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# Semiotic control in calculus: the case of limits of functions at the beginning of the university

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*This work is situated in the context of the secondary-tertiary transition in analysis. From the results of a test in the first year of university, we find low success rates in the calculation of limits of functions, which do not increase considerably when the graph of the function is given. To understand the lack of success in these tasks, we interview students and analyse the mathematical activities they develop (or not) when answering the test. For this, we use Activity Theory in Didactics of Mathematics, focusing on the notion of semiotic control as well as that of deconstructions with localisation perspectives.*

*Keywords: Analysis, limits, semiotic control, localisation perspectives, secondary-tertiary transition.*

## Introduction

For several years now, we have been setting up tests (MCQ) for students at the start of their university course. In these tests, we ask questions about the calculation of limits of functions to identify the knowledge acquired at the end of their secondary school education. In an earlier paper (Vandebrouck & Leidwanger, 2016), we identified that students did not necessarily use the expected algebraic methods to calculate the requested limits. They were however able to operate what we called *decompositions of algebraic formulas according to the local perspective* (Montoya et al., 2018). This helped to explain why some calculations were more successful than others, even if the complexity was higher and it also helped to explain some significant student errors. For this paper, we try to go further and find out about the control that students can exercise with the help of *graphical decompositions*. Indeed, mathematical control has been targeted as a key issue to study and explain the difficulties of students in Analysis in the secondary-tertiary transition (Flores González, 2021).

We have asked some of the students to compute the limits of some functions by also providing the graph of the functions in play. It has been observed that success rates are not radically higher when the graph of functions are available. Interviews were conducted with eight students to understand when and how graphical representations helped them to calculate the limits of functions.

## Theoretical framework

We situate our work in Activity Theory in Didactics of Mathematics (ATDM), Leontiev activity theory developed within French didactic (Rogalski, 2013; Vandebrouck, 2018). It considers activity as multidimensional and focusing on both cognitive and mediative dimensions, related to mathematical knowledge and interactions between the actors in the classroom respectively.

In this paper, we focus only on the cognitive dimension. We use *mathematical activity* to refer to students' activity on a specific mathematical task in a given context. We use what we call students' *mathematical activities* (and sub-activities) which are operationalisation (or not) of mathematical

knowledge (Robert, 2012). Mathematical activities are *algebraic* or *graphical* according to the register of semiotic representations to which they are related (Duval, 2017). Duval identifies treatments and conversions as crucial mathematical activities with semiotic representations. He also develops the fundamental role of *visualisation*, especially visualisation with dimensional deconstructions (Ibid.). In Vandebrouck and Leidwanger (2016), we introduced DWP (deconstructions with perspectives) as students' ability to visualise representations of functions (graphical or algebraic) according to the three perspectives of localisation: local, global and punctual perspective (Montoya et al., 2018). In this paper, we rely on *treatments and visualisations with DWP* to characterize students' mathematical activities on functions given with or without graphs. We also adopt concepts coming from Galperine (1966): orientation, *execution and control*. As we analyse answers to a MCQ (closed questions), we do not focus on orientation, only on execution of the task and control. During execution, students develop either treatment (mostly with applications of knowledge, organisation, with or without recognising sub-activities) or visualisations with DWP (with recognising sub-activities of pieces of knowledge about representations). Control permits the subject to ensure that the answer produced corresponds to the objective aimed at by the activity (Flores González et al., 2022). During control, students mostly develop visualisation, especially with DWP. Then the control is mostly a *semiotic control* (Flores González, 2021). For instance, in the case of the limits of functions, the visualisation of an asymptote on a graph of a function  $f$  (decomposition of the graph with local perspective) must be consistent with the calculation of a limit obtained with treatments on the algebraic expression of the function. Finally, we refer to the didactic contract defined as a set of rules (often implicit) that refer to the mathematical practices of the teacher and the student (Pepin, 2014).

## Research questions

Calculus, especially computation of limits of functions supposes treatment activities on algebraic representations. In Vandebrouck et Leidwanger (2016), we had identified that some students use DWP instead of rules of computations. Increasingly, students are doing algebraic calculations with the help of calculators and the graphical representations. We wonder to what extent students are able to control their algebraic treatments when graphical representations of the functions are given. What kind of mathematical activities do students develop during computations of limits? Do the graphical representations change the execution of the computation? With which kind of visualisation?

