# A COMPACT ATTRACTOR FOR ENERGY CRITICAL AND SUPER-CRITICAL NLS

Phan van Tin

# A COMPACT ATTRACTOR FOR ENERGY CRITICAL AND SUPER-CRITICAL NLS

PHAN VAN TIN

Abstract. We study the asymptotic behavior of large data solutions to nonlinear Schrödinger equations (NLS): $iu_t + \Delta u = \lambda|u|^p u + Vu$, where $\lambda = \pm 1$, $p \geqslant \frac{4}{d-2}$ and $V \in C_0^\infty(\mathbb{R}^d)$. In the case $\frac{4}{d} < p < \frac{4}{d-2}$, Tao [20, 21] proved that in radial setting, any solution which is uniformly bounded in $H^1$ can split into a term of form $e^{it\Delta}u_+$ and a remainder term which converges in $H^1(\mathbb{R}^d)$ to a compact attractor, which is invariant under the (NLS) flow. In this paper, using the method used in [20], we prove that the similar result holds in the case $p \geqslant \frac{4}{d-2}$. Specially, we work on fractional Sobolev space $H^s(\mathbb{R}^d)$ instead of $H^1(\mathbb{R}^d)$ for some $s > s_p := \frac{d}{2} - \frac{2}{p}$, which is the most difficulty in the proof.

## Contents



## 1. Introduction

We consider the following (NLS) equation:

$$\begin{cases} iu_t + \Delta u = \lambda|u|^p u + Vu, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = \phi, \end{cases} \tag{1.1}$$

where $d \geqslant 5$, $\lambda = \pm 1$, $p \geqslant \frac{4}{d-2}$, and $V \in C_0^\infty(\mathbb{R}^d)$. Let $s_p = \frac{d}{2} - \frac{2}{p}$ and $s$ satisfy $\frac{d}{2} > s > s_p$. Throughout of this paper, we assume that $p$ is even or $p > \lceil s \rceil$ (the smallest integer number larger or equal $s$) and solutions to (1.1) are uniformly bounded in $H^s(\mathbb{R}^d)$:

$$E := \sup_{t \in I_{\max}(u)} \|u(t)\|_{H^s}^2 < \infty, \tag{1.2}$$

where $I_{\max}(u)$ is the maximal time of existence of $u$. Using Theorem 1.2, under the above assumptions, we have $I_{\max}(u) = \mathbb{R}$ i.e $u$ is global. For convenience, we denote $H = H^s$. We abbreviate

*Date*: November 16, 2024.
2020 *Mathematics Subject Classification.* 35Q55.
*Key words and phrases.* Nonlinear Schrödinger equations, Compact attractor, soliton resolution conjecture, energy super-critical power.

$\|u\|_{L^p(\mathbb{R}^d)}$ by $\|u\|_{L^p}$ or sometimes by $\|u\|_p$.

We consider the following conjecture called *soliton resolution conjecture*, which states that: any global, uniformly bounded solution $u$ of dispersive equations decomposes in large time as follows:

$$u(t) = e^{it\Delta}u_+ + \sum_{j=1}^{J} R_j(t,x) + o(1),$$

where $R_j$ is a soliton and $o(1)$ converges to zero as $t \to +\infty$. There are two special cases of this conjecture:

(1) As $t \to +\infty$, $u(t) = \sum_{j=1}^{J} R_j(t,x) + o(1)$. In this case, $u(t,x)$ is called a *multi-soliton*.
(2) As $t \to +\infty$, $u(t) = e^{it\Delta}u_+ + o(1)$. In this case, $u(t,x)$ scatters forward in time.

Recently, in [9, 7], the authors proved that the soliton resolution conjecture holds for the radial critical wave equation in all odd space dimensions and in six space dimension. In [13], the authors showed the conjecture for the equivariant wave maps equation $\mathbb{R}^{1+2} \to \mathbb{S}^2$, in all equivalence class $k \in \mathbb{N}$. To our best knowledge, there is no a proof for the soliton resolution conjecture for general nonlinear Schrödinger equations even in radial setting. In [20, 21], the author proved a weaker result than the above conjecture. More precisely, the author showed that, in radial setting, if $\frac{4}{d} < p < \frac{4}{d-2}$, any global uniformly bounded solution in $H^1(\mathbb{R}^d)$ to (1.1) decomposes in large time into $u(t) = e^{it\Delta}u_+ + K + o(1)$, where $u_+ \in H^1(\mathbb{R}^d)$, and $K$ is a compact set which is invariant under the flow of (1.1). Our goal in this paper is to give a similar result to [20, 21] in the energy critical and supercritical cases i.e $p \geqslant \frac{4}{d-2}$.

In the case $V = 0$, (1.1) becomes the usual (NLS) with a single power nonlinearity:

$$iu_t + \Delta u = \lambda|u|^p u. \tag{1.3}$$

In energy critical and energy-subcritical cases (i.e $s_p \leqslant 1$), there are a lot of interested in the studying of the multi-solitons theory and the scattering theory for nonlinear Schrödinger equations. In energy supercritical cases (i.e $s_p > 1$), the situation is more complex, where we work on fractional Sobolev spaces. In [5], the authors proved local well posedness for (1.3) in high regular Sobolev space $H^s(\mathbb{R}^d)$ ($s \geqslant s_p$). In [16], the authors showed that for defocusing (NLS) (i.e $\lambda = 1$) in high dimensions ($d \geqslant 5$), any uniformly bounded solution to (1.3) in critical Sobolev space $\dot{H}^{s_p}$ is global and scatters under a suitable condition of $s_p$. Similar results were proved in [19, 17, 8]. Specially, in [18], the authors showed that there exists a blow up solutions of (1.3) with energy supercritical power even in defocusing case. Recently, in [11], we established a general profile decomposition and proved some scattering results for general nonlinear Schrödinger equations. Moreover, in [10], we extend the scattering result in [15] for $L^2$-supercritical powers, specially, our results adapt to the cases of energy-supercritical nonlinearity.

Inspired by [5], we study local and global theory for (1.1). Define

$$\rho = \frac{p+2}{1+ps/d}, \quad \gamma = \frac{4(p+2)}{p(d-2s)}.$$

It is easy to check that $(\gamma, \rho)$ is a admissible pair and satisfies the Sobolev embedding $\dot{H}^{s,\rho} \hookrightarrow L^{\rho^*}$, where $\rho^*$ is defined by

$$\frac{1}{\rho'} = \frac{p}{\rho^*} + \frac{1}{\rho}.$$

We have the following properties of $(\gamma, \rho)$:

**Proposition 1.1.** *([5, Proposition 1.5]) We have*

(i) $2 < \rho < \frac{2d}{d-2}$;
(ii) $2/\gamma = d(1/2 - 1/\rho)$, *and hence* $(\gamma, \rho)$ *is an admissible pair;*
(iii) $\rho < d/s$ *and hence* $\rho^* > \rho$;
(iv) $1/\rho' = p/\rho^* + 1/\rho$;
(v) $1/\gamma' \geqslant p/\gamma + 1/\gamma$.

As in [5], we have the following result:

**Theorem 1.2.** *Let $\phi \in H^s$ and $p, s$ be as the above. Then there exists a unique solution $u$ of (1.1) such that $u \in C(0, T_{\max}; H^s) \cap L^\gamma_{loc}(0, T_{\max}; H^{s,\rho})$. Moreover, the following properties hold:*

(i) *$u \in L^q(0, T; H^{s,r})$ for every admissible pair $(q, r)$ and every $T < T_{\max}$.*
(ii) *$u$ is unique in $L^\gamma(0, T; H^{s,\rho})$ for every $T < T_{\max}$.*
(iii) *$\|u(t)\|_{L^2} = \|\phi\|_{L^2}$ and $E(u(t)) = E(\phi)$ for all $t \in (0, T_{\max})$, where the energy $E$ is defined by*

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 + V(x)\frac{1}{2}|u(t,x)|^2 + \lambda\frac{1}{p+2}|u(t,x)|^{p+2}dx.$$

(iv) *If $T_{\max} < \infty$ then*

$$\lim_{t \to T_{\max}} \|u(t)\|_H = \infty.$$

From Theorem 1.2, we see that if solution $u$ is uniformly bounded in $H^s$ then $u$ is global. Inspired by [20], we study asymptotic behavior in large time of such solution. Our main result is the following:

**Theorem 1.3.** *Let $E > 0$. Then there exists a compact subset $\mathcal{K}_{E,rad} \subset H$ which is invariant under the flow of (1.1), and such that for every radial global solution $u$ satisfying (1.2), there exists a unique $u_+ \in H^s$ such that*

$$\lim_{t \to +\infty} dist_H(u(t) - e^{it\Delta}u_+, \mathcal{K}_{E,rad}) = 0. \tag{1.4}$$

*Here and in the sequel we write $dist_H(f, K) := \inf\{\|f - g\|_H : g \in K\}$ for the distance between $f$ and $K$.*

*Remark* 1.4. Consider the compact set $\mathcal{K}_{E,rad}$ in the above theorem. Let $u_0 \in \mathcal{K}_{E,rad}$ and $u$ be the associated solution of (1.1). Since $\mathcal{K}_{E,rad}$ is invariant under the flow of (1.1), we have $u(t) \in \mathcal{K}_{E,rad}$ for all $t \in \mathbb{R}$. Thus, $\{u(t) : t \in \mathbb{R}\}$ is pre-compact in $H^s$ and $u$ is called an *almost periodic solution*. Assume that each almost periodic solution of (1.1) equals to zero, this property is called *rigidity property* (see e.g [14, Theorem 5.1], [11, Proposition 5.1], [10, Proposition 1.5]). Then, Theorem 1.3 implies that each uniformly bounded solution of (1.1) in $H^s$ scatters in both time directions. However, in the case $s = s_p$, the situation is more complex since *smoothing effect* can be not true in this case (see Remark 3.18).

*Remark* 1.5. It seems that we could also prove a similar result of [20, Theorem 1.28] in nonradial setting. However, in this paper, we only focus on solutions in radial setting.

Let $u$ be a solution as in Theorem 1.3. From Lemma 3.20, $e^{-it\Delta}u(t)$ is weakly convergent to some $u_+ \in H^s(\mathbb{R}^d)$ as $t$ goes to infinity. Define $v(t) = u(t) - e^{it\Delta}u_+$, which is called the *weakly bound component* of $u$. We will show that $u_+$ satisfies the property (1.4).

We recall the following equivalence of precompactness and localisation:

**Proposition 1.6.** *(see e.g [20, Proposition B.1]) Let $K \subset H^s$. Then the following are equivalent:*

(i) *$K$ is precompact in $H^s$.*
(ii) *$K$ is bounded, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have the frequency localisation estimate*

$$\left\|P_{\geqslant 1/\mu_1} f\right\|_H \lesssim \mu_0$$

*and the spatial localisation estimate*

$$\int_{|x| \geqslant 1/\mu_1} |f(x)|^2 dx \lesssim \mu_0^2,$$

*for all $f \in K$.*
(iii) *$K$ is bounded, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have the frequency localisation estimates*

$$\left\|P_{\geqslant 1/\mu_1} f\right\|_H \lesssim \mu_0$$

*and*

$$\|P_{\leqslant \mu_1} f\|_H \lesssim \mu_0$$

*and the spatial localisation estimate*

$$\int_{|x|\geqslant 1/\mu_1} |f(x)|^2 + |D^s f(x)|^2 dx \lesssim \mu_0^2,$$

*for all* $f \in K$.

The above proposition is generalized by the below criterion for compact attractor:

**Proposition 1.7.** *(see e.g* [20, Proposition B.2]*) Let* $\mathcal{U}$ *be a collection of trajectories* $u : \mathbb{R}^+ \to H$. *Then the following are equivalent:*

(i) *There exists a compact set* $K \subset H$ *such that* $\lim_{t\to+\infty} dist_H(u(t), K) = 0$ *for all* $u \in \mathcal{U}$.
(ii) $\mathcal{U}$ *is asymptotically bounded in the sense that*

$$\sup_{u\in\mathcal{U}} \limsup_{t\to+\infty} \|u(t)\|_H < \infty \tag{1.5}$$

*and for any* $\mu_0 > 0$ *there exists* $\mu_1 > 0$ *such that we have the asymptotic frequency localisation estimate*

$$\limsup_{t\to+\infty} \|P_{\geqslant 1/\mu_1} u(t)\|_H \lesssim \mu_0$$

*and the spatial localisation estimate*

$$\limsup_{t\to+\infty} \int_{|x|\geqslant 1/\mu_1} |u(t,x)|^2 dx \lesssim \mu_0^2$$

*for all* $u \in \mathcal{U}$.
(iii) $\mathcal{U}$ *is asymptotically bounded in the sense of* (1.5)*, and for any* $\mu_0 > 0$ *there exists* $\mu_1 > 0$ *such that we have the asymptotic frequency localisation estimates*

$$\limsup_{t\to+\infty} \|P_{\geqslant 1/\mu_1} u(t)\|_H \lesssim \mu_0$$

*and*

$$\limsup_{t\to+\infty} \|P_{\leqslant \mu_1} u(t)\|_H \lesssim \mu_0$$

*and the following improved spatial localisation estimate*

$$\limsup_{t\to+\infty} \int_{|x|\geqslant 1/\mu_1} |u(t,x)|^2 + |D^s u(x)|^2 dx \lesssim \mu_0^2$$

*for all* $u \in \mathcal{U}$.

