A COMPACT ATTRACTOR FOR ENERGY CRITICAL AND SUPER-CRITICAL NLS

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Abstract. We study the asymptotic behavior of large data solutions to nonlinear Schrödinger equations (NLS): $iu_t + \Delta u = \lambda|u|^p u + Vu$, where $p \geqslant \frac{4}{d-2}$ and $V \in C_0^\infty(\mathbb{R}^d)$. In the case $\frac{4}{d} < p < \frac{4}{d-2}$, Tao [12] proved that in radial setting, as $t \to +\infty$, any solution u of bounded energy can split into a term of form $e^{it\Delta}u_+$ and remainder term converges in $H^1(\mathbb{R}^d)$ to a compact attractor which is invariant under the NLS flow. In the case $p \geqslant \frac{4}{d-2}$, we work on Sobolev space $H^s(\mathbb{R}^d)$ for some $s > s_p := \frac{d}{2} - \frac{2}{p}$, which is more complex than working on $H^1(\mathbb{R}^d)$. Basing on the method in [12], we prove that any radial solution with uniform bounded on $H^s(\mathbb{R}^d)$ can split as in the work of Tao.

Contents

1. Introduction

We consider the following (NLS) equation:

$$\begin{cases} iu_t + \Delta u = \lambda|u|^p u + Vu, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = \phi, \end{cases} \tag{1.1}$$

where $d \geqslant 5$, $\lambda = \pm 1$, $p \geqslant \frac{4}{d-2}$, p is even or $p > \lceil s \rceil$ and $V \in C_0^\infty(\mathbb{R}^d)$. Let $s_p = \frac{d}{2} - \frac{2}{p}$. Fix s such that $\frac{d}{2} > s > s_p$, the energy is defined by

$$E(u) = \sup_{t \in I} \|u(t)\|_{H^s}^2, \tag{1.2}$$

For convenience, denote $H = H^s$. We abbreviate $\|f\|_{L^p(\mathbb{R}^d)}$ by $\|u\|_{L^p}$ or sometimes by $\|u\|_p$.

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We consider the following conjecture which is called soliton resolution conjecture: any global uniform bounded solution u of dispersive equations can be decomposed as $t \to +\infty$ as follows:

$$u(t) = e^{it\Delta}u_+ + \sum_{j=1}^{J} R_j(t,x) + o(1),$$

where R_j are solitons and $o(1)$ converges to zero as $t \to +\infty$. There are two special cases of this conjecture:

(1) As $t \to +\infty$, $u(t) = \sum_{j=1}^{J=1} R_j(t,x)$. In this case, $u(t,x)$ is called multi-soliton.
(2) As $t \to +\infty$, $u(t) = e^{it\Delta}u_+$. In this case, $u(t,x)$ scatters forward in time.

Recently, in [6] and [5], the authors proved the soliton resolution conjecture holds for the radial critical wave equation in all odd space dimensions and in six space dimension. To the best of our knowledge, there is not a proof for the soliton resolution conjecture for general nonlinear Schrödinger equations even in radial setting. In [12] and [13], the author proved a weaker result than the above soliton resolution conjecture. More precise, the author showed that any radial uniform bounded solution of a nonlinear Schrödinger equation (NLS) can decompose as $t \to +\infty$ by $u(t) = e^{it\Delta} + K + o(1)$, where K is a compact set which is invariant under flow of NLS. Our goal of this paper is to give a similar result to [12, 13] in the energy critical and supercritical cases i.e $p \geqslant \frac{4}{d-2}$.

In energy critical and energy subcritical, there are many works relating to two above special cases of the soliton resolution conjecture for nonlinear Schrödinger equations. Let us give a short introduce for studying NLS in energy supercritical case. In [4], the authors proved local well posedness of simple power NLS ($V = 0$ in (1.1)) in high regular Sobolev space $H^s(\mathbb{R}^d)$ ($s \geqslant s_p$). In [10], the authors showed that for defocusing NLS in high dimensions ($d \geqslant 5$), any uniform bounded in critical Sobolev space $\dot{H}^{s_p}$ is global and scatters under a suitable condition of s_p. Specially, in [11], the authors showed that there exists a blow up solutions for NLS in energy supercritical setting even in defocusing case. In focusing case, there is not many studying for asymptotic behavior of solutions for large initial data in energy supercritical setting. Recently, in [7], we establish profile decomposition and prove some scattering results for general nonlinear Schrödinger equations.

Inspiring by [4], we study local and global theory for (1.1). Define

$$\rho = \frac{p+2}{1+ps/d}, \quad \gamma = \frac{4(p+2)}{p(d-2s)}.$$

It is easy to check that (γ, ρ) is a admissible pair and satisfy the Sobolev embedding $\dot{H}^{s,\rho} \hookrightarrow L^{\rho^*}$, where ρ^* such that

$$\frac{1}{\rho'} = \frac{p}{\rho^*} + \frac{1}{\rho}.$$

We have the following properties of ρ, γ:

Proposition 1.1 ([4][Proposition 1.5]). *We have*

(i) $2 < \rho < \frac{2d}{d-2}$;
(ii) $2/\gamma = d(1/2 - 1/\rho)$, *and hence* (γ, ρ) *is an admissible pair;*
(iii) $\rho < d/s$ *and hence* $\rho^* > \rho$;
(iv) $1/\rho' = p/\rho^* + 1/\rho$;
(v) $1/\gamma' \geqslant p/\gamma + 1/\gamma$.

As in [4], we have the following result.

