre-structuralist* since the role played by real numbers is not elucidated at this level of generality. A proper structuralist praxeology would deal with structures on abstract spaces, for instance the ordered field structure endowed with the order topology.

It shall be noted in the written solution of task 11 (a) that it is sufficient to link the problem to the case where  $a_n=0$  and  $(c_n)$  is converging to 0 since such a result has already been proved in the tutorials (by means of the praxeology  $\mathbf{P}_2$ ). As a result, properties of limits are used: in structuralist terms, these reflect the compatibility of limits with respect to the ordered field structure, in other words they may be regarded as pre-structuralist theorems. The modified praxeology, denoted  $\mathbf{P}_2'$ , may be called a *pre-structuralist abstract praxeology of level 2* and described as follows:

$\mathbf{T}_2$ : Prove that any real sequence that fulfils certain conditions is convergent and determine the limit;  $\boldsymbol{\tau}_2'$ : Apply properties of limits; use the conditions and standard bounding techniques;  $\boldsymbol{\vartheta}_2'$ : Properties of limits; properties of real numbers;  $\boldsymbol{\Theta}_2$ : Real Analysis.

It is the presence of task 11 (b), dedicated to the proof of the Cesàro lemma, that allows us to argue that the praxeology  $\mathbf{P}_2$  is effectively developed in the Analysis course (Figure 2). Indeed, it is another instantiation of  $\mathbf{T}_2$  and  $\boldsymbol{\tau}_2$  is applied.

**Problem 11 (b)**

Let  $(a_k)$  be a convergent sequence of real numbers so that  $\lim_{k \rightarrow \infty} a_k = a$ . Show that the sequence of the arithmetic means  $x_n := \frac{a_1+a_2+\dots+a_n}{n}$  converges to  $a$  as well.

Note:  $\epsilon - n_0$  – definition of convergence! Choose  $m_0$  for a given  $\epsilon > 0$  such that  $|a_k - a| < \frac{\epsilon}{2}$  for  $k > m_0$ . Then split it up for  $n > m_0$ :

$$|x_n - a| = \left| \frac{(a_1 - a) + (a_2 - a) + \dots + (a_{m_0} - a)}{n} + \frac{(a_{m_0+1} - a) + (a_{m_0+2} - a) + \dots + (a_n - a)}{n} \right|$$

and use the triangle inequality for an upper estimate.

**Standard solution proposed by the teacher:**

$\epsilon > 0$  is given. First we choose  $N_1$  so that for  $n > N_1$  we get

$$|a - a_k| < \frac{\epsilon}{2}.$$

Then we choose  $N > N_1$  so that

$$\left| \frac{(a_1 - a) + \dots + (a_{N_1} - a)}{N} \right| < \frac{\epsilon}{2}.$$

Then for  $k > N$  we get

$$\begin{aligned} & \left| a - \frac{a_1 + \dots + a_{N_1} + a_{N_1+1} + \dots + a_k}{k} \right| \\ & \leq \left| \frac{(a - a_1) + \dots + (a - a_{N_1})}{k} \right| \\ & \quad + \left| \frac{(a - a_{N_1+1}) + \dots + (a - a_k)}{k} \right| \\ & < \frac{\epsilon}{2} + \left| \frac{a - a_{N_1+1}}{k} \right| + \dots + \left| \frac{a - a_k}{k} \right| \\ & < \frac{\epsilon}{2} + \frac{\epsilon}{k} + \dots + \frac{\epsilon}{k} = \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot \frac{k - N_1}{k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

**Figure 2: Task 11 (b) and its standard solution**

In this case, applications of the Cesàro lemma have not been previously encountered by students. The rationale of the lemma will thus appear later, when it effectively contributes to expand the punctual

praxeology  $\mathbf{P}_1$  into a local praxeology (in the sense of ATD, obtained by unification of punctual praxeologies) by providing another technique and technology (a new criteria for convergence and tool to determine the limit).