From Proposition 1.6 and Proposition 1.7, we see that precompactness is formally equivalent to localisation in spatial and localisation in frequency. From Proposition 1.7, to prove Theorem 1.3, we only need to prove an asymptotic localisation in spatial and an asymptotic localisation in frequency of $v(t)$. The proof of asymptotic localisation in spatial of $v(t)$ in $L^2$ is similar in [20] and we only sketch the proof in this paper.

This paper is organised as follows. In Section 2, we introduce the notations and basic tools used in this paper. In Section 3, we prove local well-posedness of solutions to (1.1) in $H^s$ and prove some preliminary results used in the proof of the main result. In Section 4, we prove asymptotic localisation in frequency of $v(t)$. In Section 5, we prove asymptotic localisation in spatial of $v(t)$ and then prove the main result Theorem 1.3.

## Acknowledgement

The author wishes to thank Prof.Thomas Duyckaerts for his guidance and encouragement. The author is supported by postdoc fellowship of Labex MME-DII: SAIC/2022 No 10078.

## 2. Notation and preliminaries

Let $L = i\partial_t + \Delta$ be Schrödinger operator.

For convenience, let $f_1(u) = \lambda|u|^p u$, $f_2(u) = Vu$ and $f(u) = f_1(u) + f_2(u)$.

For each $X$ Banach space, define $\|(u,v)\|_X = \|u\|_X + \|v\|_X$.

Denote by $\mathcal{S}$ the space of Schwartz functions.

Denote by $\tau_h$ the shift operator in space defined by $\tau_h f(x) = f(x-h)$.

Denote by $\mathcal{F}(f) = \hat{f}$ the Fourier transformation in space variable and $\mathcal{F}^{-1}f$ or $\check{f}$ the inverse Fourier transformation.

For each $f \in \mathcal{S}$, define $D^s f := \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))$.

Denote by $S^s(I)$ the Strichartz space i.e

$$\|u\|_{S^s(I)} = \sup_{(q,r)\text{admissible pair}} \|u\|_{L_t^q(I)H_x^{s,r}},$$

and $N^s(I)$ the dual space of $S^s(I)$.

We shall need some small exponents

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 > 0,$$

where $\eta_0$ is small enough depending on $E$ and the other fixed parameters, $\eta_i$ is small enough depending on $E$, $\eta_0, \eta_1, \cdots, \eta_{i-1}$ for each $i \in \{1,2,3\}$.

Moreover, in Section 5, we shall need the other parameters:

$$1 > \mu_0 \gg \mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_4 > 0.$$

where $\mu_0$ is small enough depending on $E$, and for each $i \in \{1,2,3,4\}$, $\mu_i$ is small enough depending on $\mu_0, \mu_1, \cdots, \mu_{i-1}$.

**Lemma 2.1.** *There exists an admissible pair $(q_0, r_0)$ with $q_0 > 2$, exponents $2 < Q_0, Q < \frac{2d}{d-2s}$, and an exponent $1 \leqslant R < \frac{2d}{d+4}$ such that*

$$\frac{1}{r_0} + \frac{p}{Q_0} = \frac{1}{r_0'} \tag{2.1}$$

*and*

$$\frac{1}{2} + \frac{p}{Q} = \frac{1}{R}. \tag{2.2}$$

**Lemma 2.2.** *For any $u \in H$, we have*

$$\|f(u)\|_{H^{s,R}} \lesssim \|u\|_H^{p+1} + \|u\|_H.$$

*Proof.* By the fractional chain rule in Lemma 3.4 and $|f_1'(u)| \lesssim |u|^p$, we have

$$\left\|D^j f_1(u)\right\|_{L^R} \lesssim \|u\|_{L^Q}^p \left\|D^j u\right\|_{L^2},$$

for $j = 0, s$. Let $a$ be defined by $\frac{1}{R} = \frac{1}{a} + \frac{1}{2}$. Thus, $2 \leqslant a < \frac{d}{2}$. By Hölder and Sobolev embedding, for $j = 0, s$, we have

$$\begin{aligned}
\left\|D^j(f_2(u))\right\|_{L^R} &= \left\|D^j(Vu)\right\|_{L^R} \\
&\lesssim \left\|D^j(V)\right\|_{L^a} \|u\|_{L^2} + \|V\|_{L^a} \left\|D^j u\right\|_{L^2} \\
&\lesssim \|u\|_{L^2} + \left\|D^j u\right\|_{L^2} \\
&\lesssim \|u\|_H.
\end{aligned}$$

This implies the desired result. □

We have the following properties in Fourier analysis.

**Lemma 2.3.** *Let $f, g \in \mathcal{S}$. We have the following identity*

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g).$$

**Lemma 2.4.** *(Bilinear Strichartz estimate)(see e.g [22]).*
*For any time interval $I \subset \mathbb{R}$, any $t_0 \in I$ and any $0 < \delta \leqslant \frac{1}{2}$, we have*

$$\begin{aligned}\|uv\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim_{\delta,q,r,\tilde{q},\tilde{r}} &\left(\|u(t_0)\|_{\dot{H}^{-1/2+\delta}} + \left\||\nabla|^{-1/2+\delta} Lu\right\|_{L^{q'}L^{r'}(I\times\mathbb{R}^d)}\right)\\ &\times \left(\|v(t_0)\|_{\dot{H}^{(d-1)/2-\delta}} + \left\||\nabla|^{(d-1)/2-\delta} Lv\right\|_{L^{\tilde{q}'}L^{\tilde{r}'}(I\times\mathbb{R}^d)}\right),\end{aligned}$$

*for any $u, v$ and any admissible pairs $(q, r)$, $(\tilde{q}, \tilde{r})$ with $q, \tilde{q} > 2$.*

As a consequence of Lemma 2.4, we have the following result.

**Lemma 2.5.** *(see e.g [20, Corollary 4.4]) For any time interval $I \subset \mathbb{R}$, any $N, M > 0$ dyadic numbers, we have*

$$\|u_N u_M\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim \langle |I| \rangle^{\frac{1}{2}} \frac{M^{(d-1)/2-\delta} N^{-\frac{1}{2}+\delta}}{\langle N \rangle^s \langle M \rangle^s}.$$

**Lemma 2.6** (Dispersive estimate)**.** *For all $2 \leqslant r \leqslant +\infty$, $f \in L^{r'}(\mathbb{R}^d)$, and $t > 0$ we have*

$$\left\|e^{it\Delta} f\right\|_{L^r} \lesssim \frac{1}{|t|^{d(1/2-1/r)}} \|f\|_{L^{r'}}.$$

The following lemma can be useful in the further analysis.

**Lemma 2.7.** *For $1 \leqslant r \leqslant 2$, $f \in C_0^\infty(\mathbb{R}^d)$, we have*

$$\left\|e^{it\Delta} f\right\|_{L^r} \lesssim_f (1+t)^M,$$

*for all $M \in \mathbb{N}$ and $M > d/2 - d/r$.*

*Proof.* Let $a$ such that $\frac{1}{r} = \frac{1}{a} + \frac{1}{2}$, and $M$ be a natural number such that $Ma > d$. We have

$$\begin{aligned}\left\|e^{it\Delta} f\right\|_{L^r} &= \left\|\mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^r_x}\\ &= \left\|(1+|x|)^{-M}(1+|x|)^M \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^r_x}\\ &\lesssim \left\|(1+|x|)^{-M}\right\|_{L^a} \left\|(1+|x|)^M \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^2_x}\\ &\lesssim \left\|(1+|x|)^M \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^2_x}\\ &\approx \left\|e^{-it|\xi|^2}\hat{f}(\xi)\right\|_{H^M_\xi} \quad \text{(by Plancherel theorem)}\\ &\lesssim_f (t+1)^M,\end{aligned}$$

where $H^M_\xi$ denotes the Sobolev space with variable $\xi$:

$$H^M_\xi := \{v \in L^2_\xi : D^M v \in L^2_\xi\}.$$

□

**Lemma 2.8.** *Let $N$ be a dyadic number. We have*

$$P_N(fg) = P_N(P_{\gtrsim N} f g) + P_N(f P_{\gtrsim N} g).$$

*Proof.* Using Lemma 2.3, we have

$$\begin{aligned}\mathcal{F}(P_N(fg))(\xi) &\approx 1_{|\xi|\approx N}(\hat{f}*\hat{g})(\xi)\\ &\approx \int_{\mathbb{R}^d} 1_{|\xi|\approx N}\hat{f}(\tilde{\xi})\hat{g}(\xi-\tilde{\xi})d\tilde{\xi}\\ &= \int_{\mathbb{R}^d} 1_{|\xi|\approx N}(1_{|\tilde{\xi}|\gtrsim N}+1_{|\xi-\tilde{\xi}|\gtrsim N})\hat{f}(\tilde{\xi})\hat{g}(\xi-\tilde{\xi})d\tilde{\xi}\\ &= \int_{\mathbb{R}^d} 1_{|\xi|\approx N}1_{|\tilde{\xi}\gtrsim N|}\hat{f}(\tilde{\xi})\hat{g}(\xi-\tilde{\xi})d\tilde{\xi}+\int_{\mathbb{R}^d} 1_{|\xi|\approx N}1_{|\xi-\tilde{\xi}|\gtrsim N}\hat{f}(\tilde{\xi})\hat{g}(\xi-\tilde{\xi})d\tilde{\xi}\\ &= P_N(P_{\gtrsim N}fg)+P_N(fg_{\gtrsim N}).\end{aligned}$$

□

**Lemma 2.9.** *(Interchange of norms) Let $r>1$ and $f\in L^1_yL^r_x$. Then*

$$\|f\|_{L^r_xL^1_y}\leqslant\|f\|_{L^1_yL^r_x}.$$

**Lemma 2.10.** *Let $f\in L^r$, $r>1$. Then, for each $N\geqslant 1$ dyadic number, we have*

$$\|P_{\geqslant N}f\|_{L^r}\lesssim\sup_{|h|\leqslant 1/N}\|\tau_hf-f\|_{L^r}.$$

*Proof.* Assume that $N=2^j$. We have

$$P_{\geqslant N}f=\sum_{k\geqslant j}Q_kf,$$

where $Q_kf$ is defined by

$$\mathcal{F}(Q_kf)(\xi)=\eta(2^{-k}\xi)\hat{f}(\xi),$$

for smooth function $\eta\in C_0^\infty(\mathbb{R}^d)$ nonnegative, supported in $\{1/2<|\xi|<2\}$ and satisfying

$$\sum_{j=-\infty}^{\infty}\eta(2^j\xi)=1,\quad \text{on}\,\mathbb{R}\setminus\{0\}.$$

From [6, Proof of Proposition 3.1], we have $Q_kf=f*\psi_k$, where $\psi_k\in\mathcal{S}$,

$$|\psi_k(x)|\leqslant C_M2^{kd}(1+2^k|x|)^{-M},\tag{2.3}$$

for all $M$, uniformly in $k\in\mathbb{Z}$, and

$$\int_{\mathbb{R}^d}\psi_k=0.$$

We use the following norm on $\mathbb{R}^d$

$$|y|=\sup_{i=1,\cdots,d}|y_i|.$$

We have

$$\begin{aligned}\|P_{\geqslant N}f\|_{L^r} &= \left\|f*\sum_{k\geqslant j}\psi_k\right\|_{L^r}\\ &= \left\|\int_{\mathbb{R}^d}f(x-y)\sum_{k\geqslant j}\psi_k(y)dy\right\|_{L^r_x}\\ &= \left\|\int_{\mathbb{R}^d}(f(x-y)-f(x))\sum_{k\geqslant j}\psi_k(y)dy\right\|_{L^r_x}.\end{aligned}$$

We rewrite $f(x)-f(x-y)$ by

$$(f(x)-f(x-e_1/N))+(f(x-e_1/N)-f(x-(e_1+e_2)/N))+\cdots+(f(x-(e_1+\cdots+e_h)/N)-f(x-y)),$$

where $e_i$ is the unit vector chosen such that

$$|y-(e_1+\cdots+e_h)/N|<1/N.$$

We see that there are about $h \lesssim 1 + |Ny| = 1 + 2^j|y|$ terms in the above sum. Thus, by Lemma 2.9, we have

$$\|P_{\geqslant N} f\|_{L^r} \leqslant \sup_{|h|\leqslant 1/N} \|\tau_h f - f\|_{L^r} \int_{\mathbb{R}^d} (1 + 2^j|y|)|\sum_{k\geqslant j} \psi_k(y)|dy.$$

Consider the integral in the above expression. Let denote it by $A$. It suffices to show that $A$ is bounded. In the domain $|y| > 2^{-j}$, using (2.3), the integral is bounded by

$$\int_{|y|>2^{-j}} 2^j|y| \sum_{k\geqslant j} C_M 2^{kd}(2^k|y|)^{-M} dy \lesssim \int_{|y|>2^{-j}} 2^j C_M 2^{j(d-M)}|y|^{1-M} dy.$$

It is easy to see that the above integral is bounded when $M > d+1$. In the domain $|y| \leqslant 2^{-j}$, $A$ is bounded by

$$\int_{\mathbb{R}^d} |\sum_{k\geqslant j} \psi_k(y)|dy.$$

Since $\sum_{k\geqslant j} \psi_k$ is the kernel of $P_{\geqslant N}$, this is in $L^1$. This completes the proof. □

**Lemma 2.11.** *([2, Proposition 9.3]) Let $f \in H^{1,r}$, $r > 1$. Then,*

$$\|\tau_h f - f\|_{L^r} \leqslant |h| \|\nabla f\|_{L^r},$$

*for all $h > 0$.*

The following expression is very useful in our proofs.