Theorem 1.2. *Let $\phi \in H$ and p, s be as the above. Then there exists a solution of (1.1) $u \in C(0, T_{\max}; H) \cap L^{\gamma}_{loc}(0, T_{\max}; H^{s,\rho})$. Moreover, the following properties hold:*

i $u \in L^q(0,T; H^{s,r})$ *for every admissible pair (q,r) and every $T < T_{\max}$.*
ii u *is unique in $L^{\gamma}(0,T; H^{s,\rho})$ for every $T < T_{\max}$.*
iii $\|u(t)\|_{L^2} = \|\phi\|_{L^2}$ *and $E(u(t)) = E(\phi)$ for all $t \in (0, T_{\max})$.*
iv *If $T_{\max} < \infty$ then*

$$\lim_{t \to T_{\max}} \|u(t)\|_H = \infty.$$

From Theorem 1.2, we see that if solution u has bounded energy then u is global. Inspire by [12], we study asymptotic behavior of solutions with uniformly bounded energy. Our main result is the following:

Theorem 1.3. *Let $E > 0$. Then there exists a compact subset $\mathcal{K}_{E,rad} \subset H$ which is invariant under the NLS flow, and such that for every radial global solution u of energy at most E, there exists a unique radiation $u_+ \in H$ such that*

$$\lim_{t\to+\infty} dist_H(u(t) - e^{it\Delta}u_+, \mathcal{K}_{E,rad}) = 0. \tag{1.3}$$

Here and in the sequel we write $dist_H(f, K) := \inf\{\|f - g\| : g \in K\}$ for the distance between f and K.

Remark 1.4. Consider the compact set $\mathcal{K}_{E,rad}$ in the theorem 1.3. Let $u_0 \in \mathcal{K}_{E,rad}$ and u be the associated solution of (1.1). Since $\mathcal{K}_{E,rad}$ is invariant under flow of (1.1), $u(t) \in \mathcal{K}_{E,rad}$ for all $t \in \mathbb{R}$. Thus, $\{u(t) : t \in \mathbb{R}\}$ is pre-compact in H^s and u is called an *almost periodic solution*. Assume that each almost periodic solution of (1.1) must be zero, this property is called *rigidity property* (see e.g [9][Theorem 5.1]). Then Theorem 1.3 implies that each uniformly bounded solution of (1.1) scatters in both directions. However, in the case $s = s_p$, the situation is more complex since local smoothing effect can not hold in this case (see Remark 3.18).

Remark 1.5. It seems that we could also obtain a similar result in [12][Theorem 1.28] in nonradial setting. However, in this paper, we only focus on solutions in radial setting.

Let u be a solution as in Theorem 1.3. From Lemma 3.20, $e^{-it\Delta}u(t)$ is weakly convergent to some $u_+ \in H^s(\mathbb{R}^d)$ as $t \to +\infty$. Define $v(t) = u(t) - e^{it\Delta}u_+$, which be called *weak bound component* of u. We will show that u_+ satisfies the property (1.3).

We recall the following equivalence of precompactness and localisation:

Proposition 1.6. *(see e.g [12][Proposition B.1]) Let $K \subset H$. Then the following are equivalent:*

(i) *K is precompact in H.*

(ii) *K is bounded, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have frequency localisation estimate*

$$\left\|P_{\geqslant 1/\mu_1} f\right\|_H \lesssim \mu_0$$

and the spatial localisation estimate

$$\int_{|x|\geqslant 1/\mu_1} |f(x)|^2 dx \lesssim \mu_0^2,$$

for all $f \in K$.

(iii) *K is bounded, and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have frequency localisation estimate*

$$\left\|P_{\geqslant 1/\mu_1} f\right\|_H \lesssim \mu_0$$

and

$$\|P_{\leqslant \mu_1} f\|_H \lesssim \mu_0$$

and the spatial localisation estimate

$$\int_{|x|\geqslant 1/\mu_1} |f(x)|^2 + |D^s f(x)|^2 dx \lesssim \mu_0^2,$$

for all $f \in K$.

The above proposition is generalized by the below criterion for compact attractor:

Proposition 1.7. *(see e.g [12][Proposition B.2]) Let $\mathcal{U}$ be a collection of trajectories $u : \mathbb{R}^+ \to H$. Then the following are equivalent:*

(i) *There exists a compact set $K \subset H$ such that $\lim_{t\to+\infty} dist_H(u(t), K) = 0$ for all $u \in \mathcal{U}$.*

(ii) *$\mathcal{U}$ is asymptotically bounded in the sense that*

$$\sup_{u\in\mathcal{U}} \limsup_{t\to+\infty} \|u(t)\|_H < \infty \tag{1.4}$$

and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have the asymptotic frequency localisation estimate

$$\limsup_{t\to+\infty} \|P_{1/\mu_1} u(t)\|_H \lesssim \mu_0$$

and the spatial localisation estimate

$$\limsup_{t\to+\infty} \int_{|x|\geqslant 1/\mu_1} |u(t,x)|^2 dx \lesssim \mu_0^2$$

for all $u \in \mathcal{U}$.

(iii) *$\mathcal{U}$ is asymptotically bounded in the sense of (1.4), and for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that we have the asymptotic frequency localisation estimate*

$$\limsup_{t\to+\infty} \|P_{\geqslant 1/\mu_1} u(t)\|_H \lesssim \mu_0$$

and

$$\limsup_{t\to+\infty} \|P_{\leqslant \mu_1} u(t)\|_H \lesssim \mu_0$$

and the following improved spatial localisation estimate

$$\limsup_{t\to+\infty} \int_{|x|\geqslant 1/\mu_1} |u(t,x)|^2 + |D^s u(x)|^2 dx \lesssim \mu_0^2$$

for all $u \in \mathcal{U}$.

From Proposition 1.6 and Proposition 1.7, we see that precompactness are formally equivalent to localisation in spatial and frequency. From Proposition 1.7, to prove Theorem 1.3, we only need to prove asymptotic localisation in spatial and frequency of $v(t)$.

