



**HAL**  
open science

# From real to abstract analysis: the development of structuralist praxeologies in the analysis path

Thomas Hausberger, Reinhard Hochmuth

## ► To cite this version:

Thomas Hausberger, Reinhard Hochmuth. From real to abstract analysis: the development of structuralist praxeologies in the analysis path. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04406694

**HAL Id: hal-04406694**

**<https://hal.science/hal-04406694>**

Submitted on 19 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# From real to abstract analysis: the development of structuralist praxeologies in the analysis path

Thomas Hausberger<sup>1</sup> and Reinhard Hochmuth<sup>2</sup>

<sup>1</sup>University of Montpellier, France; [thomas.hausberger@umontpellier.fr](mailto:thomas.hausberger@umontpellier.fr)

<sup>2</sup>Leibniz University Hannover, Germany

*This article is a continuation of Winsløw's work on the concrete to abstract transition in analysis and Hausberger's work on structuralist praxeologies, developed in the paradigmatic context of abstract algebra. It aims at giving a foundation to the notion of structuralist praxeology in analysis. After a brief epistemological study of abstract analysis, we discuss the praxeological analyses of excerpts from a real and abstract analysis textbook used in the transition between Bachelor and Master degree programs in mathematics.*

*Keywords: Real analysis, abstract analysis, university mathematics education, Anthropological Theory of the Didactic, structuralist praxeologies.*

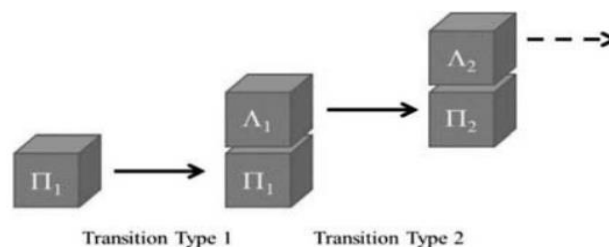
## Introduction

The starting point of this research is the model proposed by Winsløw (2008), based on the Anthropological Theory of the Didactic (ATD), that aims to describe the “transition from concrete to abstract” in real analysis. Winsløw focuses primarily on the institutional transition from high school to university, which is accompanied by ruptures in students' relationships to mathematics. He takes the analysis path as an example, so that this transition coincides, in epistemological terms, with that from calculus to real analysis. After defining real analysis as the study of real functions and the limiting processes (continuity, derivability, integration, series,...) associated with them, Winsløw distinguishes two components: a concrete analysis centred on the study of particular classes of functions, and an abstract analysis that deals with axiomatic systems (metric spaces, operators,...). While high school calculus deals mostly with the practical-technical<sup>1</sup> blocks  $\Pi_1$  of concrete analysis, more complete praxeologies are developed at university, in the sense that they are endowed with a formal technological-theoretical<sup>1</sup> block  $\Lambda_1$ , mobilizing abstract analysis concepts (convergence, continuity, neighbourhood, compactness, etc.) that provide a more rigorous justification of the praxis and organize the logos in a coherent whole. The result, in a second stage, is the appearance of abstract tasks dealing with these concepts (praxis  $\Pi_2$ ): this is the beginning of abstract analysis. As an example taken from the end of the analysis path, Winsløw gives a task of the type “show that a given linear application between two normed spaces is bounded and determine its norm”, the solution of which uses the Cauchy-Schwartz inequality and the interpretation of the sum of a certain series as a scalar product in  $\ell^2$ . In other words, solving the task requires the reformulation in the language of Hilbert spaces of elementary results of real analysis.

---

<sup>1</sup>In the sense of ATD, see theoretical framework

In later research, Winsløw et al. (2014) highlight the generality of this model (Figure 1) to describe ruptures in the development of praxeologies taught in university mathematics curricula in both analysis and linear algebra, mainly during the first 2 years. The first transition is associated with rigor and the learning of proof, while the second highlights the didactic need to anchor concepts in previously constructed praxeologies before engaging them as objects in new tasks of a higher level of abstraction. The construction of institutional praxeological models (relative to universities that serve as case studies) of the knowledge taught, on the basis of the syllabi and course documents, supports the relevance of the model as a descriptive model of didactic transposition choices made by its actors at university and as an explanatory model of certain learning difficulties observed in students. In doing so, the horizon of abstract analysis (in terms of structures, as illustrated by Winsløw's first example) is lost sight of and the dialectic between concrete and abstract that impacts praxeologies is not further clarified from an epistemological point of view.