## Methods

A compulsory MCQ was given to 690 students at the beginning of their Bachelor's degree (September 2022) in science<sup>1</sup> at the Université Paris Cité. These students will be following quite similar courses in mathematics in the first year.

This test included several questions about the limits of functions, some questions were provided with the graph and others without graph, randomly. Among the students who answered the test, eight volunteers had an interview on the computation of three of the limits. Of these volunteers, six were

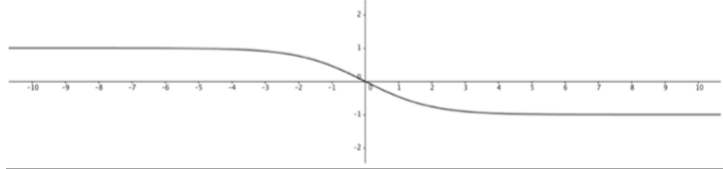
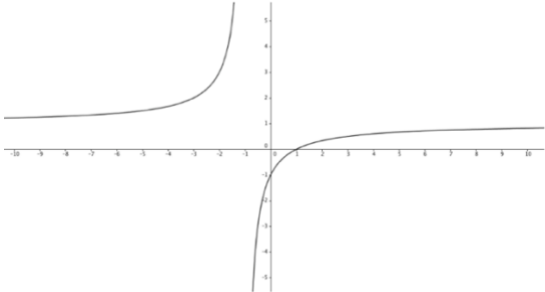
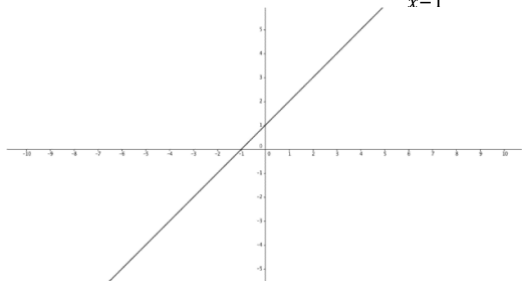
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<sup>1</sup> Degrees in Mathematics, Computer Science, Chemistry, etc. (and no Prep School).

from French high schools, two from Tunisian high schools, and all of them obtained a scientific baccalaureate in 2022. This means that they should be able to recognise indeterminate forms for the calculation of limits. They all stated that they had not done any mathematics since the end of the last school year. All eight students gave their consent for the recordings of the interviews and their written output to be used for the research. Some students had already encountered these limits in the test, others not, some with graph, others not. For the interview, the three questions provided the graph of the functions. In order to answer our research questions, the interview with the students was designed to find out their mathematical activities for the calculation of the limit, and also to find out to what extent the graph of the function influenced (or not) the mathematical activities. We recorded and transcribed the entire interview, and collected the students' written work. We also collected their answers to the test that took place just before the interview. This allowed us to make an initial comparison between the test responses and the interview responses. To analyse the transcripts and the students' written work, we used task analysis, which is described in the next section.

## Tasks analysis

**Table 1: the three questions Q1, Q2, Q3**

Q1	<p>The graphical representation of the function <math>x \mapsto \frac{e^{-x}-1}{e^{-x}+1}</math> is given</p> 	$\lim_{x \rightarrow -\infty} \frac{e^{-x}-1}{e^{-x}+1} =$ <p>Please choose an answer</p> <p><input type="radio"/> a. <math>+\infty</math></p> <p><input type="radio"/> b. 0</p> <p><input type="radio"/> c. 1</p> <p><input type="radio"/> d. -1</p>
Q2	<p>The graphical representation of the function <math>x \mapsto \frac{x-1}{x+1}</math> is given</p> 	$\lim_{x \rightarrow +\infty} \frac{x-1}{x+1} =$ <p>Please choose an answer</p> <p><input type="radio"/> a. 0</p> <p><input type="radio"/> b. 1</p> <p><input type="radio"/> c. <math>+\infty</math></p> <p><input type="radio"/> d. -1</p>
Q3	<p>The graphical representation of the function <math>x \mapsto \frac{x^2-1}{x-1}</math> is given</p> 	$\lim_{\substack{x \rightarrow 1 \\ x > 1}} \frac{x^2-1}{x-1}$ <p>Please choose an answer</p> <p><input type="radio"/> a. 2</p> <p><input type="radio"/> b. <math>+\infty</math></p> <p><input type="radio"/> c. 0</p> <p><input type="radio"/> d. 1</p>