This paper is organised as follows. In Section 2, we introduce the useful notations and basic tools in this paper. In Section 3, we prove local well-posedness of solutions of (1.1) in H^s and give some basic results in the proof of the main result. In Section 4, we prove asymptotic localisation in frequency of $v(t)$. In Section 5, we prove asymptotic localisation in spatial of $v(t)$ and then prove the main theorem 1.3. In Appendix, we prove some lemmas used in the proof of the main result.

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2. Notation and preliminaries

Let $L = i\partial_t + \Delta$ be Schrödinger operator.

For convenience, we define $f_1(u) = \lambda|u|^p u$, $f_2(u) = Vu$ and $f(u) = f_1(u) + f_2(u)$.

For each X Banach space, define $\|(u,v)\|_X = \|u\|_X + \|v\|_X$.

Denote $\mathcal{S}$ is the space of Schwartz functions. We shall need some small exponents

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 > 0,$$

where η_0 is small enough depending on E and the other fixed parameters, η_i is small enough depending on E, $\eta_0, \eta_1, \cdots, \eta_{i-1}$.

Moreover, in Section 5, we shall need some other constants:

$$1 > \mu_0 \gg \mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_4 > 0.$$

where μ_0 is small enough depending on E and the other fixed parameters, μ_i is small enough depending on $\mu_0, \mu_1, \cdots, \mu_{i-1}$.

Lemma 2.1. *There exists an admissible pair (q_0, r_0) with $q_0 > 2$, exponents $2 < Q_0, Q < \frac{2d}{d-2s}$, and an exponent $1 \leqslant R < \frac{2d}{d+4}$ such that*

$$\frac{1}{r_0} + \frac{p}{Q_0} = \frac{1}{r_0'} \tag{2.1}$$

and

$$\frac{1}{2} + \frac{p}{Q} = \frac{1}{R}. \tag{2.2}$$

Lemma 2.2. *For any $u \in H$, we have*

$$\|f(u)\|_{H^{s,R}} \lesssim \|u\|_H^{p+1} + \|u\|_H .$$

Proof. By Hölder, we have

$$\left\|D^j f_1(u(t))\right\|_{L^R} \lesssim \|u(t)\|_{L^Q}^p \left\|D^j u(t)\right\|_{L^2},$$

for $j = 0, s$. Let a be such that $\frac{1}{R} = \frac{1}{a} + \frac{1}{2}$. Thus, $2 \leqslant a \leqslant \frac{d}{2}$. By Hölder and Sobolev embedding, for $j = 0, s$, we have

$$\begin{aligned}
\left\|D^j(f_2(u))\right\|_{L^R} &= \left\|D^j(Vu)\right\|_{L^R} \\
&\lesssim \left\|D^j(V)\right\|_{L^a} \|u\|_{L^2} + \|V\|_{L^a} \left\|D^j u\right\|_{L^2} \\
&\lesssim \|u\|_{L^2} + \left\|D^j u\right\|_{L^2} \\
&\lesssim \|u\|_H .
\end{aligned}$$

This implies the desired result. □

Denote $\mathcal{F}(f) = \hat{f}$ is Fourier-transformation by space variable. We have the following basic property in Fourier-analysis.

Lemma 2.3. *Let $f, g \in \mathcal{S}$. we have the following identity:*

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g).$$

Lemma 2.4. *For any time interval $I \subset \mathbb{R}$, any $t_0 \in I$ and any $0 < \delta \leqslant \frac{1}{2}$, we have*

$$\begin{aligned}
\|uv\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim &\left(\|u(t_0)\|_{\dot{H}^{-1/2+\delta}} + \left\||\nabla|^{-1/2+\delta} Lu\right\|_{L^{q'}L^{r'}(I\times\mathbb{R}^d)}\right) \\
&\times \left(\|v(t_0)\|_{\dot{H}^{(d-1)/2-\delta}} + \left\||\nabla|^{(d-1)/2-\delta} Lv\right\|_{L^{\tilde{q}'}L^{\tilde{r}'}(I\times\mathbb{R}^d)}\right),
\end{aligned}$$

for any u, v and any admissible pairs (q, r), $(\tilde{q}, \tilde{r})$ with $q, \tilde{q} > 2$.

As a consequence, we have the following result.

Lemma 2.5. *(see e.g [12][Corollary 4.4]) For any time interval $I \subset \mathbb{R}$, any $N, M > 0$ dyadic numbers, we have*

$$\|u_N u_M\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim \langle |I| \rangle^{\frac{1}{2}} \frac{M^{(d-1)/2-\delta} N^{-\frac{1}{2}+\delta}}{\langle N \rangle^s \langle M \rangle^s}.$$

Lemma 2.6 (Dispersive estimate)**.** *For all $2 \leqslant r \leqslant +\infty$ and $f \in L^{r'}(\mathbb{R}^d)$, we have*

$$\left\|e^{it\Delta} f\right\|_{L^r} \lesssim \frac{1}{|t|^{d(1/2-1/r)}} \|f\|_{L^{r'}} .$$

Lemma 2.7. *For $1 \leqslant r \leqslant 2$, $f \in \mathcal{D}(\mathbb{R}^d)$, we have*

$$\left\|e^{it\Delta} f\right\|_{L^r} \lesssim_f (1+t)^M,$$

for all $M \in \mathbb{N}$ and $M > d/2 - d/r$.

Proof. Let a be such that

$$\frac{1}{r}=\frac{1}{a}+\frac{1}{2}.$$

Let M be a natural number such that $Ma>d$.
We have

$$\begin{aligned}\left\|e^{it\Delta}f\right\|_{L^r}&=\left\|\mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^r_x}\\&=\left\|(1+|x|)^{-M}(1+|x|)^{M}\mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^r_x}\\&\lesssim\left\|(1+|x|)^{-M}\right\|_{L^a}\left\|(1+|x|)^{M}\mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))(x)\right\|_{L^2_x}\\&\lesssim\left\|e^{-it|\xi|^2}\hat{f}(\xi)\right\|_{H^M_\xi}\\&\lesssim_f(t+1)^M.\end{aligned}$$

□

3. Local estimate

This section gives some useful tools for studying asymptotic behavior as $t\to+\infty$ of solutions of (1.1). First, we introduce some basic tools in studying of fractional derivative.