**Figure 1: Winsløw's model of transitions**

The aim of this article is to clarify these aspects, in line with the research program initiated by Hausberger (2018) around the notion of structuralist praxeology. Starting from an epistemological analysis of mathematical structuralism, Hausberger proposed a second model (see theoretical framework) where the type 1 transition is no longer conceived as a phenomenon of praxeology completion, but of the construction of structuralist praxeologies according to the structuralist methodology (as described notably by Bourbaki). These evolve and generate tasks dealing only with concepts and structures, hence the second type of transition. In the emblematic case of abstract algebra, Hausberger relies on the analysis of course documents and the analysis of forums to show the relevance of the model in a teaching-learning context.

Is Hausberger's model suitable for describing the development of praxeologies in a university analysis path, from the Bachelor to the Master of Mathematics, or are adaptations required with respect to the paradigmatic case of abstract algebra? What didactic phenomena does the lens of structuralist praxeologies reveal? These questions guide the exploratory study presented in this paper. We begin by a brief account of the historical study that we performed to elucidate the development of structuralist thinking in the scholarly practice in analysis (first section). We then present our theoretical framework and apply our model to extracts from a textbook entitled "Foundations of Real and Abstract Analysis" (Bridges, 1998). Used at the transition between Bachelor and Master degree programs, this textbook was chosen by virtue of the didactic project that underlies it: to make visible how the concepts and theorems of abstract analysis enlighten real analysis, which is first recapitulated in view of its generalization.

## Abstract analysis: a brief epistemological account

First of all, modern analysis is rooted in the 19th-century development of the foundations of calculus, which has several striking epistemological features. The 18th century concept of quantity in analysis has been replaced by the concepts of set and real numbers, presented axiomatically (Jahnke, 2003, chap. 10). The rigourisation and arithmetisation process of analysis (led by Cauchy and Weierstrass, among others) is thus also a movement of generalisation and axiomatisation due to its foundational aspects. Problems with the proofs of basic theorems concerning limiting processes such as the convergence of sequences of functions or the representation of functions by means of series (in particular Fourier series) led to foundational questions such as: how could the domain of the functions be characterized in a technically satisfactory way? This required the clarification of the notion of real number (Cantor), continuity, differentiability, integrability. Moreover, mathematicians actively sought pathological functions to delineate the limits of those concepts and challenge their associated intuitive backgrounds. The result is a new style in analysis where theorems now take the form: “Let  $A$  be a ... subset of  $\mathbf{R}$  which is ... (closed, open ... everywhere dense, measurable, simply connected, etc.) and let  $f$  be a function defined on  $A$  (or  $\overline{A}$  or ...) which is  $C^n$  or ... in  $A$  and ... on  $\overline{A}$  and let  $x_0$  be a point of  $A$  such that ..., then ...” (Jahnke, 2003, p. 188).

The study of sets of other objects than points of the real line or  $\mathbf{R}^n$  was necessary for the birth of point set topology as a branch of the emerging general/abstract/modern analysis. The calculus of variations by the Italian school (Ascoli, Volterra, Arzela) who considered sets of curves and functions (an epistemological breakthrough) played a major role and led to the early developments of yet another branch of abstract analysis: functional analysis (Jahnke, 2003, chap. 13). Both concrete applied problems (e.g. Dirichlet’s problem of potential originating in physics or biology phenomena pointed out by Volterra) and the search for a unifying generalising point of view stimulated its development. Frechet was the first to define functionals on abstract metric spaces (called classes  $E$ ) in his 1906 thesis and he called the emerging subject “analyse générale dans les espaces abstraits” (general analysis in abstract spaces). In 1914, Hausdorff began the development of topology on the basis of neighborhood axioms (and introduced the terminology metric space), which marked the development of topology as a separate discipline.