**Lemma 2.12.** *Let $f \in C^1(\mathbb{C})$. Then,*

$$f(u) - f(v) = (u - v) \int_0^1 \partial_z f(v + \theta(u - v))d\theta + \overline{u - v} \int_0^1 \partial_{\bar{z}} f(v + \theta(u - v))d\theta.$$

## 3. Local estimate

In this section, we prove some useful results to study asymptotic behavior in large time of solutions to (1.1).

### 3.1. Some useful estimates.

We need the following results for our analysis, many of them were used in [1]:

**Lemma 3.1.** *Let $-\infty < s_2 \leqslant s_1 < \infty$ and $1 < p_1 \leqslant p_2 < \infty$ with $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$. Then we have the following embeddings:*

$$\dot{H}^{s_1,p_1} \subset \dot{H}^{s_2,p_2}, \quad H^{s_1,p_1} \subset H^{s_2,p_2}.$$

**Lemma 3.2** (Product chain rule 1). *(see e.g [1, Lemma 2.2]) Let $s \geqslant 0$, $1 < r, r_2, p_1 < \infty$, $1 < r_1, p_2 \leqslant \infty$ such that $\frac{1}{r} = \frac{1}{r_i} + \frac{1}{p_i}$ $(i = 1, 2)$. Then,*

$$\|fg\|_{\dot{H}^{s,r}} \lesssim \|f\|_{r_1} \|g\|_{\dot{H}^{s,p_1}} + \|f\|_{\dot{H}^{s,r_2}} \|g\|_{p_2}.$$

As a consequence of the above lemma, we have the following estimate:

**Lemma 3.3** (Product chain rule 2). *(see e.g [1, Corollary 2.3]) Let $s \geqslant 0$, $q \in \mathbb{N}$ $(q \geqslant 1)$, $1 < r, r_k^i < \infty$, for $1 \leqslant i, k \leqslant q$: $\frac{1}{r} = \sum_{i=1}^q \frac{1}{r_k^i}$. Then*

$$\left\| \prod_{i=1}^q f_i \right\|_{\dot{H}^{s,r}} \lesssim \sum_{k=1}^q \left( \|f_k\|_{\dot{H}^{s,r_k^k}} \prod_{i\neq k} \|f_i\|_{r_k^i} \right).$$

**Lemma 3.4** (Fractional chain rule). *(see e.g [1, Lemma 2.4]) Let $G \in C^1(\mathbb{C})$, $s \in (0, 1)$, $1 < r, r_2 < \infty$, and $1 < r_1 \leqslant \infty$ satisfying $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then,*

$$\|G(u)\|_{\dot{H}^{s,r}} \lesssim \|G'(u)\|_{r_1} \|u\|_{\dot{H}^{s,r_2}}.$$

**Lemma 3.5** (Gagliardo-Nirenberg inequality). *(see [3]). Let $s_1 \leqslant s_0$, $1 \leqslant p_0, p_1, p \leqslant \infty$, $p_0 > 1$, $s = \theta s_1 + (1 - \theta)s_0$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$. Then*

$$\|u\|_{H^{s,p}} \lesssim \|u\|^{\theta}_{H^{s_1,p_1}} \|u\|^{1-\theta}_{H^{s_0,p_0}}.$$

**Lemma 3.6** (Homogeneous Gagliardo-Nirenberg inequality). *(see* [11, Lemma 2.5]*) Let* $s_1 \leqslant s_0$, $p_0 > 1$, $s = \theta s_1 + (1-\theta)s_0$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$. *Then*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|^{\theta}_{\dot{H}^{s_1,p_1}} \|u\|^{1-\theta}_{\dot{H}^{s_0,p_0}}.$$

**Lemma 3.7.** *(see* [1, Lemma 3.2]*) Let* $s > 0$, $1 < p < \infty$ *and* $v = s - \lfloor s \rfloor$. *Then*

$$\sum_{|\alpha| = \lfloor s \rfloor} \|D^\alpha f\|_{\dot{H}^v_p} \approx \|f\|_{\dot{H}^s_p},$$

*where* $\lfloor s \rfloor$ *denotes the largest integer smaller or equal* $s$ *(the integer part of* $s$*).*

**Lemma 3.8** (Leibniz rule). *Let* $f \in C^k$ *be a real variable function and* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{N}^d$ *such that* $|\alpha| \leqslant k$. *Then* $D^\alpha f(u)$ *is written as follows:*

$$D^\alpha f(u) = \sum_{h=1}^{|\alpha|} \sum_{\Lambda^h_\alpha} C_{\alpha,h} f^{(h)}(u) \prod_{i=1}^{h} D^{\beta_i} u, \tag{3.1}$$

*where* $C_{\alpha,h} \in \mathbb{N}$ *and* $\Lambda^h_\alpha = \{(\beta_1, \cdots, \beta_h) :$ *for each* $i$: $\beta_i \in \mathbb{N}^d$, $\beta_1 + \cdots + \beta_h = \alpha, |\beta_i| \geqslant 1\}$.

3.2. **Local theory of** (1.1) **in** $H^s$. In this subsection, we establish local theory for (1.1) in $H^s$ and prove Theorem 1.2. Define

$$\mathcal{G}(f(u(t))) = -i \int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

We may rewrite (1.1) by

$$u(t) = S(t)\phi + \mathcal{G}(f(u(t))).$$

Before proving Theorem 1.2, we prove some useful results.

**Lemma 3.9.** *(*[5, Lemma 4.1]*) We have for any* $(q, r)$ *admissible:*

$$\|\mathcal{G}(f(u))\|_{L^q(0,T)\dot{H}^{s,r}(\mathbb{R}^d)} \lesssim T^\delta \|u\|^{p+1}_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} + T^{\frac{\gamma-2}{\gamma}} \|u\|_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} \tag{3.2}$$

*and*

$$\begin{aligned}\|\mathcal{G}(f(u)) - \mathcal{G}(f(v))\|_{L^q(0,T)L^r(\mathbb{R}^d)} &\lesssim T^\delta \left( \|u\|^p_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} + \|v\|^p_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} \right) \|u - v\|_{L^\gamma(0,T)L^\rho(\mathbb{R}^d)} \\ &\quad + T^{\frac{\gamma-2}{\gamma}} \|u - v\|_{L^\gamma(0,T)L^\rho(\mathbb{R}^d)}, \end{aligned} \tag{3.3}$$

*where* $\delta = 1 - \frac{p+2}{\gamma}$.

*Proof.* Use the proof of Lemma 3.11 and the fact that $\dot{H}^{s,\rho} \hookrightarrow L^{\rho^*}$. We note that in the proof, we only need the condition $p$ even or $p > \lfloor s \rfloor$ (instead of $p > \lceil s \rceil$). □

Moreover, we have the following unique result.

**Proposition 3.10.** *(*[5, Proposition 4.2]*) Let* $\phi \in H^s$, *and suppose* $u \in L^\gamma(0,T;H^{s,\rho})$ *is a solution of* (1.1). *Then* $u \in L^q(0,T;H^{s,r}) \cap C([0,T];H^s)$ *for every admissible pair* $(q,r)$. *Furthermore, if* $v \in L^\gamma(0,T;H^{s,\rho})$ *is also a solution of* (1.1), *then* $u = v$.

*Proof.* By Lemma 3.9 and Strichartz, it is easy to prove that $u \in L^q(0,T;H^{s,r}) \cap C([0,T];H^s)$ for each admissible pair $(q,r)$.
Assume that $u(t) \neq v(t)$ for some $t \in [0,T]$. Let $t_0 = \inf\{t \in [0,T] : u(t) \neq v(t)\}$. Using Lemma 3.9, we have, for all $t \in [t_0, T]$:

$$\begin{aligned}\|u - v\|_{L^\gamma(t_0,t);L^\rho} &= \|\mathcal{G}(f(u)) - \mathcal{G}(f(v))\|_{L^\gamma(t_0,t);L^\rho} \\ &\leqslant C(t-t_0)^\delta \left( \|u\|^p_{L^\gamma(t_0,t);\dot{H}^{s,p}} + \|v\|^p_{L^\gamma(t_0,t);\dot{H}^{s,p}} \right) \|u - v\|_{L^\gamma(t_0,t);L^\rho} \\ &\quad + \|u - v\|_{L^\gamma(t_0,t);L^\rho} (t-t_0)^{\frac{\gamma-2}{\gamma}}.\end{aligned}$$

Let $t$ be close to $t_0$, we give a contradiction. Thus, $u(t) = v(t)$ for all $t \in [0,T]$, which is the desired result. □

*Proof Theorem 1.2.* We use a fixed point argument to prove the existence of solutions to (1.1). Let $M>0$, we define

$$\mathcal{X}=\mathcal{X}(T,M)=\{u\in L^{\gamma}(0,T;H^{s,\rho}):\|u\|_{L^{\gamma}(0,T;\dot{H}^{s,\rho})}\leqslant M\}.$$

Endowed with the metric

$$d(u,v)=\|u-v\|_{L^{\gamma}(0,T;L^{\rho})},$$

$\mathcal{X}$ is a complete metric space. We prove that the map

$$\mathcal{K}u=S(\cdot)\phi+\mathcal{G}(f(u)).$$

is a contraction mapping on $\mathcal{X}$ for suitable constants $T,M$. Let $u\in\mathcal{X}$. Using (3.2), we have if $\|S(\cdot)\phi\|_{L^{\gamma}(0,T;\dot{H}^{s,p})}+CT^{\delta}M^{p+1}+CT^{\frac{\gamma-2}{\gamma}}M\leqslant M$ then $\mathcal{K}u\in\mathcal{X}$. From (3.3), if $2CT^{\delta}M^{p}+CT^{\frac{\gamma-2}{\gamma}}<1$, then $\mathcal{K}$ is a contraction mapping on $\mathcal{X}$. By Strichartz, $\mathcal{K}$ is a contractor on $\mathcal{X}$ if

$$C\|\phi\|_{\dot{H}^{s}}+2CT^{\delta}M^{p+1}+CT^{\frac{\gamma-2}{\gamma}}M\leqslant M. \tag{3.4}$$

Combining with Proposition 3.10, there exists a unique maximal solution to (1.1) such that $u\in C((0,T_{\max});H)\cap L^{\gamma}_{\mathrm{loc}}((0,T_{\max});H^{s,\rho})$. Assume $T_{\max}<\infty$. Using (3.4) and by classical argument (see e.g [4]), we have

$$\lim_{t\to T_{\max}}\|u(t)\|_{\dot{H}^{s}}=\infty,$$

which completes the desired result. □

**Lemma 3.11.** *Let $p$ be even or $p>\lceil s\rceil$ and $u,v\in S^{s}(I\times\mathbb{R}^{d})$. We have the following estimates:*

$$\begin{aligned}\|f_1(u)-f_1(v)\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}&\lesssim|I|^{1/q_0'-1/q_0}\Big(\|u-v\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\\&\quad+\|u-v\|_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}\Big).\\\|f_2(u)-f_2(v)\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}&\lesssim_V\|u-v\|_{L^{q_0}H^{s,r_0}}|I|^{1/q_0'-1/q_0}.\end{aligned}$$

*Proof.* It suffices to show the first estimate. The second estimate is proved by using Hölder and the fact that $V\in C_0^{\infty}(\mathbb{R}^d)$. We only consider the case $p>\lceil s\rceil$; in the case $p$ even the conclusion is easily proved by using Hölder.
By Hölder,

$$\|f_1(u)-f_1(v)\|_{L^{q_0'}H^{s,r_0'}}\lesssim\|f_1(u)-f_1(v)\|_{L^{q_0}H^{s,r_0'}}|I|^{1/q_0'-1/q_0}.$$

Thus, we only need to prove that

$$\begin{aligned}\|f_1(u)-f_1(v)\|_{L^{q_0}H^{s,r_0'}}&\lesssim\|u-v\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\\&\quad+\|u-v\|_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}.\end{aligned}$$

We have

$$f_1(u)-f_1(v)=(u-v)\int_0^1\partial_z f_1(v+\theta(u-v))d\theta+\overline{u-v}\int_0^1\partial_{\overline{z}}f_1(v+\theta(u-v))d\theta. \tag{3.5}$$

Hence, by Hölder and using $|\partial_z f_1(u)|+|\partial_{\overline{z}}f_1(v)|\lesssim|u|^p$,

$$\begin{aligned}\|f_1(u)-f_1(v)\|_{L^{q_0}L^{r_0'}}&\lesssim\|u-v\|_{L^{\infty}L^{Q_0}}\sup_{\theta\in[0,1]}\||v+\theta(u-v)|^p\|_{L^{q_0}L^{a}},\quad(\text{ where }1/a+1/Q_0=1/r_0')\\&\lesssim\|u-v\|_{L^{\infty}L^{Q_0}}(\||u|^p\|_{L^{q_0}L^{a}}+\||v|^p\|_{L^{q_0}L^{a}})\\&\lesssim\|u-v\|_{L^{\infty}L^{Q_0}}\|(u,v)\|^{p-1}_{L^{\infty}L^{Q_0}}\|(u,v)\|_{L^{q_0}L^{r_0}}.\end{aligned}$$

Thus, it remains to show that

$$\begin{aligned}\|f_1(u)-f_1(v)\|_{L^{q_0}\dot{H}^{s,r_0'}}&\lesssim\|u-v\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\\&\quad+\|u-v\|_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^{\infty}L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)}.\end{aligned}$$

By Hölder, it suffices to show:

$$\|f_1(u)-f_1(v)\|_{\dot{H}^{s,r_0'}}\lesssim\|u-v\|_{\dot{H}^{s,r_0}}\|(u,v)\|^{p}_{L^{Q_0}}+\|u-v\|_{L^{Q_0}}\|(u,v)\|^{p-1}_{L^{Q_0}}\|(u,v)\|_{\dot{H}^{s,r_0}}. \tag{3.6}$$