3.1. Basic tools of fractional derivative.

We need the following results on fractional derivative for our analysis:

Lemma 3.1. *Let $-\infty<s_2\leqslant s_1<\infty$ and $1<p_1\leqslant p_2<\infty$ with $s_1-\frac{d}{p_1}=s_2-\frac{d}{p_2}$. Then we have the following embedding:*

$$\dot{H}^{s_1,p_1}\subset\dot{H}^{s_2,p_2},\quad H^{s_1,p_1}\subset H^{s_2,p_2}.$$

Lemma 3.2 (Product chain rule 1)**.** *(see e.g [1][Lemma 2.2]) Let $s\geqslant 0$, $1<r,r_2,p_1<\infty$, $1<r_1,p_2\leqslant\infty$ such that $\frac{1}{r}=\frac{1}{r_i}+\frac{1}{p_i}$ $(i=1,2)$. Then,*

$$\|fg\|_{\dot{H}^{s,r}}\lesssim\|f\|_{r_1}\|g\|_{\dot{H}^{s,p_1}}+\|f\|_{\dot{H}^{s,r_2}}\|g\|_{p_2}.$$

As a consequence of the above lemma, we have the following:

Lemma 3.3 (Product chain rule 2)**.** *(see e.g [1][Corollary 2.3]) Let $s\geqslant 0$, $q\in\mathbb{N}$ $(q\geqslant 1)$, $1<r,r_k^i<\infty$, for $1\leqslant i,k\leqslant q$: $\frac{1}{r}=\sum_{i=1}^q\frac{1}{r_k^i}$. Then*

$$\left\|\prod_{i=1}^q f_i\right\|_{\dot{H}^{s,r}}\lesssim\sum_{k=1}^q\left(\|f_k\|_{\dot{H}^{s,r_k^k}}\prod_{i\neq k}\|f_i\|_{r_k^i}\right).$$

Lemma 3.4 (Fractional chain rule)**.** *(see e.g [1][Lemma 2.4]) Let $G\in C^1(\mathbb{C})$, $s\in(0,1)$, $1<r,r_2<\infty$, and $1<r_1\leqslant\infty$ satisfying $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$,*

$$\|G(u)\|_{\dot{H}^{s,r}}\lesssim\|G'(u)\|_{r_1}\|u\|_{\dot{H}^{s,r_2}}.$$

Lemma 3.5 (Gagliardo-Nirenberg inequality)**.** *(see [2]). Let $s_1\leqslant s_2$, $p_0>1$, $s=\theta s_1+(1-\theta)s_2$, $\frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_0}$. Then*

$$\|u\|_{H^{s,p}}\lesssim\|u\|^{\theta}_{H^{s_1,p_1}}\|u\|^{1-\theta}_{H^{s_2,p_0}}.$$

Lemma 3.6 (Homogeneous Gagliardo-Nirenberg inequality)**.** *Let $s_1\leqslant s_2$, $p_0>1$, $s=\theta s_1+(1-\theta)s_2$, $\frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_0}$. Then*

$$\|u\|_{\dot{H}^{s,p}}\lesssim\|u\|^{\theta}_{\dot{H}^{s_1,p_1}}\|u\|^{1-\theta}_{\dot{H}^{s_2,p_0}}.$$

Lemma 3.7. *(see [1][Lemma 3.2]) Let $s>0$, $1<p<\infty$ and $v=s-\lfloor s\rfloor$. Then*

$$\sum_{|\alpha|=\lfloor s\rfloor}\|D^\alpha f\|_{\dot{H}^v_p}\approx\|f\|_{\dot{H}^s_p},$$

where $\lfloor s\rfloor$ is largest integer below s.

Lemma 3.8 (Leibniz rule). *Let $f \in C^k$ be a real variable function and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| \leqslant k$. Then $D^\alpha f(u)$ is written as follows:*

$$D^\alpha f(u) = \sum_{h=1}^{|\alpha|} \sum_{\Lambda_\alpha^h} C_{\alpha,h} f^{(h)}(u) \prod_{i=1}^{h} D^{\beta_i} u, \tag{3.1}$$

where $C_{\alpha,h} \in \mathbb{N}$ and $\Lambda_\alpha^h = \{(\beta_1, \cdots, \beta_h) : \text{ for each } i:\ \beta_i \in \mathbb{N}^d,\ \beta_1 + \cdots + \beta_h = \alpha, |\beta_i| \geqslant 1\}$.

3.2. **Local theory in H^s.** In this subsection, we prove local theory for solutions in H^s of (1.1) Theorem 1.2. Define

$$\mathcal{G}(f(u(t))) = -i \int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

We may rewrite (1.1) by

$$u(t) = S(t)\phi + \mathcal{G}(f(u(t))).$$

Before proving Theorem 1.2, we need some tools.

Lemma 3.9 ([4][Lemma 4.1]). *We have for any (q, r) admissible:*

$$\|\mathcal{G}(f(u))\|_{L^q(0,T)\dot{H}^{s,r}(\mathbb{R}^d)} \lesssim T^\delta \|u\|^{p+1}_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} + T^{\frac{\gamma-2}{\gamma}} \|u\|_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} \tag{3.2}$$

and

$$\begin{aligned}\|\mathcal{G}(f(u)) - \mathcal{G}(f(v))\|_{L^q(0,T)L^r(\mathbb{R}^d)} &\lesssim T^\delta \left(\|u\|^p_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} + \|v\|^p_{L^\gamma(0,T)\dot{H}^{s,\rho}(\mathbb{R}^d)} \right) \|u - v\|_{L^\gamma(0,T)L^\rho(\mathbb{R}^d)} \\ &\quad + T^{\frac{\gamma-2}{\gamma}} \|u - v\|_{L^\gamma(0,T)L^\rho(\mathbb{R}^d)}, \end{aligned} \tag{3.3}$$

where $\delta = 1 - \frac{p+2}{\gamma}$.