The Göttingen school led by Hilbert gave a foundation to the theory of linear integral equations (originating from the problem of potential). Hilbert used the analogy with the eigenvalue theory of linear algebra to deal with operators with symmetric kernels. The transition from the finite to the infinite relied on set theory and opened up the new realm of what Schoenflies called Hilbert spaces in 1908, and which were later axiomatised in their modern form in the 1920s by von Neumann in connection to the development of an adequate formalism for quantum mechanics. Riesz, building upon Frechet’s concept of distance, established the connections between the French school of real function theory (Lebesgue) and the Göttingen school (in modern terms, he established an isomorphism between the spaces  $\ell^2$  and  $L^2$ , the Riesz-Fischer theorem). Riesz also introduced the new, linear, normed function space  $L^p$  which is not a Hilbert space for  $p \neq 2$  and pointed the way to the Hahn-Banach theorem (on the extension of linear functionals). The abstract axiomatic presentation of complete linear normed spaces was achieved by Banach (thus the Polish school) in his 1920 thesis. Topological tools (based on Baire’s category theory) were used frequently in the later

development of functional analysis, for instance the open mapping theorem proved by Banach. Three monographs based on axiomatic presentations by Banach, von Neumann, and the American Stone in 1932 marked the establishment of functional analysis as an important and solid mathematical subdiscipline.

To summarize, the generalisation of concepts (boundedness, distance, convergence, completeness, scalar product, compactness, continuity,...) from the  $n$ -dimensional Euclidean space  $\mathbf{R}^n$  and functions defined on it to infinite-dimensional function spaces and their operators was a main mathematical motivation to the development of functional analysis, which needed to pass from the finite to the infinite. The axiomatic definitions of the spaces, with  $\mathbf{R}^n$  as a reference providing geometrical insights, allowed to elucidate the logical relations among a diversity of new general concepts (e.g. convergence which became diversified). Properties such as boundedness and compactness separate from each other and new concepts appear with no classical counterpart in  $\mathbf{R}^n$  (separability, Hahn-Banach extension). Conceptual analogies with linear algebra and real functions, together with generalised geometric intuition and approximation principles, allow mathematicians to think about and process the passage from the finite to the infinite (Jahnke, 2003, chap. 13).

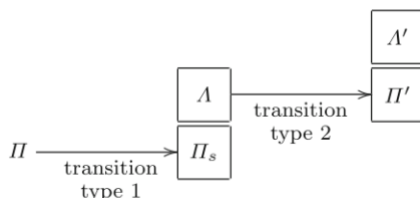
The French group Bourbaki, inspired by van der Waerden's textbook *Moderne Algebra* (published in 1930), set out to apply the structuralist methodology developed by the German algebraists to all fields of mathematics. Bringing to the forth the theme of unity of mathematics, they systematised the perspective of structures in a treatise *Eléments de Mathématique* whose first part, in several volumes, bore the subtitle "The fundamental structures of Analysis". The first volume dedicated to *General Topology* appeared in 1940 and the first volume in functional analysis, entitled *Topological vector spaces* in 1953.

## Theoretical framework

The structuralist methodology, described by Bourbaki in its Manifesto (the Architecture of Mathematics), is the epistemological anchor-point of Hausberger's model, which provides an interpretation in the praxeological language of ATD. According to ATD (Chevallard & Bosch, 2020), every human activity consists in the coordination of a praxis and a logos, hence the key notion of a praxeology, represented by a quadruple  $[T/\tau/\theta/\Theta]$ . Its practical-technical block (or know-how) consists of a type of tasks  $T$  together with a corresponding technique  $\tau$  (useful to carry out the tasks  $t \in T$ ). The technological-theoretical block (or know-why) comprises the technology  $\theta$ , a discourse on the technique, and the theory  $\Theta$ , the ultimate level of justification.