By using (3.5) (with noting that the contribution of the first term and the second term are similar) and Lemma 3.2, we have

$$\begin{aligned}\|f_1(u)-f_1(v)\|_{\dot H^{s,r_0'}} \lesssim \|u-v\|_{L^{Q_0}} \left\|\int_0^1 |\nabla|^s \partial_z f_1(v+\theta(u-v))d\theta\right\|_{L^a} \\ + \|u-v\|_{\dot H^{s,r_0}} \left\|\int_0^1 \partial_z f_1(v+\theta(u-v))d\theta\right\|_{L^b} \quad (\text{where } 1/b+1/r_0=1/r_0').\end{aligned}$$

Since $|\partial_z f_1(u)| \lesssim |u|^p$, we have

$$\left\|\int_0^1 \partial_z f_1(v+\theta(u-v))d\theta\right\|_{L^b} \lesssim \||u|^p+|v|^p\|_{L^b} \lesssim \|(u,v)\|_{L^{Q_0}}^p ,$$

hence the second term is acceptable. Thus, to prove (3.6), it suffices to show that, for $g=\partial_z f_1$, for each $u$:

$$\||\nabla|^s g(u)\|_{L^a} \lesssim \|u\|_{L^{Q_0}}^{p-1} \|u\|_{\dot H^{s,r_0}} . \tag{3.7}$$

Since $p>\lceil s\rceil$, the function $g$ belongs to $C^{\lceil s\rceil}$ and satisfies

$$|g^{(k)}(u)| \lesssim |u|^{p-k}, \quad \text{for each } 0\leqslant k\leqslant \lceil s\rceil . \tag{3.8}$$

By Lemma 3.7 and Leibniz rule Lemma 3.8, it suffices to show that for each $\alpha\in\mathbb{N}^d$, $|\alpha|=\lfloor s\rfloor$, $1\leqslant h\leqslant|\alpha|$, each $1\leqslant i\leqslant h$: $\beta_i\in\mathbb{N}^d$, $|\beta_i|\geqslant 1$: $\sum_{i=1}^h\beta_i=\alpha$:

$$\left\|g^{(h)}(u)\prod_{i=1}^h D^{\beta_i}u\right\|_{\dot H^{v,a}} \lesssim \|u\|_{L^{Q_0}}^{p-1}\|u\|_{\dot H^{s,r_0}} , \quad (\text{where } v=s-\lfloor s\rfloor). \tag{3.9}$$

It is easy to check that $h\leqslant|\alpha|=\lfloor s\rfloor\leqslant\lceil s\rceil<p$. Using Lemma 3.3 and (3.8), we have

$$\begin{aligned}\left\|g^{(h)}(u)\prod_{i=1}^h D^{\beta_i}u\right\|_{\dot H^{v,a}} &\lesssim \left\|g^{(h)}(u)\right\|_{\dot H^{v,m_0}}\prod_{i=1}^h\|D^{\beta_i}u\|_{L^{m_i}} \\ &\quad + \sum_{k=1}^h \left\||u|^{p-h}\right\|_{L^{\tilde m_0}}\left\|D^{\beta_k}u\right\|_{\dot H^{v,\tilde m_k}}\prod_{i=1,i\neq k}^h\|D^{\beta_i}u\|_{L^{m_i}} ,\end{aligned} \tag{3.10}$$

where $m_i$ and $\tilde m_i$ for $i=1,\cdots,h$ are chosen such that

$$\begin{aligned}&\frac{1}{m_i}=\frac{\theta_i}{r_0}+\frac{1-\theta_i}{Q_0}, \quad (\text{where } \theta_i=|\beta_i|/s)\\ &\frac{1}{m_0}=\frac{1}{a}-\sum_{i=1}^h\frac{1}{m_i},\\ &\tilde m_0=\frac{Q_0}{p-h},\\ &\frac{1}{\tilde m_k}=\frac{1}{a}-\frac{p-h}{Q_0}-\sum_{i=1,i\neq k}^h\frac{1}{m_i}=\frac{(|\beta_k|+v)/s}{r_0}+\frac{1-(|\beta_k|+v)/s}{Q_0},\end{aligned}$$

By the choice of $m_i$ and $\tilde m_i$, using Lemma 3.6, we have

$$\begin{aligned}\|D^{\beta_i}u\|_{L^{m_i}} &\lesssim \|u\|_{\dot H^{s,r_0}}^{\theta_i}\|u\|_{L^{Q_0}}^{1-\theta_i},\\ \|D^{\beta_k}u\|_{\dot H^{v,\tilde m_k}} &\lesssim \left\|D^{|\beta_k|+v}u\right\|_{L^{\tilde m_k}}\\ &\lesssim \|u\|_{\dot H^{s,r_0}}^{\tilde\theta_k}\|u\|_{L^{Q_0}}^{1-\tilde\theta_k}, \quad (\text{where } \tilde\theta_k=(|\beta_k|+v)/s).\end{aligned}$$

Thus, the second term in (3.10) is acceptable.

Consider the first term. If $s\notin\mathbb{N}$ then $\lceil s\rceil\geqslant h+1$ (since $h\leqslant|\alpha|=\lfloor s\rfloor$), using Lemma 3.4, we have

$$\begin{aligned}\left\|g^{(h)}(u)\right\|_{\dot H^{s,m_0}} &\lesssim \left\|g^{(h+1)}(u)\right\|_{L^m}\|u\|_{\dot H^{v,n}} \quad (\text{ where } 1/m+1/n=1/m_0)\\ &\lesssim \left\||u|^{p-h-1}\right\|_{L^m}\|u\|_{\dot H^{v,n}} .\end{aligned}$$

Choosing $m$ such that $(p-h-1)m = Q_0$ then $1/n = 1/m_0 - 1/m = \frac{v/s}{r_0} + \frac{\lfloor s \rfloor/s}{Q_0}$. Using Gagliardo-Nirenberg inequality Lemma 3.6, we have

$$\|u\|_{\dot{H}^{v,n}} \lesssim \|u\|^{v/s}_{\dot{H}^{s,r_0}} \|u\|^{\lfloor s \rfloor/s}_{L^{Q_0}}.$$

Thus, the first term in (3.10) is acceptable in the case $s \notin \mathbb{N}$. We next assume that $s \in \mathbb{N}$. It is easy to check that

$$\frac{1}{m_0} = \frac{1}{a} - \sum_{i=1}^{h} \frac{1}{m_i} = \frac{Q_0}{p-h}.$$

Using (3.8), we see that the first term in (3.10) is acceptable in the case $s \in \mathbb{N}$. Thus, (3.9) holds and the proof is completed.

□

*Remark* 3.12. In the proof of the above lemma, we need the condition $p > \lceil s \rceil$.

**Lemma 3.13** (Perturbation lemma)**.** *Let $u_0 \in H^s$, let $I$ be a compact time interval containing $t_0$, $A, \tilde{\mu}_1 > 0$, and $v : I \to H$ be the strong solution of the following NLS:*

$$iv_t + \Delta = f(v) + G$$

*where $v, G$ satisfy*

$$\|v\|_{L^{q_0}H^{s,r_0} \cap C^0 H(I \times \mathbb{R}^d)} + \|v(t_0) - u_0\|_H \lesssim A$$

*and*

$$\left\| e^{i(t-t_0)\Delta}(v(t_0) - u_0) \right\|_{L^{q_0}H^{s,r_0} \cap L^\infty L^{Q_0}(I \times \mathbb{R}^d)} \lesssim \tilde{\mu}_1$$

*and*

$$\|G\|_{L^{q_0'}H^{s,r_0'}(I \times \mathbb{R}^d)} \lesssim \tilde{\mu}_1.$$

*Then, if $\tilde{\mu}_1$ is small enough depending on $A$, $|I|$ then there exists a solution $u : I \to H$ of (1.1) with initial data $u(t_0) = u_0$ such that*

$$\|u - v\|_{L^{q_0}H^{s,r_0} \cap L^\infty L^{Q_0}(I \times \mathbb{R}^d)} \lesssim_{A,|I|} \tilde{\mu}_1.$$

*and*

$$\|u - v\|_{C^0 H} \lesssim_{A,|I|} 1.$$

*Proof.* By local theory, $u$ exists on $I$. Let $w = u - v$, then $w$ solves the following equation

$$iw_t + \Delta w = f(w+v) - f(v) - G; \quad w(t_0) = v(t_0) - u_0.$$

Define $X = \|w\|_{L^{q_0}H^{s,r_0} \cap L^\infty L^{Q_0}(I \times \mathbb{R}^d)}$. We have

$$X \lesssim \tilde{\mu}_1 + \|f(v+w) - f(v)\|_{L^{q_0'}H^{s,r_0'}}.$$

By Lemma 3.11, we have

$$X \lesssim \tilde{\mu}_1 + |I|^{1/q_0' - 1/q_0} X((X+A)^p + 1),$$

where we have used $\|v\|_{L^\infty L^{Q_0}(I \times \mathbb{R}^d)} \lesssim \|v\|_{L^\infty H^s(I \times \mathbb{R}^d)} \lesssim A$. If $|I|$ is small enough (depending only on $A$) and $\tilde{\mu}_1$ is small enough depending on $I, A$, we conclude that $X \lesssim \tilde{\mu}_1$. Moreover,

$$\|u - v\|_{C^0 H(I \times \mathbb{R}^d)} \lesssim A + \|f(u) - f(v)\|_{L^{q_0'}H^{s,r_0'}} \lesssim_A 1.$$

Divide $I$ into $|I|/C(A)$ intervals $I_k$ such that $|I_k|$ is small enough for each $k$. Similarly as the above, on each $I_k$, we have

$$\|w\|_{L^{q_0}H^{s,r_0} \cap L^\infty L^{Q_0}(I_k \times \mathbb{R}^d)} \lesssim \tilde{\mu}_1$$

and

$$\|w\|_{C^0 H(I_k \times \mathbb{R}^d)} \lesssim_A 1.$$

By an inductive argument, the claim follows. □

3.3. **Fixed time estimate.** From (1.2) and our convention to suppress dependence on $E$ we have

$$\|u(t)\|_H \lesssim 1 \quad \text{for all } t \in [0,+\infty) \tag{3.11}$$

and hence by Sobolev embedding we have

$$\|u(t)\|_{L^q} \lesssim 1 \quad \text{for all } 2 \leqslant q \leqslant \frac{2d}{d-2s} \text{ and } t \in [0,+\infty). \tag{3.12}$$

From Lemma 2.2, we have

$$\|f(u)\|_{H^{s,R}} \lesssim 1 \quad \text{for all } t \in [0,+\infty). \tag{3.13}$$

3.4. **Local in time estimates.** Next we prove a local-in-time Strichartz estimate.

**Lemma 3.14.** *(Local Strichartz control). For any time interval $I \subset [0,+\infty)$ and any admissible pair $(q,r)$ we have*

$$\|u\|_{L^q H^{s,r}(I)} \lesssim \langle |I| \rangle^{\frac{1}{q}} \tag{3.14}$$

*as well as the nonlinearity estimate*

$$\|f(u)\|_{L^{q_0'} H^{s,r_0'}(I)} \lesssim \langle |I| \rangle^{\frac{1}{q_0'}}. \tag{3.15}$$

*Proof.* By Theorem 1.2, the existence and uniqueness of solution $u$ are proved. Moreover, $u$ is global and $u \in L^q_{loc} H^{s,r}(\mathbb{R}\times\mathbb{R}^d)$ for any $(q,r)$ admissible. By subdividing $I$, it suffices to prove this lemma in the case when $|I|$ is much smaller than 1. Define

$$X = \|u\|_{L^{q_0}(I) H^{s,r_0}(\mathbb{R}^d)}.$$

By Strichartz and Hölder inequality, we have

$$\begin{aligned} X &\lesssim \|u\|_{L^\infty H} + \||u|^p u\|_{L^{q_0'}(I) H^{s,r_0'}} + \|Vu\|_{L^1(I) H^s} \\ &\lesssim 1 + \||u|^p u\|_{L^{q_0'}(I) H^{s,r_0'}} + |I| \, \|u\|_{L^\infty H^s} \\ &\lesssim 1 + |I|^{\frac{1}{q_0'} - \frac{1}{q_0}} \, \|u\|^p_{L^\infty L^{Q_0}} X \\ &\lesssim 1 + |I|^{\frac{1}{q_0'} - \frac{1}{q_0}} X, \quad (\text{ since } \|u\|_{L^\infty L^{Q_0}} \lesssim \|u\|_{L^\infty H} \lesssim 1) \\ &\lesssim 1 + \frac{1}{2} X, \end{aligned}$$

if $|I|$ is small enough. This give $X = O(1)$ and then $\|f(u)\|_{L^{q_0'} H^{s,r_0'}} = O(1)$. This completes the proof. □

The following result is useful to prove asymptotic localisation frequency of $v$:

**Proposition 3.15.** *(Smoothing effect). We have the following estimate*

$$\|P_N f(u)\|_{L^{q_0'}(I) L^{r_0'}} \lesssim \langle N \rangle^{-s-\eta_1} \langle |I| \rangle^{\frac{1}{q_0'}}. \tag{3.16}$$

*Proof.* It suffices to prove Proposition 3.15 for $|I| \leqslant 1$. Indeed, assume that Proposition 3.15 holds for $|I| \leqslant 1$. Divide $I$ into $O(|I|)$ subintervals $I_k$ such that $|I_k| \approx 1$. On each $I_k$, we have

$$\|P_N f(u)\|_{L^{q_0'}(I_k) L^{r_0'}} \lesssim \langle N \rangle^{-s-\eta_1}.$$

Summing the above inequality in $k$, we obtain (3.16). By Lemma 3.14, we may also assume $N \geqslant 1$. For convenience, we shall omit the domain $I \times \mathbb{R}^d$ in all norms in this proof. Our first task is to show that

$$\|D^s P_N f_1(u)\|_{L^{q_0'} L^{r_0'}} \lesssim N^{-\eta_1}, \tag{3.17}$$

where $f_1(u) = \lambda |u|^p u$. For convenience, in the proof of (3.17), we abbreviate $f_1$ by $f$.