Proof. Use the proof of Lemma 3.11 and note that $\dot{H}^{s,\rho} \hookrightarrow L^{\rho^*}$. Remark that in this proof, we only need the condition p even or $p > \lfloor s \rfloor$. □

Moreover, we have the following unique result.

Proposition 3.10 ([4][Proposition 4.2]). *Let $\phi \in H^s$, and suppose $u \in L^\gamma(0,T;H^{s,\rho})$ is a solution of (1.1). Then $u \in L^q(0,T;H^{s,r}) \cap C([0,T];H^s)$ for every admissible pair (q,r). Furthermore, if $v \in L^\gamma(0,T;H^{s,\rho})$ is also a solution of (1.1), then $u = v$.*

Proof. By Lemma 3.9 and Strichartz, it is easy to prove that $u \in L^q(0,T;H^{s,r}) \cap C([0,T];H^s)$. Assume $u(t) \neq v(t)$ for some $t \in [0,T]$. Let $t_0 = \inf\{t \in [0,T], u(t) \neq v(t)\}$. Using Lemma 3.9, we have, for all $t \in [t_0, T]$:

$$\begin{aligned}\|u - v\|_{L^\gamma(t_0,t);L^\rho} &= \|\mathcal{G}(f(u)) - \mathcal{G}(f(v))\|_{L^\gamma(t_0,t);L^\rho} \\ &\leqslant C(t-t_0)^\delta \left(\|u\|^p_{L^\gamma(t_0,t);\dot{H}^{s,p}} + \|v\|^p_{L^\gamma(t_0,t);\dot{H}^{s,p}} \right) \|u - v\|_{L^\gamma(t_0,t);L^\rho} \\ &\quad + \|u - v\|_{L^\gamma(t_0,t);L^\rho} (t-t_0)^{\frac{\gamma-2}{\gamma}}.\end{aligned}$$

Let t be close to t_0, we give a contradiction. Thus, $u(t) = v(t)$ for all $t \in [0,T]$, which is the desired result. □

Proof Theorem 1.2. We use fixed point argument to prove existence of solution of (1.1). Let $M > 0$, we define

$$\mathcal{X} = \mathcal{X}(T,M) = \{u \in L^\gamma(0,T;H^{s,\rho}) : \|u\|_{L^\gamma(0,T;\dot{H}^{s,\rho})} \leqslant M\}.$$

Endowed with the metric

$$d(u,v) = \|u - v\|_{L^\gamma(0,T;L^\rho)},$$

$\mathcal{X}$ is a complete metric space.
We wish to prove that the map

$$\mathcal{F}u = S(\cdot)\phi + \mathcal{G}(f(u)).$$

is a contradiction map on $\mathcal{X}$ for suitable constants T, M. Let $u \in \mathcal{X}$. Using (3.2), we have if $\|S(\cdot)\phi\|_{L^\gamma(0,T;\dot{H}^{s,\rho})} + CT^\delta M^{p+1} + CT^{\frac{\gamma-2}{\gamma}} M \leqslant M$ then $\mathcal{F}u \in \mathcal{X}$. From (3.3), if $2CT^\delta M^p + CT^{\frac{\gamma-2}{\gamma}} < 1$ then $\mathcal{F}$ is a strict contradiction on $\mathcal{X}$. By Strichartz, $\mathcal{F}$ is a contradiction from $\mathcal{X}$ to $\mathcal{X}$ if

$$C\|\phi\|_{\dot{H}^s} + 2CT^\delta M^{p+1} + CT^{\frac{\gamma-2}{\gamma}} M \leqslant M. \tag{3.4}$$

Combining with Proposition 3.10, there exists a unique maximal solution of (1.1) $u \in C(0,T_{\max};H)\cap L^{\gamma}_{\rm loc}(0,T_{\max};H^{s,\rho})$. Assume $T_{\max}<\infty$. Using (3.4) and by classical argument (see e.g [3]), we have

$$\lim_{t\to T_{\max}} \|u(t)\|_{\dot H^s}=\infty,$$

which completes the desired result. □

Lemma 3.11. *Let p be even or $p\geqslant \lceil s\rceil$ and $u,v\in S^s(I\times\mathbb{R}^d)$. We have the following estimates:*

$$\|f_1(u)-f_1(v)\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}\lesssim |I|^{1/q_0'-1/q_0}\Big(\|u-v\|_{L^{q_0}\dot H^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^p_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}$$
$$+\|u-v\|_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}\Big).$$
$$\|f_2(u)-f_2(v)\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}\lesssim_V \|u-v\|_{L^{q_0}H^{s,r_0}}|I|^{1/q_0'-1/q_0}.$$

Proof. It suffices to show the first estimate. The second estimate is proved by using Hölder and $V\in\mathcal{D}(\mathbb{R}^d)$. We only consider the case $p\geqslant\lceil s\rceil$; in the case p even the conclusion is easily proved by using Hölder.
By Hölder,

$$\|f_1(u)-f_1(v)\|_{L^{q_0'}H^{s,r_0'}}\lesssim \|f_1(u)-f_1(v)\|_{L^{q_0}H^{s,r_0'}}|I|^{1/q_0'-1/q_0}.$$

Thus, we only need to prove that

$$\|f_1(u)-f_1(v)\|_{L^{q_0}H^{s,r_0'}}\lesssim \|u-v\|_{L^{q_0}\dot H^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^p_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}$$
$$+\|u-v\|_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}.$$