The three questions we are focusing on concern the calculation of limits in  $-\infty$ , in  $+\infty$  and in one point (see Table 1). We make a task analysis assuming that the students do not have the graph at their disposal. In this context, students must first recognise that Q1 and Q2 are algebraic indeterminate forms (*recognising algebraic sub-activities*). In both cases, the expected activities are *treatments in*

the algebraic register, especially application of the “dominant term rule” which is supposed to be available in students’ repertory of knowledge: to factorise by  $\exp(-x)$  for Q1 and by  $x$  for Q2 in numerator and denominator in order to remove the indeterminate form. However, there exists a specific adaptation for Q1 because  $x$  goes to  $-\infty$  and therefore  $\exp(-x)$  goes to  $+\infty$ . That is to say that there is a composition of limits for Q1. Moreover, for Q2, students can only apply a simpler rule: to factorise by the term of highest degree because it is a rational fraction. By factoring in this way and then simplifying the algebraic expressions, it remains a direct application of the algebraic rules about operation on limits to find 1 in both cases (answers a and b). With this task analysis, we understand that Q1 is more complex than Q2 because of the adaptations: composition of limits and so the more complex form in Q1 than the rational fractions in Q2. For question Q3, students must also recognise an indeterminate form  $0/0$ , which students do not easily recognise. Often for them,  $0/0$  gives 0 (answer c). To remove the indeterminate form, there is no generic rule. This increases the complexity of the task compared to Q1 and Q2. Students must recognise the remarkable identity  $x^2-1=(x-1)(x+1)$ , then simplify to remove the indeterminate form and find 2 (answer a). With this analysis, we understand that Q3 is more complex than Q1 and Q2 because of the two adaptations which call for recognising activities: the specific indeterminate form  $0/0$  and the remarkable identity.

From the graph, students can use a local perspective (graphical visualisation) with recognition of the horizontal asymptote  $y=1$ , at the neighbourhoods of  $-\infty$  for Q1 and  $+\infty$  for Q2. For Q3, students can use a punctual perspective by obtaining the image of  $x=1$  as the function is extendable by continuity. These procedures can be carried out as execution or control activities.

### Results on students’ activities

Table 2 gives the success percentages of the students with and without graphs. In the last line, we give the number of students (among 690) who had this question in their test.

**Table 2: success percentage of students with and without graphs (G)**

$\lim_{x \rightarrow -\infty} \frac{\exp(-x)-1}{\exp(-x)+1} = 1$ (Q1)		$\lim_{x \rightarrow +\infty} \frac{x-1}{x+1} = 1$ (Q2)		$\lim_{x \rightarrow 1^+} \frac{x^2-1}{x-1} = 2$ (Q3)	
Without G	With G	Without G	With G	Without G	With G
40,45%	65,54%	68,38%	73,42%	18,99%	46,32%
173	168	155	194	179	177

The success percentages without graph are 40,45% (Q1), 68,38% (Q2), 18,99% (Q3) and this is coherent with our tasks’ analyses. However, when the graphs are available it should be expected that the success percentages are almost 100%, especially because it is a multiple-choice test. The success rates for the limits at  $+\infty$  and  $-\infty$  should be closed because the graphical visualisation with local perspective at infinity (DWP) seems to be of the same nature.

The table 3 shows the 8 students’ activities during the interviews for the calculation of each three limits when the graph is available. During execution or control, activities can be algebraic or graphical and activities can be treatments or visualisations with DWP. In this table, (1) means not spontaneous – (2) local perspective in  $+\infty$  instead of  $-\infty$  for Q1 – (3) the control is done with external factors (didactic contract for instance) – (4) algebraic control using symmetry of the algebraic form of Q1.

This synthetic table shows that most of the students develop graphical activities to find the limits at  $+\infty$  and  $-\infty$  (first column). However, execution with visualisation is not spontaneous for E1, and furthermore, E4 and E8 do not do it. In addition, E5 and E6 make errors in the graphical execution when they adopt the local perspective in Q1: they are looking toward  $+\infty$  instead of  $-\infty$ . Then E1 and E5 develop algebraic activities (execution) to find limit of Q2 (third column). Concerning control, it is possible to develop graphical activities when the execution has been done with algebraic activities (E4 and E8). This is not spontaneous for E4, and E8 makes a mistake by adopting the local perspective on the graph Q1 (fourth column). Only three students are able to control their graphical executions with activities in the algebraic register (E1, E2, E3). This control is semiotic and uses visualisation with local perspective for two of them (E1, E2). Three students are not able to control at all (E5, E6, E7). Regarding the students' activities for solving Q3, 3 students are not able to develop either algebraic or graphical activities (E6, E7, E8). Most of the other students solve the task using graphical activities with visualisation, and only one student (E3) is able to control with algebraic rules.