In the praxeological terms of ATD, the structuralist method consists in the passage from a praxeology  $\mathbf{P} = [T/?/?/\Theta_{\text{particular}}]$  where it is unclear which technique to apply to perform a type of tasks  $T$  concerning particular objects, to a structuralist praxeology  $\mathbf{P}_s = [T^g/\tau/\theta/\Theta_{\text{structure}}]$  where, modulo generalization of the type of tasks ( $T^g$ ), the theory of a given type of structure guides the mathematician in solving the problem. The first structuralist transition thus consists in the passage from a praxis  $II = [T/?]$  to a structuralist praxis  $II_s = [T^g/\tau]$  endowed with a structuralist logos  $\mathcal{A}$  (figure 2). In this transition, Hausberger (2018) distinguishes two structuralist levels of praxeologies: at level 1, structures act as a vocabulary and appear mainly through definitions (e.g. a task of type  $T$  is solved by showing, by hand, that the definition is satisfied); at level 2, the technique mobilises

general results about structures and structuralist technologies come into play. These consist in structure theorems (how structures canonically decompose into sub-substructures), structural stability of conceptual properties under operations on structures, isomorphism theorems, expressed abstractly in the language of Set Theory or conceptualized on a meta-level in terms of Category Theory. Although the structuralist principles may be described transversally to all domains of mathematics, structuralist technologies still need to be specified in the context of analysis to account for the structuralist dimensions of the latter field.



**Figure 2: Structuralist transitions**

In the application of the structuralist methodology, the stage is reached when structuralist statements are no more contextualized to specific objects or domains of objects (e.g. functions in one real variable) but concern abstractly defined classes of objects (e.g. generic functions between generic metric spaces). Although generic objects such as real functions or sequences/series already appear in early analysis courses in the context of abstract tasks of what we called pre-structuralist praxeologies (Laukert et al., 2023), the properties of functions and their domain ( $\mathbf{R}$  or  $\mathbf{R}^n$ ) that play a role are now fully elucidated in terms of topological concepts and concepts of functional analysis (theories of Hilbert, Banach and metric spaces). This process is achieved through the second type of transition, situated at the institutional transition of Bachelor and Master degree programs. Figure 2 highlights that the new purely abstract praxis  $II'$  is anchored on reasoning with concepts that take their origin and rationale in the logos of previously developed structuralist praxeologies. The latter are fundamental to reduce the level of abstraction and integrate the objects of  $II'$  (highly abstract concepts) as a form of concrete knowledge.

### **Praxeological analyses of excerpts of the real and abstract analysis textbook**

What makes the presentation in the textbook special, and in this respect different from usual real analysis textbooks, is that the arguments in real analysis are already oriented towards corresponding structuralist arguments: the logos block of core real analysis praxeologies is already shaped by corresponding structuralist concepts and practices. It would not actually be necessary in terms of mathematical content, but it is intended to make the following step towards abstract analysis easier for the students. This central didactic gesture is pointed out in the preface by the author as follows:

I have tried, wherever possible, to present proofs so that they translate mutatis mutandis into their counterparts in a more abstract setting, such as that of a metric space (for results in Chapter 1) or a topological space (for results in Chapter 3). On the other hand, some results first appear as exercises in one context before reappearing as theorems in another [...] I hope that this procedure of double exposure will enable students to grasp the material more firmly. (Bridges, 1998, p. xi)

We will thus base our observations on chapters 1 (Analysis on the Real Line) and 3 (Analysis in Metric Spaces). Instead of metric spaces, other topics such as Hilbert spaces, which generalises the inner product in  $\mathbf{R}^n$ , would also be possible. Our goal is to illustrate the application of our model and provide preliminary results. Therefore, we are not going to detail the analysis methodology (choice of extracts, modelling difficulties,...) because we are not aiming at systematicity.

In Chap. 1, concepts such as distance, sequences, convergence, compactness and continuity of functions are presented in the context of real numbers. Regarding our model (Fig. 2), we consider the praxeologies established in Chap. 1 as an intermediate stage in the transition step of type 1. Indeed, compared to school analysis, the practical blocks are already based on an abstract axiomatic foundation (of the real numbers) and conceptual definitions are formulated on this basis. The logos block already appears fully developed. Only against the background of the treatment of metric spaces (see below) does it then become clear that they can be identified mostly as concretisations in the context of real numbers of more abstract praxeologies involving the metric space structure. This is a reason why we refer to the corresponding praxeologies of real analysis as pre-structuralist (see also Laukert et al., 2023) and we see them as precursors of the structuralist praxeologies that compose the middle stage of our model.