We have

$$f_1(u)-f_1(v)=(u-v)\int_0^1\partial_z f_1(v+\theta(u-v))d\theta+\overline{u-v}\int_0^1\partial_{\bar z}f_1(v+\theta(u-v))d\theta. \tag{3.5}$$

By Hölder and using $|\partial_z f_1(u)|+|\partial_{\bar z}f_1(v)|\lesssim |u|^p$,

$$\|f_1(u)-f_1(v)\|_{L^{q_0}L^{r_0'}}\lesssim \|u-v\|_{L^\infty L^{Q_0}}\sup_{\theta\in[0,1]}\||v+\theta(u-v)|^p\|_{L^{q_0}L^a},\quad (\text{ where } 1/a+1/Q_0=1/r_0')$$
$$\lesssim \|u-v\|_{L^\infty L^{Q_0}}(\||u|^p\|_{L^{q_0}L^a}+\||v|^p\|_{L^{q_0}L^a})$$
$$\lesssim \|u-v\|_{L^\infty L^{Q_0}}\|(u,v)\|^{p-1}_{L^\infty L^{Q_0}}\|(u,v)\|_{L^{q_0}L^{r_0}}.$$

Thus, it remains to show that

$$\|f_1(u)-f_1(v)\|_{L^{q_0}\dot H^{s,r_0'}}\lesssim \|u-v\|_{L^{q_0}\dot H^{s,r_0}(I\times\mathbb{R}^d)}\|(u,v)\|^p_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}$$
$$+\|u-v\|_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|^{p-1}_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\|(u,v)\|_{L^{q_0}\dot H^{s,r_0}(I\times\mathbb{R}^d)}.$$

By Hölder, it suffices to show:

$$\|f_1(u)-f_1(v)\|_{\dot H^{s,r_0'}}\lesssim \|u-v\|_{\dot H^{s,r_0}}\|(u,v)\|^p_{L^{Q_0}}+\|u-v\|_{L^{Q_0}}\|(u,v)\|^{p-1}_{L^{Q_0}}\|(u,v)\|_{\dot H^{s,r_0}}. \tag{3.6}$$

By using (3.5) (with noting that the contribution of the first term and the second term are similar) and product rule Lemma 3.2, we have

$$\|f_1(u)-f_1(v)\|_{\dot H^{s,r_0'}}\lesssim \|u-v\|_{L^{Q_0}}\Big\|\int_0^1|\nabla|^s\partial_z f_1(v+\theta(u-v))d\theta\Big\|_{L^a}$$
$$+\|u-v\|_{\dot H^{s,r_0}}\Big\|\int_0^1\partial_z f_1(v+\theta(u-v))d\theta\Big\|_{L^b}\quad(\text{where } 1/b+1/r_0=1/r_0').$$

Since $|\partial_z f_1(u)|\lesssim|u|^p$, we have

$$\Big\|\int_0^1\partial_z f_1(v+\theta(u-v))d\theta\Big\|_{L^b}\lesssim \||u|^p+|v|^p\|_{L^b}\lesssim\|(u,v)\|^p_{L^{Q_0}},$$

hence the second term is acceptable. Thus, to prove (3.6), it suffices to show that, for $g=\partial_z f_1$, for each u:

$$\||\nabla|^s g(u)\|_{L^a}\lesssim\|u\|^{p-1}_{L^{Q_0}}\|u\|_{\dot H^{s,r_0}}. \tag{3.7}$$

Since $p > \lceil s \rceil$, the function g belongs to $C^{\lceil s \rceil}$ and satisfies

$$|g^{(k)}(u)| \lesssim |u|^{p-k}, \quad \text{for each } 0 \leqslant k \leqslant \lceil s \rceil . \tag{3.8}$$

By Lemma 3.7 and Leibniz rule Lemma 3.8, we only need to prove for each $\alpha \in \mathbb{N}^d$, $1 \leqslant h \leqslant |\alpha|$, each $1 \leqslant i \leqslant h$: $\beta_i \in \mathbb{N}^d$, $|\beta_i| \geqslant 1$: $\sum_{i=1}^h \beta_i = \alpha$:

$$\left\| g^{(h)}(u) \prod_{i=1}^h D^{\beta_i} u \right\|_{\dot{H}^{v,a}} \lesssim \|u\|_{L^{Q_0}}^{p-1} \|u\|_{\dot{H}^{s,r_0}} , \quad (\text{where } v = s - \lfloor s \rfloor). \tag{3.9}$$

We have $h \leqslant |\alpha| \leqslant \lfloor s \rfloor \leqslant \lceil s \rceil < p$. Using Lemma 3.3 and (3.8), we have

$$\begin{aligned} \left\| g^{(h)}(u) \prod_{i=1}^h D^{\beta_i} u \right\|_{\dot{H}^{v,a}} \lesssim{}& \left\| g^{(h)}(u) \right\|_{\dot{H}^{v,m_0}} \prod_{i=1}^h \left\| D^{\beta_i} u \right\|_{L^{m_i}} \\ &+ \sum_{k=1}^h \left\| |u|^{p-h} \right\|_{L^{\tilde{m}_0}} \left\| D^{\beta_k} u \right\|_{\dot{H}^{v,\tilde{m}_k}} \prod_{i=1, i\neq k}^h \left\| D^{\beta_i} u \right\|_{L^{m_i}} , \end{aligned} \tag{3.10}$$

where m_i and $\tilde{m}_i$ for $i = 0, \cdots, h$ are chosen such that

$$\begin{aligned} \frac{1}{m_i} &= \frac{\theta_i}{r_0} + \frac{1-\theta_i}{Q_0}, \quad (\text{where } \theta_i = |\beta_i|/s) \\ \frac{1}{m_0} &= \frac{1}{a} - \sum_{i=1}^h \frac{1}{m_i}, \\ \tilde{m}_0 &= \frac{Q_0}{p-h}, \\ \frac{1}{\tilde{m}_k} &= \frac{1}{a} - \frac{p-h}{Q_0} - \sum_{i=1, i\neq k}^h \frac{1}{m_i} = \frac{(|\beta_k| + v)/s}{r_0} + \frac{1 - (|\beta_k| + v)/s}{Q_0}, \end{aligned}$$