Consider the case $p > \lceil s \rceil$. Using Lemma 3.7 and Lemma 3.8, we only need to prove that

$$\left\| P_N \left( f^{(h)}(u) \prod_{i=1}^h D^{\beta_i} u \right) \right\|_{L^{q_0'} \dot{H}^{v,r_0'}} \lesssim N^{-\eta_1}, \tag{3.18}$$

where $1 \leqslant h \leqslant \lfloor s \rfloor$, $\beta_i, \alpha \in \mathbb{N}^d$, $|\beta_i| \geqslant 1$, $|\alpha| = \lfloor s \rfloor$ and $\sum_{i=1}^h \beta_i = \alpha$.
By Lemma 2.8, we have

$$\left\| P_N \left( f^{(h)}(u) \prod_{i=1}^h D^{\beta_i} u \right) \right\|_{L^{q_0'} \dot{H}^{v,r_0'}} \lesssim \left\| P_{\gtrsim N} f^{(h)}(u) \prod_{i=1}^h D^{\beta_i} u \right\|_{L^{q_0'} \dot{H}^{v,r_0'}} \tag{3.19}$$

$$+ \sum_{k=1}^h \left\| f^{(h)}(u) (P_{\gtrsim N} D^{\beta_k} u) \prod_{i=1; i\neq k}^h D^{\beta_i} u \right\|_{L^{q_0'} \dot{H}^{v,r_0'}}, \tag{3.20}$$

where $P_{\gtrsim N} = P_{\geqslant \frac{N}{1+\lfloor s \rfloor}}$. By Lemma 3.3, the right hand side of (3.19) is controlled by the following

$$\left\| P_{\gtrsim N} f^{(h)}(u) \right\|_{L^{q_1} \dot{H}^{v,r_1}} \prod_{i=1}^h \left\| D^{\beta_i} u \right\|_{L^{q_1^{0,i}} L^{r_1^{0,i}}} \tag{3.21}$$

$$+ \left\| P_{\gtrsim N} f^{(h)}(u) \right\|_{L^{q_2} L^{r_2}} \sum_{k=1}^h \left\| D^{\beta_k} u \right\|_{L^{q_2^{k,k}} \dot{H}^{v,r_2^{k,k}}} \prod_{i=1, i\neq k}^h \left\| D^{\beta_i} u \right\|_{L^{q_2^{k,i}} L^{r_2^{k,i}}}, \tag{3.22}$$

where the above parameters satisfy, for each $k \in [1, h]$,

$$\frac{1}{q_0'} = \frac{1}{q_1} + \sum_{i=1}^h \frac{1}{q_1^{0,i}}, \tag{3.23}$$

$$\frac{1}{r_0'} = \frac{1}{r_1} + \sum_{i=1}^h \frac{1}{r_1^{0,i}}, \tag{3.24}$$

$$\frac{1}{q_0'} = \frac{1}{q_2} + \sum_{i=1}^h \frac{1}{q_2^{k,i}}, \tag{3.25}$$

$$\frac{1}{r_0'} = \frac{1}{r_2} + \sum_{i=1}^h \frac{1}{r_2^{k,i}}. \tag{3.26}$$

In (3.21) and (3.22), the parameters are chosen such that, for each $i \in [1, h]$, $i \neq k$:

$$\frac{2}{q_1^{0,i}} + \frac{d}{r_1^{0,i}} = \frac{d}{2} - (s - |\beta_i|), \tag{3.27}$$

$$\frac{2}{q_2^{k,i}} + \frac{d}{r_2^{k,i}} = \frac{d}{2} - (s - |\beta_i|), \tag{3.28}$$

$$\frac{2}{q_2^{k,k}} + \frac{d}{r_2^{k,k}} = \frac{d}{2} - (s - |\beta_k| - v). \tag{3.29}$$

It implies that the pairs $(q_1, r_1)$, $(q_2, r_2)$ in (3.21) and (3.22) satisfy:

$$\frac{2}{q_1} + \frac{d}{r_1} = \frac{d}{2} - s_1 + (p - h)\left(\frac{d}{2} - s\right), \tag{3.30}$$

$$\frac{2}{q_2} + \frac{d}{r_2} = \frac{d}{2} - s_2 + (p - h)\left(\frac{d}{2} - s\right), \tag{3.31}$$

where $s_1 = \lfloor s \rfloor - \left(2 - p\left(\frac{d}{2} - s\right)\right)$ and $s_2 = s - \left(2 - p\left(\frac{d}{2} - s\right)\right)$.

Using Sobolev embedding Lemma 3.1 and Lemma 3.14, the right hand of (3.19) is bounded by the following

$$\left\| P_{\gtrsim N} f^{(h)}(u) \right\|_{L^{q_1} \dot{H}^{v,r_1}} + \left\| P_{\gtrsim N} f^{(h)}(u) \right\|_{L^{q_2} L^{r_2}}. \tag{3.32}$$

Define $u_{\mathrm{lo}} = P_{<N^{\eta_0}} u$, $u_{\mathrm{hi}} = u - u_{\mathrm{lo}}$. Using Lemma 2.10, the second term in (3.32) is bounded by

$$\begin{aligned}
&\left\|P_{\gtrsim N} f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_2} L^{r_2}} + \left\|P_{\gtrsim N}(f^{(h)}(u) - f^{(h)}(u_{\mathrm{lo}}))\right\|_{L^{q_2} L^{r_2}} \\
&\lesssim \sup_{|\tilde{h}|\leqslant 1/N} \left\|\tau_{\tilde{h}} f^{(h)}(u_{\mathrm{lo}}) - f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_2} L^{r_2}} + \left\|f^{(h)}(u) - f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_2} L^{r_2}} \\
&\lesssim \sup_{|\tilde{h}|\leqslant 1/N} \left\|\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}\right\|_{L^{q_2^1} L^{r_2^1}} \left\|(\tau_{\tilde{h}} u_{\mathrm{lo}}, u_{\mathrm{lo}})\right\|_{L^{q_2^2} L^{r_2^2}}^{p-h} + \left\|u - u_{\mathrm{lo}}\right\|_{L^{\tilde{q}_2^1} L^{\tilde{r}_2^1}} \left\|(u_{\mathrm{lo}}, \tau_{\tilde{h}} u_{\mathrm{lo}})\right\|_{L^{\tilde{q}_2^2} L^{\tilde{r}_2^2}}^{p-h} \\
&\lesssim \left\| \frac{1}{N} \left\|\nabla u_{\mathrm{lo}}\right\|_{L_x^{r_2^1}} \right\|_{L_t^{q_2^1}} + \left\|u_{\mathrm{hi}}\right\|_{L^{\tilde{q}_2^1} L^{\tilde{r}_2^1}},
\end{aligned}$$

where the above parameters satisfy the following

$$\begin{aligned}
\frac{2}{q_2^2} + \frac{d}{r_2^2} = \frac{2}{\tilde{q}_2^2} + \frac{d}{\tilde{r}_2^2} = \frac{d}{2} - s, \\
\frac{2}{q_2^1} + \frac{d}{r_2^1} = \frac{2}{\tilde{q}_2^1} + \frac{d}{\tilde{r}_2^1} = \frac{d}{2} - s_2,
\end{aligned}$$

and hence, by Lemma 3.14,

$$\left\|(\tau_{\tilde{h}} u_{\mathrm{lo}}, u_{\mathrm{lo}})\right\|_{L^{q_2^2} L^{r_2^2}}, \left\|(u_{\mathrm{lo}}, \tau_{\tilde{h}} u_{\mathrm{lo}})\right\|_{L^{\tilde{q}_2^2} L^{\tilde{r}_2^2}} \lesssim 1.$$

Thus, the second term in (3.32) is controlled by

$$\begin{aligned}
&\frac{N^{\eta_0}}{N} \left\|u_{\mathrm{lo}}\right\|_{L^{q_2^1} L^{r_2^1}} + \left\|u_{\mathrm{hi}}\right\|_{L^{\tilde{q}_2^1} L^{\tilde{r}_2^1}} \lesssim \frac{N^{\eta_0}}{N} \left\|D^{s_2} u\right\|_{S^0} + \left\|D^{s_2} u_{\mathrm{hi}}\right\|_{S^0} \\
&\lesssim \frac{N^{\eta_0}}{N} + N^{\eta_0(s_2 - s)} \lesssim \frac{N^{\eta_0}}{N} + N^{-\eta_0\left(2 - p\left(\frac{d}{2} - s\right)\right)} \lesssim N^{-\eta_1}.
\end{aligned}$$

We next estimate the first term in (3.32). If $s \in \mathbb{N}$ then $v = 0$ and hence, we estimate the first term in (3.32) by similar argument to estimate the second term. Assume that $s \notin \mathbb{N}$. Then, $p > \lceil s \rceil = \lfloor s \rfloor + 1 \geqslant h + 1$ and hence, $f \in C^{(h+2)}(\mathbb{C})$. For $u_{\mathrm{lo}}$ as the above, we have

$$\begin{aligned}
\left\|P_{\gtrsim N} f^{(h)}(u)\right\|_{L^{q_1} \dot{H}^{v, r_1}} &\lesssim \left\|P_{\gtrsim N} f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_1} \dot{H}^{v, r_1}} + \left\|P_{\gtrsim N}(f^{(h)}(u) - f^{(h)}(u_{\mathrm{lo}}))\right\|_{L^{q_1} \dot{H}^{v, r_1}} \\
&\lesssim \left\|P_{\gtrsim N} f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_1} \dot{H}^{v, r_1}} + \left\|f^{(h)}(u) - f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_1} \dot{H}^{v, r_1}}.
\end{aligned}$$

For convenience, we denote by $A, B$ the first and the second term in the above expression. To estimate $A$, using Lemma 2.10, Lemma 2.12, Lemma 3.2, Lemma 3.4 and Lemma 2.11, we have

$$\begin{aligned}
A &\lesssim \sup_{\tilde{h}\leqslant 1/N} \left\|\tau_{\tilde{h}} f^{(h)}(u_{\mathrm{lo}}) - f^{(h)}(u_{\mathrm{lo}})\right\|_{L^{q_1} \dot{H}^{v, r_1}} \\
&\lesssim \sup_{\tilde{h}\leqslant 1/N} \left\|\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}\right\|_{L^{q_1^1} \dot{H}^{v, r_1^1}} \sup_{\theta\in[0,1]} \left\|f^{(h+1)}(u_{\mathrm{lo}} + \theta(\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}))\right\|_{L^{q_1^2} L^{r_1^2}} \\
&\quad + \sup_{\tilde{h}\leqslant 1/N} \left\|\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}\right\|_{L^{q_2^1} L^{r_2^1}} \sup_{\theta\in[0,1]} \left\|f^{(h+1)}(u_{\mathrm{lo}} + \theta(\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}))\right\|_{L^{q_2^2} \dot{H}^{v, r_2^2}} \\
&\lesssim \sup_{\tilde{h}\leqslant 1/N} \left\|\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}\right\|_{L^{q_1^1} \dot{H}^{v, r_1^1}} \sup_{\theta\in[0,1]} \left\|f^{(h+1)}(u_{\mathrm{lo}} + \theta(\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}))\right\|_{L^{q_1^2} L^{r_1^2}} \\
&\quad + \sup_{\tilde{h}\leqslant 1/N} \left\|\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}\right\|_{L^{q_2^1} L^{r_2^1}} \sup_{\theta\in[0,1]} \left\|f^{(h+2)}(u_{\mathrm{lo}} + \theta(\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}}))\right\|_{L^{q_2^3} L^{r_2^3}} \left\|u_{\mathrm{lo}} + \theta(\tau_{\tilde{h}} u_{\mathrm{lo}} - u_{\mathrm{lo}})\right\|_{L^{q_2^4} \dot{h}^{v, r_2^4}} \\
&\lesssim \frac{1}{N} \left\|D^{1+v} u_{\mathrm{lo}}\right\|_{L^{q_1^1} L^{r_1^1}} \left\|u_{\mathrm{lo}}\right\|_{L^{(p-h) q_1^2} L^{(p-h) r_1^2}}^{p-h} + \frac{1}{N} \left\|\nabla u_{\mathrm{lo}}\right\|_{L^{q_2^1} L^{r_2^1}} \left\|u_{\mathrm{lo}}\right\|_{L^{(p-h-1) q_2^3} L^{(p-h-1) r_2^3}}^{p-h-1} \left\|u_{\mathrm{lo}}\right\|_{L^{q_2^4} \dot{H}^{v, r_2^4}} \\
&\lesssim \frac{1}{N} \left\|D^{1+s_2} u_{\mathrm{lo}}\right\|_{S^0} \lesssim \frac{1}{N} N^{\eta_0(1+s_2)} \ll N^{-\eta_1},
\end{aligned}$$

where the undefined parameters in the above estimates are chosen such that

$$\begin{aligned}
&\frac{2}{q_1^2}+\frac{d}{r_1^2}=(p-h)\left(\frac{d}{2}-s\right),\\
&\frac{2}{q_1^1}+\frac{d}{r_1^1}=\frac{d}{2}-s_1,\\
&\frac{2}{q_2^3}+\frac{d}{r_2^3}=(p-h-1)\left(\frac{d}{2}-s\right),\\
&\frac{2}{q_2^4}+\frac{d}{r_2^4}=\frac{d}{2}-s+v,\\
&\frac{2}{q_2^1}+\frac{d}{r_2^1}=\frac{d}{2}-s_2,
\end{aligned}$$

and hence by Sobolev embedding $\|u_{\rm lo}\|_{L^{(p-h)q_1^2}L^{(p-h)r_1^2}}, \|u_{\rm lo}\|_{L^{(p-h-1)q_2^3}L^{(p-h-1)r_2^3}}, \|u_{\rm lo}\|_{L^{q_2^4}\dot H^{v,r_2^4}} \lesssim \|D^S u_{\rm lo}\|_{S^0}$. This implies that the term $A$ is acceptable.