The metric space structure is introduced in the beginning of Chap. 3. Concepts such as openness, closeness, convergence of sequences and their limit value or continuity are then defined as verbatim generalisations of the corresponding notions from Chap. 1 where they have been introduced in the context of real numbers using the absolute value:

In order to define the notions of *open set*, *interior point*, *interior of a set*, *neighbourhood*, *cluster point*, *closure*, and *closed set* for a metric space  $X$ , in the corresponding definition for subsets of  $\mathbf{R}$  we replace

- the open interval  $(x - r, x + r)$  by its analogue, the open ball  $B(x, r)$  in  $X$ , and
- the inequality  $|x - y| < r$  by the inequality  $\rho(x, y) < r$ .

**Figure 4: generalisation in terms of metric spaces (Bridges, 2018, p. 130)**

Accordingly, there are several tasks where students have to generalise proofs from the real analysis chapter: “prove that any constant mapping between metric spaces is continuous”, “prove that a contractive mapping is continuous”, or the proof that the sum, difference, max, min, product of real-valued functions defined on a metric space  $X$  and continuous at a point  $a$  are again continuous at  $a$  (p. 136). The last task, phrased in the same terms except the omission of  $X$ , was assigned in Chap. 1 as an application of properties of limits. The pre-structuralist local praxeology  $\mathbf{P}_{ps}$  around the type of tasks T “prove that a real function is continuous” and the unifying logos of which contains definitions and properties of limits and continuity thus gives birth to a full structuralist praxeology  $\mathbf{P}_s$  in which  $\mathbf{R}$  is replaced by  $X$ . Structuralist technologies are now explicit in the form of theorems established through the assigned abstract tasks. This achieves the type 1 transition (with respect to the targeted praxeology), which is facilitated for the learner by the close relationship between  $\mathbf{P}_{ps}$  and  $\mathbf{P}_s$ . It also prepares the type 2 transition by an emphasis on abstract tasks.

For reasons of space, we will limit our account of further relationships between the contents of chapters 1 and 3 with a focus on compactness. In Chap. 1, we can read: “A bounded closed interval in  $\mathbf{R}$  is also called a compact interval”. Later in the chapter, the concept of open covering of a subset  $S$  of  $\mathbf{R}$  is defined and the Heine-Borel-Lebesgue Theorem is stated and proved: “Every open cover of a compact interval  $I$  in  $\mathbf{R}$  contains a finite subcover of  $I$ ”. It is noteworthy that this result is introduced by the following meta-discourse (p. 47): “deeper results about continuity depend on two fundamental properties of the real line, described in our next two theorems. [...] Although there exists shorter proofs, the one we present is adapted to prove a more general result in Chap. 3”.

Open coverings are the essential ingredient to define compactness in the general context of metric spaces: “a metric space  $X$  is called compact if every open cover of  $X$  contains a finite subcover” (p. 147). The following task revisits the original real analysis definition: “prove that a subset of the Euclidean space  $\mathbf{R}^n$  is compact if and only if it is bounded and closed” (p. 148). This example shows very well how the dialectic between the particular and the general is addressed in the textbook. Furthermore, three types of structuralist assertions (including proofs) can be identified in the context of the concept of compactness. A first type is about generalising theorems that were already formulated and proved in the real analysis context: “a compact subset of a metric space is separable and bounded”, “a compact set in a metric space is closed” (p. 147). A second type concerns statements that highlight the specificities of the real analysis context: “a normed space is finite-dimensional if and only if its unit ball is totally bounded, in which case that ball is compact” (p. 191). Finally, a third type of assertions are proven which are not only new, but also simplify (in the sense of the structuralist methodology) proofs of statements that are easy to formulate in the context of real analysis but are cumbersome to prove: “if  $f$  is a continuous mapping of a compact metric space  $X$  into a metric space  $Y$ , then  $f(X)$  is a compact set” (p. 148). Indeed, this result provides an immediate argument for the compactness of the circle line in  $\mathbf{R}^2$ , since it can easily be identified as the image of a continuous map from  $[0, 2\pi]$  to  $\mathbf{R}^2$ . If all three types of structuralist assertions we have highlighted have in common that they are facets of the relationship between the particular and the general, i.e. between contents of real and abstract analysis, only the first facet is fully explicit in the textbook.