### Analysis and discussion of Problem 28

There are four tasks in this problem: except task 28 (b), they all deal with piecewise defined functions (by explicit formulas) whose continuity is to be proved or investigated. The most difficult (task (d)) is the popular example  $f(x) = 1/q$  for an irreducible rational  $x=p/q$  and  $f=0$  for irrationals. The density of  $\mathbf{Q}$  and  $\mathbf{R}\setminus\mathbf{Q}$  in  $\mathbf{R}$  plays a role, which draws a slight connection to task 28 (b):

Let  $D \subseteq R$ . A subset  $E \subseteq D$  is dense in  $D$  if for each  $d \in D$  there is a subsequence  $(x_k)$  from  $E$  such that  $\lim_{k \rightarrow \infty} x_k = d$ .

Let  $f, g : D \rightarrow R$  be continuous functions such that  $f(x) = g(x)$  for all  $x \in E$ , while  $E$  is dense in  $D$ . Show that  $f(x) = g(x)$  for all  $d \in D$ .

#### Standard solution proposed by the teacher:

Let  $x \in D$ . We choose a sequence  $(x_n)$  from  $E$  such that  $\lim_{n \rightarrow \infty} x_n = x$ . The continuity of  $f$  and  $g$  provides

$$f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = g(x)$$

**Figure 3: Task 28 (b) and its standard solution**

This task is about proving a theorem that states that two continuous functions, which are defined on the same domain and are equal on a dense subset of that domain, are equal all over their domain. One would use this theorem, for instance, to conclude that there is only one continuous extension of  $2^x$  to the real numbers if it were defined on the rational numbers only. Just as in problem 11, we first consider the rationale of the task and model the following praxeology:

**T<sub>3</sub>**: Show that two explicitly given real functions  $f$  and  $g$  are equal; **τ<sub>3</sub>**: Show that  $f$  and  $g$  are equal on a dense subset (under the assumption of continuity); **θ<sub>3</sub>**: Theorem, that two real continuous functions equal on a dense subset are equal; **Θ<sub>3</sub>**: Definitions of continuity and density.

We would like to call this praxeology  $\mathbf{P}_3$  a *pre-structuralist concrete praxeology of level 2* since it is completely contextualised to explicit real functions and involves the pre-structuralist theorem **θ<sub>3</sub>**. Making the underlying structures further explicit would lead to the following structuralist praxeology  $\mathbf{P}_4 = \mathbf{P}_3^g$  and proper structuralist theorem **θ<sub>4</sub>**:

**T<sub>4</sub>**: Show that two given continuous functions on a metric space  $E$  are equal; **τ<sub>4</sub>**: Show that  $f$  and  $g$  are equal on a dense subset; **θ<sub>4</sub>**: Theorem, that two continuous functions on a metric space equal on a dense subset are equal; **Θ<sub>4</sub>**: Definitions of continuity and density.

In fact, tasks of type **T<sub>3</sub>** appear neither in the textbook nor in the problem sets of the homework assignments and tutorial. Our model thus reveals a lack of coherence in the problem set, which may hinder the development of pre-structuralist praxeologies. As a result, students would not perceive the rationale of the abstract task 28 (b) which remains isolated whereas University teachers are well aware of the structuralist dimension of the task fully revealed by  $\mathbf{P}_4$ .

## Conclusion and outlook

Assignments in introductory Analysis courses constitute a bridge in the transition from School to University Mathematics. On the one hand, they take up topics and contents of School Mathematics, but treat them based on formal definitions. Assertions are formulated on this basis and proven using logical principles. Their transitional character is not only evident in the topics, such as the treatment of the derivative or integration in the sense of the Riemann integral (or related concepts), but also in the fact that they do not yet consider the most general mathematical structures possible, such as those from Topology or Functional Analysis. However, the assignments we have analysed allow us to identify a pre-structuralist character of the praxeologies developed, and to illustrate it by means of exemplary tasks.

One structuralist aspect we identified in the exemplary tasks (problem 11 (a)) are the particular/general and the concrete/abstract dialectics. This aspect is used to develop a pre-structuralist praxeology. It is done by presenting an abstract task, preceded (in the tutorials) by a contextualised task that gives the abstract task its *raison d'être*. Therefore, a shift takes place from a particular object (explicitly given sequence) to a general object (general sequence) which also causes a transition in the technique and logos such that the technique is justified by a pre-structuralist technology (formal definition of convergence of sequences). This allows us to identify this praxeology ( $\mathbf{P}_2$ ) as a pre-structuralist praxeology of level 1. It readily generalises to a proper structuralist praxeology if the field structure and topological structure on  $\mathbf{R}$  are clarified. Whenever this kind of generalisation is missing in the praxeologies that contain structuralist aspects we have called them pre-structuralist. This points at the transitional character of these praxeologies and indicates that the development of structuralist praxeologies is a process.