Similarly, to estimate the term $B$, we have

$$\begin{aligned}
B &\lesssim \left\|f^{(h)}(u)-f^{(h)}(u_{\rm lo})\right\|_{L^{q_1}\dot H^{v,r_1}}\\
&\lesssim \|u-u_{\rm lo}\|_{L^{q_1^3}\dot H^{v,r_1^3}}\sup_{\theta\in[0,1]}\left\|f^{(h+1)}(u_{\rm lo}+\theta(u-u_{\rm lo}))\right\|_{L^{q_1^4}L^{r_1^4}}\\
&+\quad \|u-u_{\rm lo}\|_{L^{q_2^5}L^{r_2^5}}\sup_{\theta\in[0,1]}\left\|f^{(h+1)}(u_{\rm lo}+\theta(u-u_{\rm lo}))\right\|_{L^{q_2^6}\dot H^{v,r_2^6}}\\
&\lesssim \|u-u_{\rm lo}\|_{L^{q_1^3}\dot H^{v,r_1^3}}\sup_{\theta\in[0,1]}\left\|f^{(h+1)}(u_{\rm lo}+\theta(u-u_{\rm lo}))\right\|_{L^{q_1^4}L^{r_1^4}}\\
&+\quad \|u-u_{\rm lo}\|_{L^{q_2^5}L^{r_2^5}}\sup_{\theta\in[0,1]}\left\|f^{(h+2)}(u_{\rm lo}+\theta(u-u_{\rm lo}))\right\|_{L^{q_2^7}L^{r_2^7}}\|u_{\rm lo}+\theta(u-u_{\rm lo})\|_{L^{q_2^8}\dot H^{v,r_2^8}}\\
&\lesssim \|u_{\rm hi}\|_{L^{q_1^3}\dot H^{v,r_1^3}}\||u|+|u_{\rm lo}|\|^{p-h}_{L^{(p-h)q_1^4}L^{(p-h)r_1^4}}\\
&\quad+\|u_{\rm hi}\|_{L^{q_2^5}L^{r_2^5}}\||u|+|u_{\rm lo}|\|^{p-h-1}_{L^{(p-h-1)q_2^7}L^{(p-h-1)r_2^7}}(\|u\|_{L^{q_2^8}\dot H^{v,r_2^8}}+\|u_{\rm lo}\|_{L^{q_2^8}\dot H^{v,r_2^8}})\\
&\lesssim \|D^{s_2}u_{\rm hi}\|_{S^0}\lesssim N^{\eta_0(s_2-s)}\lesssim N^{-\eta_1},
\end{aligned}$$

where the undefined parameters in the above expressions satisfy

$$\begin{aligned}
&\frac{2}{q_1^4}+\frac{d}{r_1^4}=(p-h)\left(\frac{d}{2}-s\right),\\
&\frac{2}{q_1^3}+\frac{d}{r_1^3}=\frac{d}{2}-s_1,\\
&\frac{2}{q_2^7}+\frac{d}{r_2^7}=(p-h-1)\left(\frac{d}{2}-s\right),\\
&\frac{2}{q_2^8}+\frac{d}{r_2^8}=\frac{d}{2}-s+v,
\end{aligned}$$

and by Sobolev embedding we have $\||u|+|u_{\rm lo}|\|_{L^{(p-h)q_1^4}L^{(p-h)r_1^4}}, \||u|+|u_{\rm lo}|\|_{L^{(p-h-1)q_2^7}L^{(p-h-1)r_2^7}}, \|u\|_{L^{q_2^8}\dot H^{v,r_2^8}}, \|u_{\rm lo}\|_{L^{q_2^8}\dot H^{v,r_2^8}}$ $\|D^s u\|_{S^0}+\|D^s u_{\rm lo}\|_{S^0}\lesssim 1$. Thus, the term $B$ is acceptable. Combining with the boundedness of $A$, we conclude that the right hand side of (3.19) is acceptable.

We next estimate (3.20). Using the fact that $|f^{(h)}(u)| \lesssim |u|^{p-h+1}$, $|f^{(h+1)}(u)| \lesssim |u|^{p-h}$, Lemma 3.4, Lemma 3.14 and Hölder inequality, we have

$$
\begin{aligned}
(3.33)\qquad & \left\| f^{(h)}(u)(P_{\gtrsim N} D^{\beta_k} u) \prod_{i=1;i\neq k}^{h} D^{\beta_i} u \right\|_{L^{q_0'} \dot{H}^{v,r_0'}} \\
& \lesssim \|u\|_{L^{q_3^1} L^{r_3^1}}^{p-h+1} \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{q_3^k} \dot{H}^{v,r_3^k}} \prod_{i=1;i\neq k}^{h} \left\| D^{\beta_i} u \right\|_{L^{q_3^i} L^{r_3^i}} \\
& \quad + \|u\|_{L^{q_3^1} L^{r_3^1}}^{p-h+1} \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{\tilde{q}_3^k} L^{\tilde{r}_3^k}} \left\| D^{\beta_i} u \right\|_{L^{\tilde{q}_3^i} \dot{H}^{v,\tilde{r}_3^i}} \prod_{j=1;j\neq\{k,i\}}^{h} \left\| D^{\beta_j} u \right\|_{L^{\tilde{q}_3^j} L^{\tilde{r}_3^j}} \\
& \quad + \|u\|_{L^{\tilde{q}_4^0} L^{\tilde{r}_4^0}}^{p-h} \|u\|_{L^{\tilde{q}_4^1} \dot{H}^{v,\tilde{r}_4^1}} \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{\tilde{q}_4^k} L^{\tilde{r}_4^k}} \prod_{i=1;i\neq k}^{h} \left\| D^{\beta_i} u \right\|_{L^{\tilde{q}_4^i} L^{\tilde{r}_4^i}},
\end{aligned}
$$

where the about parameter satisfy

$$
\begin{gathered}
\frac{2}{q_3^1} + \frac{d}{r_3^1} = \frac{d}{2} - s, \quad \frac{2}{q_3^i} + \frac{d}{r_3^i} = \frac{d}{2} - (s - |\beta_i|), \quad \frac{2}{q_3^k} + \frac{d}{r_3^k} = \frac{d}{2} - \lfloor s \rfloor + \left(2 - p\left(\frac{d}{2} - s\right)\right) + |\beta_k|, \\
\frac{2}{\tilde{q}_3^i} + \frac{d}{\tilde{r}_3^i} = \frac{d}{2} - (s - |\beta_i| - v), \quad \frac{2}{\tilde{q}_3^j} + \frac{d}{\tilde{r}_3^j} = \frac{d}{2} - (s - |\beta_j|), \quad \frac{2}{\tilde{q}_3^k} + \frac{d}{\tilde{r}_3^k} = \frac{d}{2} - s + \left(2 - p\left(\frac{d}{2} - s\right)\right) + |\beta_k| \\
\frac{2}{\tilde{q}_4^0} + \frac{d}{\tilde{r}_4^0} = \frac{d}{2} - s, \quad \frac{2}{\tilde{q}_4^1} + \frac{d}{\tilde{r}_4^1} = \frac{d}{2} - (s - v), \quad \frac{2}{\tilde{q}_4^i} + \frac{d}{\tilde{r}_4^i} = \frac{d}{2} - (s - |\beta_i|), \\
\frac{2}{\tilde{q}_4^k} + \frac{d}{\tilde{r}_4^k} = \frac{d}{2} - s + \left(2 - p\left(\frac{d}{2} - s\right)\right) + |\beta_k|.
\end{gathered}
$$

Thus, using Sobolev embedding Lemma 3.1, (3.33) is bounded by

$$
\begin{aligned}
& \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{q_3^k} \dot{H}^{v,r_3^k}} + \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{\tilde{q}_3^k} L^{\tilde{r}_3^k}} + \left\| P_{\gtrsim N} D^{\beta_k} u \right\|_{L^{\tilde{q}_4^k} L^{\tilde{r}_4^k}} \\
& \lesssim \left\| P_{\gtrsim N} D^{s-\left(2-p\left(\frac{d}{2}-s\right)\right)} u \right\|_{S^0} \lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)} \lesssim N^{-\eta_1},
\end{aligned}
$$

which is acceptable. This implies that the term (3.20) is acceptable. Then, (3.18) and hence (3.17) holds. The case $p$ even is much more simple and is left for the reader.

Our second task is to show the following estimate:

$$
(3.34)\qquad \|D^s P_N(Vu)\|_{L^{q'} L^{r'}} \lesssim N^{-\eta_1},
$$

for all admissible pair $(q, r)$. To prove (3.34), we use the following results.

**Lemma 3.16.** $\left\| P_{\gtrsim N} V \right\|_{L^r} \lesssim_{V,k} N^{-k}$ *for all* $k \geqslant 0$.

**Lemma 3.17.** *(See [12][Corollary 1.1]) Let* $p, p_1, p_2$ *be such that* $1 < p, p_1, p_2 < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *The the following holds for all* $f, g \in \mathcal{S}$

$$
\|D^s(fg) - D^s f g - D^s g f\|_{L^p} \leqslant C \|Df\|_{L^{p_1}} \left\| D^{s-1} g \right\|_{L^{p_2}}.
$$

Denote $P_{<N^{\eta_0}} V = V_{\text{lo}}$ and $V_{\text{hi}} = V - V_{\text{lo}}$. We have

$$
\begin{aligned}
\|D^s P_N(Vu)\|_{L^{r'}} &= \|D^s P_N(V_{\text{lo}} u)\|_{L^{r'}} + \|D^s P_N(V_{\text{hi}} u)\|_{L^{r'}} \\
&= \left\| D^s P_N(V_{lo} P_{>N/10} u) \right\|_{L^{r'}} + \|D^s P_N(V_{\text{hi}} u)\|_{L^{r'}}
\end{aligned}
$$

Consider the second term. Using Lemma 3.16, Lemma 3.17 and Hölder inequality, we have

$$\begin{aligned}\|D^s P_N(V_{\rm hi}u)\|_{L^{q'}L^{r'}} &= \|D^s P_N(V_{\rm hi}u)\|_{L^{q'}L^{r'}}\\ &\leqslant \|D^s(V_{\rm hi}u)\|_{L^{q'}L^{r'}}\\ &\leqslant \|D^s(V_{\rm hi})u\|_{L^{q'}L^{r'}} + \|V_{\rm hi}D^s u\|_{L^{q'}L^{r'}} + \|DV_{\rm hi}\|_{L^{q_1}L^{r_1}}\left\|D^{s-1}u\right\|_{L^qL^r}\\ &\lesssim \|D^sV_{\rm hi}\|_{L^{q_1}L^{r_1}}\|u\|_{L^qL^r} + \|V_{\rm hi}\|_{L^{q_1}L^{r_1}}\|D^s u\|_{L^qL^r}\\ &\quad + \|DV_{\rm hi}\|_{L^{q_1}L^{r_1}}\left\|D^{s-1}u\right\|_{L^qL^r}\\ &\lesssim_V N^{-\eta_0}.\end{aligned}$$

where $(q_1, r_1)$ satisfies $\frac{1}{q_1} = \frac{1}{q'} - \frac{1}{q}$ and $\frac{1}{r_1} = \frac{1}{r'} - \frac{1}{r}$. Thus, this term is acceptable.
Consider the first term. Using Lemma 3.17 and Hölder inequality, we have

$$\begin{aligned}\left\|D^s P_N(V_{\rm lo}P_{>N/10}u)\right\|_{L^{q'}L^{r'}} &\leqslant \left\|D^s(V_{\rm lo}P_{>N/10}u)\right\|_{L^{q'}L^{r'}}\\ &\leqslant \left\|D^sV_{\rm lo}P_{>N/10}u\right\|_{L^{q'}L^{r'}} + \left\|V_{\rm lo}D^s(P_{>N/10}u)\right\|_{L^{q'}L^{r'}}\\ &\quad + \|DV_{\rm lo}\|_{L^{q_1}L^{r_1}}\left\|D^{s-1}(P_{>N/10}u)\right\|_{L^qL^r}\\ &\leqslant \|D^sV_{\rm lo}\|_{L^{q_1}L^{r_1}}\left\|P_{>N/10}u\right\|_{L^qL^r} + \left\|V_{\rm lo}D^s(P_{>N/10}u)\right\|_{L^{q'}L^{r'}}\\ &\quad + \|DV_{\rm lo}\|_{L^{q_1}L^{r_1}}\left\|D^{s-1}(P_{>N/10}u)\right\|_{L^qL^r}.\end{aligned}$$