In the preface, the author classifies the 750 exercises contained in the textbook in 4 categories: “applications and extensions of the main propositions and theorems; results that fill in gaps in proofs or that prepare for proofs later in the book; pointers towards new branches of the subject; deep and difficult challenges for the very best students”. This typology is centred on the theoretical development of abstract analysis and generalisation processes. Indeed, the following assigned task illustrates the transition step of type 2: “prove that if  $f$  is a continuous one-one mapping of a compact metric space  $X$  onto a metric space  $Y$ , then the inverse mapping is continuous”. The type of tasks is still T, but the technique uses the topological criteria for continuity, so that knowledge on compact subsets is required to fulfil the task. By contrast, the typology lacks a category of tasks related to the investigation of new classes of concrete problems: applications of results are mainly conceived of in relation to theoretical extension. In our view, this hinders the type 2 transition due to a feeble concrete-abstract dialectic within Chap. 3. Referring again to the middle stage of our model, we didn't encounter the contextualised level 2 structuralist praxeologies  $\mathbf{P}_s$  that we expected to see and which could have contributed to the rationale of the assigned abstract tasks.



## Conclusion and outlook

We note that the praxeologies of real analysis in Chap 1 already have an axiomatic foundation and take into account conceptual definitions that build on them. The same applies to elementary arithmetic, founded on the Peano axioms, without the perspective of Ring Theory. Nevertheless, a difference with abstract algebra resides in the quite different possibilities for structuralist generalisations of real analysis concepts besides metric (or normed, etc.) spaces, such as distributions or Hilbert space theory. Beyond the elaboration of structuralist aspects of real analysis, their justification is to make problems solvable that can actually be formulated in terms of real analysis but cannot be fully developed in its context, such as Dirichlet problems for elliptic partial differential operations and much more. Considering only metric spaces, such a strong justification appears to be missing. Since Hilbert spaces and also Sobolev spaces turn out to be metric spaces, the results regarding metric spaces can be transferred to those examples. The didactic issue arises to make coherent choices between theoretical exposition of concepts and structures and classes of problems (such as partial differential equations) to motivate and apply the structuralist insights.

Our model has unveiled a realm of structures that generalises real analysis and whose abstract face at the master's level largely rivals abstract algebra, posing the same didactic problems of developing a concrete-abstract dialectic conducive to learning. Reciprocally, the advent of structures explains the setting up of pre-structuralist praxeologies in real analysis, which serve as precursors to structuralist praxeologies in abstract analysis. Praxeological analyses of other textbooks is necessary in order to consolidate our model and deepen the study of the transpositive choices around the two types of structuralist transitions we have highlighted.

## References

- Chevallard, Y., & Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 53–61). Springer. [https://doi.org/10.1007/978-3-030-15789-0\\_100034](https://doi.org/10.1007/978-3-030-15789-0_100034)
- Bridges, D. (1998). *Foundations of real and abstract analysis*. GTM 174. Springer.
- Hausberger, T. (2018). Structuralist praxeologies as a research program on the teaching and learning of abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, 4(1), 74–93. <https://doi.org/10.1007/s40753-017-0063-4>
- Jahnke., H. (2003). *A History of Analysis*. American Mathematical Society.
- Laukert, L., Hausberger, T., & Hochmuth, R. (2023). Calculus at the school to university transition: early stages of a structuralist perspective in real analysis. Proceedings of CERME13.
- Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle [Turning theory into tasks: the transition from concrete to abstract in real analysis]. In A. Rouchier & I. Bloch (Eds.), *Actes de la XIIIème école d'été en didactique des mathématiques* (cd-rom). La Pensée Sauvage.
- Winsløw, C., Barquero, B., De Vleeschouwer, M., & Hardy, N. (2014). An institutional approach to university mathematics education: from dual vector spaces to questioning the world. *Research in Mathematics Education*, 16(2), 95–111. <https://doi.org/10.1080/14794802.2014.918345>