**Table 3: Students' activities during the interviews**

Student	Limits at infinite (Q1-Q2)						Limit at a point (Q3)	
	Execution Graphic visualisat	Execution Algebraic treatem.	Execution Algebraic visualisat.	Control Graphic visualisat.	Control Algebraic treatem.	Control Algebraic visualisat.	Execution Graphic visualisat.	Control Algebraic treatem.
E1	x (1)		x			x	x (3)	
E2	x					x (4)	x (3)	
E3	x				x		x	x
E4		x		x (1)			x	
E5	x (2)		x (1)				x	
E6	x (2)							
E7	x							
E8		x		x (2)				

### Some examples of students' mathematical activities during the interview

#### Graphical execution then algebraic control

Student E3 starts with graphical executions and visualisation with a local perspective. He controls his/her answers with algebraic treatments for Q1, Q2 and Q3 (see Figure 1).

M (Macarena): How do you find the limit? (Q1)

Student E3: Graphically, when  $x$  tends to  $-\infty$ , the asymptote has the equation  $y=1$ .

M: And without the graph how do you do it?

Student E3: I will add 1. This will give 2 on  $+\infty$ , and it tends towards  $+\infty$ , so towards 0. The result is 1.

M: How did you find the limit? (Q3)

Student E3: Graphically, as  $x$  tends to  $1+$  it is 1 but greater than 1, the orthogonal projection cuts the line in 2. But it can be done algebraically.

His controls are based on putting into operation the rules of algebraic calculation to prove the values of the limits. E3 is the only student who develops the activities expected in these situations. Moreover, in the test, even when the graph was not given, the student could still successfully complete the task.

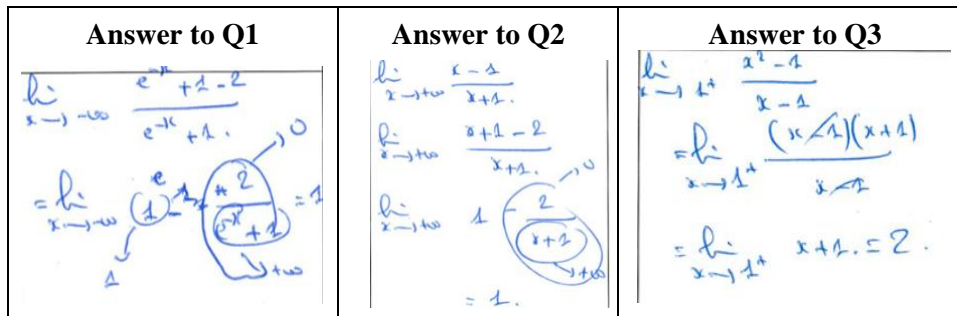


Figure 1: Student E3's written answers to the three questions

### Algebraic then graphical execution with algebraic visualisation control only on infinite limits

Student E1 develops mathematical activities in the graphical register for executions (infinite or finite limit) even if for Q1 the visualisation was spontaneously in the algebraic register. Then he does a control with algebraic treatments, but this activity includes visualisations once more (deconstruction of the algebraic formula with local perspective: it is an operationalisation of the notion of equivalence of the numerator and the denominator).

M: How do you find the limit? (Q1)

Student E1: Here I put 1, but I don't know exactly why, it's more instinctive. It's not mathematical what I'm going to say... Ah! It goes towards  $-\infty$ . Afterwards, **graphically, we can see** that when it goes towards  $-\infty$ , it goes towards 1.

M: Could you explain this differently?

Student E1: The bigger the numbers get, the more the difference is what you think in quotes... **Between -1 and +1 there is less difference, the numbers get closer and closer, and it gets closer to 1.**

However, this visualisation with local perspective is not operational for the finite limit (Q3), and the student proceeds by an execution with graphic visualisation using the punctual perspective. Finally, he cannot control his execution other than with the didactic contract (MCQ).