The first term and the third term are acceptable since $\left\|P_{>N/10}u\right\|_{L^qL^r} \lesssim N^{-s}\left\|D^s(P_{>N/10}u)\right\|_{L^qL^r} \lesssim N^{-s}$ and $\left\|D^{s-1}(P_{>N/10}u)\right\|_{L^qL^r} \lesssim N^{-1}\left\|D^s(P_{>N/10}u)\right\|_{L^qL^r} \lesssim N^{-1}$. Consider now the second term. First, we have

$$\left\|V_{\rm lo}D^s(P_{>N/10}u)\right\|_{L^{\tilde q'}L^{\tilde r'}(I\times\mathbb{R}^d)} \lesssim 1,$$

for all $\tilde q, \tilde r$ is sufficiently close to $q, r$.
Moreover, using Lemma 2.5 for $0 < \delta < \frac{1}{2}$, we have

$$\begin{aligned}\left\|V_{\rm lo}D^s(P_{>N/10}u)\right\|_{L^2_{t,x}(I\times\mathbb{R}^d)} &\leqslant \sum_{\substack{M<N^{\eta_0}\text{ dyadic},\\ \tilde M>N/10\text{ dyadic}}} \|V_M D^s u_{\tilde M}\|_{L^2_{t,x}(I\times\mathbb{R}^d)}\\ &\lesssim \sum_{\substack{M<N^{\eta_0}\text{ dyadic},\\ \tilde M>N/10\text{ dyadic}}} \frac{M^{(d-1)/2-\delta}\tilde M^{-1/2+\delta}}{\langle M\rangle^s}\\ &\lesssim \sum_{M<N^{\eta_0}\text{ dyadic}} M^{(d-1)/2-\delta}N^{-1/2+\delta}\\ &\lesssim N^{\eta_0((d-1)/2-\delta)}N^{-1/2+\delta},\end{aligned}$$

this is an acceptable term if we choose $\eta_0, \delta \ll 1$. By interpolation, this implies that the term $\left\|V_{\rm lo}D^s(P_{>N/10}u)\right\|_{L^{q'}L^{r'}}$ is also acceptable. Thus, (3.34) holds. Hence, $\|D^sP_Nf(u)\|_{L^{q_0'}(I)L^{r_0'}(\mathbb{R}^d)} \lesssim N^{-\eta_1}$ and the proof of Proposition 3.15 is completed. □

*Remark* 3.18. In the above proof, the condition $s > s_p$ is useful since we should have $2 > p\left(\frac{d}{2} - s\right)$. In the critical case $s = s_p$, the situation is more complex and the smoothing effect can be not true.

*Remark* 3.19. In the above proof, the case $p$ even is more simple. To deal with this case, we don't need to use Lemma 2.10, Lemma 2.11 to estimate the term of form $\left\|P_{\gtrsim N}f^{(h)}(u)\right\|_{L^q\dot H^{\tilde v,r}}$, where $\tilde v = 0$ or $\tilde v = v$. It suffices to use the product chain rule for this purpose since $f^{(h)}(u)$ is a polynomial in $u, \overline{u}$.

**Lemma 3.20.** *([20, Lemma 5.1]) Let $E > 0$ and $u$ be a global solution of (1.1) such that $\sup_{t\in\mathbb{R}}\|u(t)\|_H \leqslant E$. Then, $e^{-it\Delta}u(t)$ is weakly convergent to $u_+ \in H$ as $t \to +\infty$.*

**Proposition 3.21.** *([20, Proposition 5.2]) Let $E > 0$ and $u$ be a global solution of (1.1) such that $\sup_{t\in\mathbb{R}}\|u(t)\|_H \leqslant E$. Then there exists a unique decomposition*

$$u(t) = e^{it\Delta}u_+ + v(t), \tag{3.35}$$

*where $u_+ \in H$ with*

$$\|u_+\|_H^2 \leqslant E \tag{3.36}$$

*and*

$$\|v(t)\|_H^2 \leqslant 4E, \tag{3.37}$$

*for all* $t \geqslant 0$. *We also have the following identities*

$$v(t) = e^{it\Delta}(u(0) - u_+) - i\int_0^t e^{i(t-t')\Delta} f(u(t'))dt' \tag{3.38}$$

*and*

$$v(t) = \underset{T\to+\infty}{iw\text{-}lim} \int_t^T e^{i(t-t')\Delta} f(u(t'))dt'. \tag{3.39}$$

*Remark* 3.22. Assume that the solution $u$ in Proposition 3.21 is almost periodic in the sense that $\{u(t) : t \in \mathbb{R}^+\}$ is pre-compact in $H^s$. Then $u_+ = 0$. Indeed, since $e^{-it\Delta}u(t)$ weakly converges to $u_+$ in $H^s$, $e^{-it\Delta}u(t)$ also weakly converges to $u_+$ in $L^2$. It suffices to show that for all $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\lim_{t\to\infty} \left\langle u(t), e^{it\Delta}\phi \right\rangle_{L^2} = 0.$$

Let $\varepsilon > 0$. By assumption, there exists $C_\varepsilon > 0$ such that

$$\|u(t,x)\|^2_{L^2(|x|\geqslant C_\varepsilon)} \leqslant \varepsilon, \quad \forall t \in \mathbb{R}^+.$$

We have, for $t$ large enough,

$$\begin{aligned}
\left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2}\right|^2 &\lesssim \left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2(|x|\geqslant C_\varepsilon)}\right|^2 + \left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2(|x|\leqslant C_\varepsilon)}\right|^2 \\
&\lesssim \|u(t)\|^2_{L^2(|x|\geqslant C_\varepsilon)} \|\phi\|^2_{L^2} + \|u(t)\|^2_{L_t^\infty L_x^2} \int_{|x|\leqslant C_\varepsilon} |e^{it\Delta}\phi|^2 dx \\
&\lesssim \varepsilon \|\phi\|^2_{L^2} + t^{-d} C_\varepsilon^d \|\phi\|^2_{L^1} \\
&\lesssim \varepsilon.
\end{aligned}$$

This implies that $e^{-it\Delta}u(t)$ weakly converges to 0 in $L^2$, hence, $u_+ = 0$. This completes the desired result.

**Lemma 3.23.** *(*[20, Lemma 5.6]*) Let* $T \in \mathbb{R}$ *and* $u, v$ *be as in Proposition 3.21. Then for* $t_0$ *large enough (depending on* $T, u$*), we have*

$$S(T)v(t_0) = v(T + t_0) + o(1)$$

*where* $S(t)$ *is the flow of* (1.1) *and* $o(1)$ *goes to zero in* $H$ *norm as* $t_0 \to +\infty$.

*Proof.* Fix $T$, and let $t_0$ large enough. Define $I = [t_0, t_0 + T]$. We see that $v$ solves the following equation:

$$Lv = f(v) + [f(v + e^{it\Delta}u_+) - f(v)].$$

As $t_0 \to +\infty$, we have

$$\left\|e^{it\Delta}u_+\right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \to 0.$$

Moreover,

$$\left\|e^{it\Delta}u_+\right\|_{C_t^0 H(I\times\mathbb{R}^d)} \lesssim 1.$$

By Lemma 3.14 and triangle inequality, we have

$$\|v\|_{L^{q_0}H^{s,r_0}\cap C_t^0 H(I\times\mathbb{R}^d)} \lesssim_T 1.$$

Let

$$X = \left\|f(v + e^{it\Delta}u_+) - f(v)\right\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}.$$

By Lemma 3.11, we have

$$\begin{aligned}
X \lesssim |T|^{1/q_0'-1/q_0} \Big( &\left\|e^{it\Delta}u_+\right\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)} + \left\|e^{it\Delta}u_+\right\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)} \left\|(u, e^{it\Delta}u_+)\right\|^p_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \\
&+ \left\|e^{it\Delta}u_+\right\|_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \left\|(u, e^{it\Delta}u_+)\right\|^{p-1}_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \left\|(u, e^{it\Delta}u_+)\right\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)} \Big),
\end{aligned}$$

which goes to zero as $t_0 \to +\infty$. Thus, by perturbation Lemma 3.13, the claim follows. □

## 4. Frequency localisation

In this section, we prove the following asymptotic localisation frequency of the weakly bound component $v$ of $u$:

**Proposition 4.1.** *([20, Proposition 6.1]) Let $E > 0$, $u$ be a global solution of (1.1) such that $\sup_{t\in\mathbb{R}} \|u(t)\|_H \leqslant E$ and $v$ be the weakly bound component of $u$. Then, we have*

$$\limsup_{t\to+\infty} \sup_{N\leqslant 1} N^{-\eta_3} \|P_{\leqslant N} v(t)\|_H \lesssim 1 \tag{4.1}$$

*and*

$$\limsup_{t\to+\infty} \sup_{N\geqslant 1} N^{\eta_3} \|P_{\geqslant N} v(t)\|_H \lesssim 1. \tag{4.2}$$

*Proof.* The main idea is to use the Duhamel formulas (3.38) and (3.39).
Fix $\varepsilon > 0$. Let $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ such that

$$u(0) - u_+ = u_\varepsilon + O_H(\varepsilon^2).$$

Hence, from (3.38), we have

$$v(t) = e^{it\Delta} u_\varepsilon - i\int_0^t e^{i(t-t')\Delta} f(u(t'))dt' + O_H(\varepsilon^2). \tag{4.3}$$

To prove (4.1), it suffices to show that

$$\|P_{\leqslant N} v(t)\|_H \lesssim 1, \tag{4.4}$$

for $t$ large enough (can depend on $E, u, \varepsilon$ but uniform in $N$) and $N \leqslant 1$.

Using (4.3) and (3.39), we have

$$\begin{aligned}
\|P_{\leqslant N} v(t)\|_H^2 &= \left|\langle P_{\leqslant N} v(t), P_{\leqslant N} v(t)\rangle_H\right| \\
&\leqslant \Big\langle \operatorname*{iw-lim}_{T\to+\infty} \int_t^T e^{i(t-t')\Delta} P_{\leqslant N} f(u(t'))dt', \\
&\quad P_{\leqslant N} e^{it\Delta} u_\varepsilon - i\int_0^t e^{i(t-t'')\Delta} P_{\leqslant N} f(u(t''))dt'' \Big\rangle_H + O(\varepsilon^2) \\
&\leqslant \int_t^T \left|\left\langle e^{i(t-t')\Delta} P_{\leqslant N} f(u(t')), P_{\leqslant N} e^{it\Delta} u_\varepsilon \right\rangle_H\right| dt' \\
&\quad + \left|\int_t^T \int_0^t Y_N(t', t'')dt''dt'\right| + O(\varepsilon^2),
\end{aligned}$$

for $T$ large enough and $Y_N$ is defined by

$$Y_N(t', t'') = \left\langle e^{i(t-t')\Delta} P_{\leqslant N} f(u(t')), e^{i(t-t'')\Delta} P_{\leqslant N} f(u(t'')) \right\rangle_H.$$

For the first integral, we have

$$\left\| P_{\leqslant N} e^{it'\Delta} u_\varepsilon \right\|_{H^{s,R'}} \lesssim_{u_\varepsilon} \frac{1}{|t'|^{d(1/R-1/2)}}.$$

Since $R < \frac{2d}{d+4}$, the exponent of $|t'|$ is larger than 2, hence, the first integral is bounded by $O(\varepsilon^2)$ if $t, T$ $(t < T)$ are large enough (not depend on $N$).
The second integral is bounded by

$$\int_t^\infty \int_0^t |Y_N(t', t'')|\, dt'dt''.$$

Using Lemma 2.2, we have

$$\begin{aligned}|Y_N(t',t'')| &= \left|\left\langle P_{\leqslant N} f(u(t'))), e^{i(t'-t'')\Delta} P_{\leqslant N} f(u(t''))\right\rangle_H\right| \\ &\lesssim \|P_{\leqslant N} f(u(t'))\|_{H^{s,R}} \left\|e^{i(t'-t'')\Delta} P_{\leqslant N} f(u(t'))\right\|_{H^{s,R'}} \\ &\lesssim \|P_{\leqslant N} f(u(t'))\|_{L^R} \|P_{\leqslant N} f(u(t''))\|_{L^R} \frac{1}{|t'-t''|^{d(1/R-1/2)}} \\ &\lesssim \frac{1}{|t'-t''|^{d(1/R-1/2)}}.\end{aligned}$$

Moreover, using Lemma 2.2, Bernstein's inequality, we have

$$\begin{aligned}|Y_N(t',t'')| &\lesssim \|P_{\leqslant N} f(u(t'))\|_{H^s} \|P_{\leqslant N} f(u(t''))\|_{H^s} \\ &\lesssim \|P_{\leqslant N} f(u(t'))\|_{L^2} \|P_{\leqslant N} f(u(t''))\|_{L^2} \\ &\lesssim N^{d(2/R-1)} \|f(u(t'))\|_{L^R} \|f(u(t''))\|_{L^R} \\ &\lesssim N^{d(2/R-1)}.\end{aligned}$$

Combining all the above, we have

$$\int_t^\infty \int_0^t |Y_N(t',t'')|\, dt' dt'' \lesssim \int_t^\infty \int_0^t \min\left(\frac{1}{t'-t''}, N^2\right)^{d(1/R-1/2)} dt'' dt'.$$

Since $R < \frac{2d}{d+4}$, the exponent $d(1/R-1/2) > 2$, hence, the claim (4.4) follows.