By the choice of m_i and $\tilde{m}_i$, using Gagliardo-Nirenberg inequality Lemma 3.6, we have

$$\begin{aligned} \left\| D^{\beta_i} u \right\|_{L^{m_i}} &\lesssim \|u\|_{\dot{H}^{s,r_0}}^{\theta_i} \|u\|_{L^{Q_0}}^{1-\theta_i} , \\ \left\| D^{\beta_k} u \right\|_{\dot{H}^{v,\tilde{m}_k}} &\lesssim \left\| D^{|\beta_k|+v} u \right\|_{L^{\tilde{m}_k}} \\ &\lesssim \|u\|_{\dot{H}^{s,r_0}}^{\tilde{\theta}_k} \|u\|_{L^{Q_0}}^{1-\tilde{\theta}_k} , \quad (\text{where } \tilde{\theta}_k = (|\beta_k| + v)/s). \end{aligned}$$

Thus, the second term in (3.10) is acceptable.
Consider the first term. If $s \notin \mathbb{N}$ then $\lceil s \rceil \geqslant h+1$ (since $h \leqslant |\alpha| = \lfloor s \rfloor$), using fractional chain rule Lemma 3.4, we have

$$\begin{aligned} \left\| g^{(h)}(u) \right\|_{\dot{H}^{s,m_0}} &\lesssim \left\| g^{(h+1)}(u) \right\|_{L^m} \|u\|_{\dot{H}^{v,n}} \quad (\text{ where } 1/m + 1/n = 1/m_0) \\ &\lesssim \left\| |u|^{p-h-1} \right\|_{L^m} \|u\|_{\dot{H}^{v,n}} . \end{aligned}$$

Choosing m such that $(p-h-1)m = Q_0$ then $1/n = 1/m_0 - 1/m = \frac{v/s}{r_0} + \frac{\lfloor s \rfloor / s}{Q_0}$. Using Gagliardo-Nirenberg inequality Lemma 3.6, we have

$$\|u\|_{\dot{H}^{v,n}} \lesssim \|u\|_{\dot{H}^{s,r_0}}^{v/s} \|u\|_{L^{Q_0}}^{\lfloor s \rfloor / s} .$$

Thus, the first term in (3.10) is acceptable. Then, (3.9) holds and the proof is completed.

□

Remark 3.12. In the proof of the above lemma, the condition $p > \lceil s \rceil$ is needed.

Lemma 3.13 (Perturbation lemma)**.** *Let $u_0 \in H$, let I be a compact time interval containing t_0, let $A, \tilde{\mu}_1 > 0$, and $v : I \to H$ be strong solution of the following NLS:*

$$iv_t + \Delta = f(v) + G$$

where v, G satisfy

$$\|v\|_{L^{q_0} H^{s,r_0} \cap C^0 H(I \times \mathbb{R}^d)} + \|v(t_0) - u_0\|_H \lesssim A$$

and

$$\left\|e^{i(t-t_0)\Delta}(v(t_0)-u_0)\right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\lesssim\tilde{\mu}_1$$

and

$$\|G\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}\lesssim\tilde{\mu}_1.$$

Then, if $\tilde{\mu}_1$ is small enough depending on A, $|I|$ there exists a solution $u:I\to H$ of (1.1) with initial data $u(t_0)=u_0$ such that

$$\|u-v\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\lesssim_{A,|I|}\tilde{\mu}_1.$$

and

$$\|u-v\|_{C^0H}\lesssim_{A,|I|}1.$$

Proof. By local theory, u exists on I. Let $w=u-v$, then w solves the equation

$$iw_t+\Delta w=f(w+v)-f(v)-G;\quad w(0)=v(t_0)-u_0.$$

Define $X=\|w\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)}$. We have

$$X\lesssim\tilde{\mu}_1+\|f(v+w)-f(v)\|_{L^{q_0'}H^{s,r_0'}}.$$

By Lemma 3.11, we have

$$X\lesssim\tilde{\mu}_1+|I|^{1/q_0'-1/q_0}X((X+A)^p+1).$$

If $|I|\leqslant C(A)$ (depending only on A) and $\tilde{\mu}_1$ is small enough depending on I,A, we conclude that $X\lesssim\tilde{\mu}_1$ on I. Moreover,

$$\|u-v\|_{C^0H(I\times\mathbb{R}^d)}\lesssim A+\|f(u)-f(v)\|_{L^{q_0'}H^{s,r_0'}}\lesssim_A 1.$$

Dividing I into $|I|/C(A)$ interval I_k, on each I_k, we have

$$\|w\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I_k\times\mathbb{R}^d)}\lesssim\tilde{\mu}_1$$

and

$$\|w\|_{C^0H(I_k\times\mathbb{R}^d)}\lesssim_A 1.$$

By inductive argument, the claim follows. □

3.3. **Fixed time estimate.** From (1.2) and our convenience to suppress dependence on E we have

$$\|u(t)\|_H\lesssim 1\quad\text{for all } t\in[0,+\infty) \tag{3.11}$$

and hence by Sobolev embedding we have

$$\|u(t)\|_{L^q}\lesssim 1\quad\text{for all } 2\leqslant q\leqslant\frac{2d}{d-2s}\text{ and } t\in[0,+\infty). \tag{3.12}$$

From Lemma 2.2, we have

$$\|f(u)\|_{H^{s,R}}\lesssim 1\quad\text{for all } t\in[0,+\infty). \tag{3.13}$$

3.4. **Local-in-time estimates.** Next we prove a local-in-time Strichartz estimate.