M: How do you find the limit? (Q3)

Student E1: We are looking for the limit of this (the function) when  $x$  tends to 1, and since the function is defined for  $x=1$ , the limit will be the value of  $f(1)$ , **here  $f(1)=2$  graphically**. And then in **the answers we are proposed 2**, so it joins what we read graphically.

### Execution with algebraic treatments but then graphical control only at infinite limit

Two students (E4 and E8) are in this category. We show the example of E4 which allows execution when there is no basis for algebraic orientation.

M: How do you find the limit? (Q1)

Student E4: To calculate the limit, we will have to put it in another form because it is an indeterminate form. So, we will factor by the dominant term. So here we know that the limit at  $-\infty$  of the exponential is 0. So, we're going to have the limit which is equal to 1, because it ( $-e^x$ ) tends towards 0, and it ( $e^x$ ) also tends towards  $-\infty$ .

These students orient and execute their calculations by algebraic treatments, as if the graphs were not provided. They put into operation the rules of calculation for the infinite limit (Q1 and Q2). Nevertheless, the calculation for the finite limit (Q3) being more complex, these two students

encounter difficulties: they apply the rule of action of infinite limit with the same calculation rules (factoring the dominant terms), which is not operational in this case:

- M: How do you find the limit? (Q3)  
Student E4: Here you have to factor by the dominant term, the one with the highest power. Here, we factor at the bottom too... (she writes). We study... ah... bah no!... I've made a mess of things, I think. But, in this case, um... I need to use the graph for this. But normally we find the same thing...

Only E4 is able to reorient and execute his task with visualisation in the graphical register. For E8, the visualisation in the neighbourhood of  $x=2$  is not possible and the graph does not help at all (for execution or control). As soon as the algebraic knowledge is insufficient to find the limit, the student cannot catch up.

### **Execution based on fragile visualisations, without control**

Student E5 performs visualisations in the algebraic or graphical register. Nevertheless, these visualisations with the local perspective are fragile because the student spontaneously looks to the right for the limit in  $-\infty$  (Q1). For the limit in  $+\infty$  (Q2), the student favours visualisation on the algebraic formula:

- M: How do you find the limit? (Q1)  
Student E5: I just look at the graph, and when I see that  $x$  tends to  $-\infty$ , I see that it tends to  $-1$ .  
M: Did you have a good look?  
Student E5: Yes (pause)... I was wrong,  $x$  tends to  $-\infty$ . So, it's  $1$  (...).

On the other hand, two students (E6 and E7) do a graphical execution (visualisation with local perspective) only for Q1 and Q2. However, for them, there is no possibility of control using the algebraic register in Q3.

- M: How do you find the limit? (Q3)  
Student E7: I would have said  $0$  positive ( $0+$ ). We look at the function when it approaches  $1$ , I said something wrong. So, we look at where it is in  $1$ , in  $1$  we see that it's between  $1$  and  $2$ . So we can't use the graph. But I wouldn't know how to answer you...

### **Conclusions**

From the results of the MCQ, even though success rates are increasing (by 25% for Q1, 5% for Q2 and 27% for Q3), the students' answers do not improve significantly when the graph of the function is given. The small sample of eight students surveyed may explain some of the students' difficulties and allows us to understand this limited increase: not all students are able to execute spontaneously the limit at  $+\infty$  and  $-\infty$  with graphical activities (visualisation with DWP), they may even make mistakes (wrong perspective when calculating the limit at  $-\infty$  for example), and few of them are able to use semiotic control. Some students find limits more easily with algebraic treatments (E4, E8) or algebraic visualisation with DWP (E1, E5) and in these cases the semiotic control with the graphic representation is not so immediate or with mistakes. Moreover, for the finite limit (Q3) only 5 out of 8 students managed to read the limit with the graph, but they couldn't control their execution with algebraic treatments (except E3). We can conclude that the semiotic control and the execution with the graph, when the graph is given, are not spontaneous, and many students make mistakes in the visualisation activity with DWP. This may explain a default of *coordination* between semiotic registers (Duval, 2017), and reinforces our arguments about the lack of mathematical control at the



secondary-tertiary transition. Furthermore, even if the graph is provided, some students still reason in a purely algebraic manner: knowing that they are in a MCQ, is this a phenomenon of the didactic contract?

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