We now prove (4.2). As before, we have

$$\|P_{\geqslant N} v(t)\|_H^2 \lesssim \varepsilon^2 + \left|\int_t^T \int_0^t Z_N(t',t'') dt'' dt'\right|,$$

for $T$ large enough, where

$$Z_N(t',t'') = \left\langle e^{i(t-t')\Delta} P_{\geqslant N} f(u(t')), e^{i(t-t'')\Delta} P_{\geqslant N} f(u(t''))\right\rangle_H.$$

Thus, it suffices to show that

$$\left|\int_t^T \int_0^t Z_N(t',t'') dt'' dt'\right| \lesssim N^{-\eta_3}.$$

Consider the region where $t' \geqslant t + N^{\eta_2}$. We have

$$|Z_N(t',t'')| \lesssim |t'-t''|^{-d(1/R-1/2)}.$$

Since $d(1/R-1/2) > 2$, the contribution of this term is

$$\lesssim \int_{t'>t+N^{\eta_2}} \int_{t''<t} |t'-t''|^{-d(1/R-1/2)} dt'' dt' = O(N^{-\eta_3}),$$

where we use $N^{-\eta_2} \ll N^{-\eta_3}$. Similar argument for the region $t'' \leqslant t - N^{\eta_2}$. It remains to prove that

$$\left|\int_t^{t+N^{\eta_2}} \int_{\max(0,t-N^{\eta_2})}^t Z_N(t',t'') dt'' dt'\right| \lesssim N^{-\eta_3}.$$

By Hölder inequality, the left hand side is bounded by

$$\begin{aligned}&\|P_{\geqslant N} f(u(t'))\|_{L^{q_0'}_{t'} H^{s,r_0'}_x((t,t+N^{\eta_2})\times\mathbb{R}^d)} \times \\ &\left\|\int_{\max(0,t-N^{\eta_2})}^t e^{i(t'-t'')\Delta} P_{\geqslant N} f(u(t'')) dt''\right\|_{L^{q_0}_{t'} H^{s,r_0}_x((t,t+N^{\eta_2})\times\mathbb{R}^d)}.\end{aligned}$$

By Strichartz, the left hand side is bounded by

$$\|P_{\geqslant N} f(u)\|^2_{L^{q_0'} H^{s,r_0'}((\max(t-N^{\eta_2},0),t+N^{\eta_2})\times\mathbb{R}^d)},$$

which by Proposition 3.15 and dyadic decomposition can be bounded by

$$O(N^{2\eta_2/q_0'}N^{-2\eta_1}),$$

this is acceptable. The proof is completed.

□

## 5. Spatial localisation

By Proposition 1.7, it suffices to show the asymptotic localisation of $v$ in spatial in $L^2$. The proof is the same as in [20] except that we work on fractional derivative in this paper. Thus, in this section, we only sketch the main step in the proof of asymptotic localisation in spatial of $v$ and then the main result Theorem 1.3. We refer the reader to [20] for a complete proof. First, we have the following result:

**Theorem 5.1.** *([20, Theorem 7.1]) Let $E > 0$, $0 < \mu_0 < 1$. There exist $J, \mu_4$ (depending only on $E, \mu_0$) with the following property: For any forward-global solution $u$ of (1.1) such that $\sup_{t\in\mathbb{R}} \|u(t)\|_H \leqslant E$, there exists functions $x_1, x_2, \cdots, x_J : R^+ \to \mathbb{R}$, such that*

$$\limsup_{t\to+\infty} \int_{\inf_{1\leqslant j\leqslant J} |x-x_j(t)|\geqslant 1/\mu_4} |v(t,x)|^2 dx \lesssim \mu_0^2. \tag{5.1}$$

5.1. **First step: $L_x^\infty$ spatial localisation at fixed times.** Define $v_{\mathrm{med}} = P_{\mu_2<\cdot<1/\mu_2} v$. From Proposition 4.1, we have, for $t$ large enough:

$$\|v(t) - v_{\mathrm{med}}(t)\|_H \lesssim \mu_2^{\eta_3}. \tag{5.2}$$

Thus,

$$u(t) = e^{it\Delta}u_+ + v_{\mathrm{med}}(t) + O_H(\mu_2^{\eta_3}). \tag{5.3}$$

As in [20], there exists $J$ depending on $E, \mu_0$ and functions $x_1(t), x_2(t), \cdots, x_J(t)$ such that

$$|v_{\mathrm{med}}(t,x)| < \mu_3^{1/\eta_1} \quad \text{whenever} \quad \inf_{1\leqslant j\leqslant J} |x - x_j(t)| \geqslant \mu_3^{-1}. \tag{5.4}$$

Moreover, for each $t > 0$ and $j \neq k$, $x_j(t) \equiv x_k(t)$ or $|x_j(t) - x_k(t)| > \mu_3^{-1}$.

5.2. **Second step: $L_x^\infty$ spatial localisation on a time interval.** Fix $t_0$ a sufficient late time and $I = [t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$. Let $D : \mathbb{R}^d \to \mathbb{R}^+$ be the distance function $D(x) = \inf_{1\leqslant j\leqslant J} |x - x_j(t_0)|$; thus $|v_{\mathrm{med}}(t_0, x)| < \mu_3^{1/\mu_1}$ whenever $D \geqslant \mu_3^{-1}$.

Let $\chi : \mathbb{R}^d \to \mathbb{R}^+$ be a smooth cutoff function which equals to one when $D(x) \leqslant 2\mu_3^{-1}$, vanishes when $D(x) \geqslant 3\mu_3^{-1}$ and obeys the bounds $\nabla^k \chi = O_k(\mu_3^k)$ for $k \geqslant 0$. We can let $\chi(x) = \chi_1(\mu_3 x)$ and chose suitable function $\chi_1$. We have the following result:

**Lemma 5.2.** *([20, Lemma 7.5]) We have*

$$\limsup_{t_0\to+\infty} \left\| e^{i(t-t_0)\Delta}((1-\chi)u(t_0)) \right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \lesssim \mu_2^{\eta_3}.$$

We have the following spatial decay:

**Lemma 5.3.** *([20, Lemma 7.6]) We have*

$$\limsup_{t_0\to+\infty} \left\| 1_{D>\mu_3^{-2}} u \right\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} \mu_2^{\eta_3}.$$

As a consequence, we have the following result:

**Corollary 5.4.** *([20, Corollary 7.7]) For $t_0$ large enough and any $I' \subset I$, we have*

$$\left\| 1_{D>\mu_3^{-3}} \int_{I'} e^{i(t_0-t')\Delta} f(u(t'))dt' \right\|_{L_x^2(\mathbb{R}^d)} \lesssim_{\mu_1} \mu_2^{\eta_3}.$$

5.3. **Third step: $L^2_x$ localisation at fixed time.** Using Corollary 5.4, we have the following result:

**Proposition 5.5.** *([20, Proposition 7.9]) Let $\chi_3 : \mathbb{R}^d \to \mathbb{R}^+$ be a smooth cutoff function which equals 1 when $D \geqslant 2\mu_3^{-3}$, equals 0 when $D \leqslant \mu_3^{-3}$ and $\chi_3$ has bounded derivatives. Then for $t_0$ large enough, we have*

$$\chi_3 v(t_0) = O_{L^2}(\mu_1^c),$$

*for some $c > 0$.*

Theorem 5.1 is followed from Proposition 5.5. As a consequence of Theorem 5.1, we have the following asymptotic spatial localisation of $v$, which is the main result in this section:

**Proposition 5.6.** *([20, Theorem 8.1]) Let $E > 0$ and $0 < \mu_0 < 1$. Then there exists $\mu_5$ depending on $E, \mu_0$ (and on the other fixed parameters) with the following properties: For any global solution $u$ of (1.1) such that $\sup_{t\in\mathbb{R}} \|u(t)\|_H \leqslant E$, we have*

$$\limsup_{t\to+\infty} \int_{|x|\geqslant 1/\mu_5} |v(t,x)|^2 dx \lesssim \mu_0^2, \tag{5.5}$$

*where $v$ is the weak bound component of $u$.*

Using asymptotic localisation in frequency and spatial of $v$ Proposition 4.1 and Proposition 5.6, we give the proof of the main result:

*Proof of Theorem 1.3.* By Proposition 1.7, Proposition 4.1 and Proposition 5.6, there exists a compact set $K \subset H$ such that

$$\lim_{t\to+\infty} \operatorname{dist}_H(v(t), K) = 0$$

for all radial global solution $u$ which is uniformly bounded in $H$.

Let $\mathcal{K}_{E,\mathrm{rad}}$ be the closure of the set of all limit points $\lim_{t_n\to+\infty} v(t_n)$, where $t_n$ ranges over sequences of times which converge to infinity. Then, $\mathcal{K}_{E,\mathrm{rad}}$ is close and hence is a compact subset of $K$. Let us prove that $\mathcal{K}_{E,\mathrm{rad}}$ is also an attractor. Indeed, let $t_n \to +\infty$. Since $\operatorname{dist}_H(v(t_n), K) \to 0$ as $n \to +\infty$, there exists $k_n \in K$ such that $\|v(t_n) - k_n\|_H \to 0$. By the compactness of $K$, for each sub-sequence of $(t_n)$ there exists $\varphi \in K$ and a sub-sequence of $(t_n)$ (for convenience still denotes by $(t_n)$) such that $\|v(t_n) - \varphi\|_H \to 0$. Thus, $\varphi \in \mathcal{K}_{E,\mathrm{rad}}$ and hence, $\lim_{t\to+\infty} \operatorname{dist}_H(v(t), \mathcal{K}_{E,\mathrm{rad}}) = 0$. By local theory and Lemma 3.23, $\mathcal{K}_{E,\mathrm{rad}}$ is invariant under $S(t)$ for $t$ small enough and hence for all $t$ large also.

Finally, we show the uniqueness of $u_+$. If $\tilde{u}_+$ satisfies the same properties of $u_+$ then $\{e^{it\Delta}(u_+ - \tilde{u}_+), t \geqslant 0\}$ is pre-compact in $H$. Moreover, for $2 < p \leqslant \frac{2d}{d-2s}$,

$$\left\|e^{it\Delta}(u_+ - \tilde{u}_+)\right\|_{L^p} \to 0, \quad \text{as } t \to +\infty.$$

Combining the above

$$\left\|e^{it\Delta}(u_+ - \tilde{u}_+)\right\|_H \to 0 \quad \text{as } t \to +\infty.$$

Thus, $u_+ = \tilde{u}_+$, which completes the proof. □

## References


[1] J. An and J. Kim. Local well-posedness for the inhomogeneous nonlinear Schrödinger equation in $H^s(\mathbb{R}^n)$. *Nonlinear Anal. Real World Appl.*, 59:Paper No. 103268, 21, 2021.

[2] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. New York, NY: Springer, 2011.

[3] H. Brezis and P. Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: the full story. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 35(5):1355–1376, 2018.

[4] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[5] T. Cazenave and F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$. *Nonlinear Anal.*, 14(10):807–836, 1990.

[6] F. M. Christ and M. I. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 100(1):87–109, 1991.

[7] C. Collot, T. Duyckaerts, C. Kenig, and F. Merle. Soliton resolution for the radial quadratic wave equation in six space dimensions, 2022.

[8] B. Dodson, C. Miao, J. Murphy, and J. Zheng. The defocusing quintic NLS in four space dimensions. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 34(3):759–787, 2017.
[9] T. Duyckaerts, C. Kenig, and F. Merle. Soliton resolution for the radial critical wave equation in all odd space dimensions. *Acta Math.*, 230(1):1–92, 2023.
[10] T. Duyckaerts and P. van Tin. Mass-energy scattering criterion for double power schrödinger equations, 2024.
[11] T. Duyckaerts and P. Van Tin. Profile decomposition and scattering for general nonlinear schrödinger equations. *Journal of Differential Equations*, 410:113–170, 2024.
[12] K. Fujiwara, V. Georgiev, and T. Ozawa. Higher order fractional Leibniz rule. *J. Fourier Anal. Appl.*, 24(3):650–665, 2018.
[13] J. Jendrej and A. Lawrie. Soliton resolution for energy-critical wave maps in the equivariant case. *to appear in J. Amer. Math. Soc.*, 2021.
[14] C. E. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case. *Invent. Math.*, 166(3):645–675, 2006.
[15] R. Killip, T. Oh, O. Pocovnicu, and M. Vişan. Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. *Arch. Ration. Mech. Anal.*, 225(1):469–548, 2017.
[16] R. Killip and M. Visan. Energy-supercritical NLS: critical $\dot{H}^s$-bounds imply scattering. *Commun. Partial Differ. Equations*, 35(6):945–987, 2010.
[17] C. Lu and J. Zheng. The radial defocusing energy-supercritical NLS in dimension four. *J. Differ. Equations*, 262(8):4390–4414, 2017.
[18] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel. On blow up for the energy super critical defocusing nonlinear Schrödinger equations. *Invent. Math.*, 227(1):247–413, 2022.
[19] C. Miao, J. Murphy, and J. Zheng. The defocusing energy-supercritical NLS in four space dimensions. *J. Funct. Anal.*, 267(6):1662–1724, 2014.
[20] T. Tao. A (concentration-)compact attractor for high-dimensional non-linear Schrödinger equations. *Dyn. Partial Differ. Equ.*, 4(1):1–53, 2007.
[21] T. Tao. A global compact attractor for high-dimensional defocusing nonlinear Schrödinger equations with potential. *Dyn. Partial Differ. Equ.*, 5(2):101–116, 2008.
[22] M. Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.*, 138(2):281–374, 2007.

(Phan Van Tin) LAGA (UMR 7539),
Institut Galilée, Université Sorbonne Paris Nord,
99 avenue Jean-Baptiste Clément,
93430 Villetaneuse, France
*Email address*, Phan Van Tin: `vantin.phan@math.univ-paris13.fr`