Further, we could find additional tasks (problem 11 (b)) that are of the same type and use the same techniques as in problem 11(a), introducing yet another concept (Cesàro means). This is an indicator that the corresponding praxeology ( $\mathbf{P}_2$ ) is indeed developed.

Finally, we have shown that when abstract tasks (problem 28) are not provided with additional concrete tasks exemplifying the rationale of the abstract task, the particular/general and the concrete/abstract dialectic is not evoked, and the corresponding pre-structuralist concrete praxeology is not developed. Here, it is pre-structuralist in relation to metric spaces.

In fact, we have shown that our pre-structuralist praxeologies  $\mathbf{P}_2$  and  $\mathbf{P}_3$  both connect to school mathematics ( $\mathbf{P}_1$  and the issue of extending functions from  $\mathbf{Q}$  to  $\mathbf{R}$ ). We would need to interview the teachers to identify if they pointed out such connections. We nevertheless could consider that the assigned abstract tasks aim at justifying claims and techniques from high-school, or at unifying these techniques in a coherent technological discourse, to build local praxeologies (in the sense of ATD). This is what Bosch et al. (2004) call the phenomenon of completion of praxeologies in relation to the school to university transition. In other words, pre-structuralist praxeologies contribute to achieve such a completion, and they do so by initiating the first steps of the structuralist methodology described in our theoretical framework.

We don't claim that the aspects which we identified in our data are exhaustive. Further analyses of tasks might reveal the need to specify these aspects in more detail or to consider additional aspects.



Since we have focused primarily on the transition from School to University the next step is to examine the role of pre-structuralist praxeologies in the transition from the introductory Real Analysis to advanced Analysis. Together with the structuralist praxeologies which have been discussed for advanced Analysis (see Hausberger & Hochmuth, 2023) this would complete the structuralist view on the entire learning and teaching of Analysis.

## Acknowledgment

The work is funded by the Ministry of Science and Culture of Lower Saxony, Germany, within the PhD program “LernMINT: Data-assisted teaching in the STEM subjects”, by the Deutsch-Französische Hochschule (DFH), and by the German Academic Exchange Service (DAAD).

## References

- Bosch, M., Fonseca, C., & Gascón, J. (2004). *Incompletitud de las organizaciones matemáticas locales en las instituciones escolares* [Incompleteness of local mathematical organisations in school institutions]. *Recherches En Didactique Des Mathématiques*, 24(2–3), 205–250.
- Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). Teaching and Learning of Calculus. *ICME Topical Survey*. Springer. <https://doi.org/10.1007/978-3-319-32975-8>
- Chevallard, Y., & Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 53–61). Springer. [https://doi.org/10.1007/978-3-030-15789-0\\_100034](https://doi.org/10.1007/978-3-030-15789-0_100034)
- Hausberger, T. (2018). Structuralist praxeologies as a research program on the teaching and learning of abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, 4(1), 74–93. <https://doi.org/10.1007/s40753-017-0063-4>
- Hausberger, T., & Hochmuth, R. (2023). From Real to Abstract Analysis: the development of structuralist praxeologies in the analysis path. Paper submitted to CERME13 (TWG14).
- Hochmuth, R., Broley, L., & Nardi, E. (2021). Transitions to, across and beyond university. In V. Durand-Guerrier et al. (Eds.), *Research and Development in University Mathematics Education: overview produced by INDRUM* (pp.193–215). Routledge.
- Jovignot, J., & Hausberger, T. (2022). Transitions in abstract algebra throughout the Bachelor: the concept of ideal in ring theory as a gateway to mathematical structuralism. In M. Trigueros, B. Barquero, & J. Peters (Eds.), *Proceedings of INDRUM2022* (pp. 60-68). University of Hanover and INDRUM
- Winsløw, C. (2008). *Transformer la théorie en tâches: la transition du concret à l'abstraite en analyse réelle* [Turning theory into tasks: the transition from concrete to abstract in real analysis]. In A. Rouchier & I. Bloch (Eds.), *Actes de la XIIIème école d'été en didactique des mathématiques* (cd-rom). La Pensée Sauvage.