Lemma 3.14. *(Local Strichartz control). For any time interval $I\subset[0,+\infty)$ and any admissible pair of exponents (q,r) we have*

$$\|u\|_{L^qH^{s,r}(I)}\lesssim\langle|I|\rangle^{\frac{1}{q}} \tag{3.14}$$

as well as the nonlinearity estimate

$$\|f(u)\|_{L^{q_0'}H^{s,r_0'}(I)}\lesssim\langle|I|\rangle^{\frac{1}{q_0}}. \tag{3.15}$$

Proof. By Theorem 1.2, there exist and uniqueness of solution u are proved. Moreover, u is global and $u\in L^q_{loc}H^{s,r}(\mathbb{R}\times\mathbb{R}^d)$ for any (q,r) admissible. By subdividing I it suffices to prove that if $|I|\lesssim 1$ then $\|u\|_{L^qH^{s,r}(I)}\lesssim 1$ and $\|f(u)\|_{L^{q_0'}H^{s,r_0'}(I)}\lesssim 1$. Define

$$X=\|u\|_{L^{q_0}(I)H^{s,r_0}(\mathbb{R}^d)}.$$

By Strichartz and Hölder inequality, we have

$$\begin{aligned} X &\lesssim \|u\|_{L^\infty H} + \||u|^p u\|_{L^{q_0'}(I)H^{s,r_0'}} + \|Vu\|_{L^1L^2} \\ &\lesssim 1 + \||u|^p u\|_{L^{q_0'}(I)H^{s,r_0'}} + \|V\|_{L^1L^\infty}\|u\|_{L^\infty L^2} \\ &\lesssim 1 + |I|^{\frac{1}{q_0'}-\frac{1}{q_0}}\|u\|^p_{L^\infty L^{Q_0}} X \\ &\lesssim 1 + |I|^{\frac{1}{q_0'}-\frac{1}{q_0}} X, \quad (\text{ where we use } \|u\|_{L^\infty L^{Q_0}} \lesssim \|u\|_{L^\infty H} \lesssim 1) \\ &\lesssim 1 + \frac{1}{2}X, \end{aligned}$$

if we make $|I|$ small enough. This give $X = O(1)$ and then $\|f(u)\|_{L^{q_0'}H^{s,r_0'}} = O(1)$. This completes our proof. □

The following result is useful to prove asymptotic localisation frequency of v:

Proposition 3.15. *(Local smoothing effect). We have the following result*

$$\|P_N f(u)\|_{L^{q_0'}(I)L^{r_0'}} \lesssim \langle N\rangle^{-s-\eta_1}\langle |I|\rangle^{\frac{1}{q_0'}}. \tag{3.16}$$

Proof. It suffices to prove Proposition 3.15 for $|I| \leqslant 1$. Indeed, assume Proposition 3.15 holds for $|I| \leqslant 1$. Divide I into $|I|$ subintervals I_k such that $|I_k| \approx 1$. On each I_k, we have

$$\|P_N f(u)\|_{L^{q_0'}(I_k)L^{r_0'}} \lesssim \langle N\rangle^{-s-\eta_1}.$$

Summing the above inequality in k, we obtain (3.16). By Lemma 3.14, we may also assume $N \geqslant 1$. For convenience, we shall omit the domain $I \times \mathbb{R}^d$ in all norms in this proof. Consider the case p is not even and $p > \lceil s\rceil$. Our first task is to show that

$$\|D^s P_N f_1(u)\|_{L^{q_0'}L^{r_0'}} \lesssim N^{-\eta_1}, \tag{3.17}$$

where $f_1(u) = \lambda|u|^p u$. For convenience, in the proof of (3.17), we abbreviate f_1 by f.

Using Lemma 3.7 and Lemma 3.8, we only need to prove that

$$\left\| P_N\left(f^{(h)}(u)\prod_{i=1}^h D^{\beta_i}u\right)\right\|_{L^{q_0'}\dot{H}^{v,r_0'}} \lesssim N^{-\eta_1}, \tag{3.18}$$

where $1 \leqslant h \leqslant \lfloor s\rfloor$, $\beta_i, \alpha \in \mathbb{N}^d$, $|\beta_i| \geqslant 1$, $|\alpha| = \lfloor s\rfloor$ and $\sum_{i=1}^h \beta_i = \alpha$.
We have

$$\left\| P_N\left(f^{(h)}(u)\prod_{i=1}^h D^{\beta_i}u\right)\right\|_{L^{q_0'}\dot{H}^{v,r_0'}} \lesssim \left\| P_{\gtrsim N} f^{(h)}(u)\prod_{i}^h D^{\beta_i}u\right\|_{L^{q_0'}\dot{H}^{v,r_0'}} \tag{3.19}$$

$$+ \sum_{k=1}^h \left\| f^{(h)}(u)(P_{\gtrsim N}D^{\beta_k}u)\prod_{i=1;i\neq k}^h D^{\beta_i}u\right\|_{L^{q_0'}\dot{H}^{v,r_0'}}, \tag{3.20}$$

where $P_{\gtrsim N} = P_{\geqslant \frac{N}{1+\lfloor s\rfloor}}$. By Lemma 3.3, (3.19) is controlled by the following

$$\begin{aligned} &\left\|P_{\gtrsim N}f^{(h)}(u)\right\|_{L^{q_1}\dot{H}^{v,r_1}}\prod_i^h \left\|D^{\beta_i}u\right\|_{L^{q_i^0}L^{r_i^0}} \\ &+ \left\|P_{\gtrsim N}f^{(h)}(u)\right\|_{L^{q_2}L^{r_2}}\sum_{k=1}^h \left\|D^{\beta_k}u\right\|_{L^{q_k^k}\dot{H}^{v,r_k^k}}\prod_{i=1,i\neq k}^h \left\|D^{\beta_i}u\right\|_{L^{q_i^k}L^{r_i^k}} \\ &\lesssim \left\|P_{\gtrsim N}u\right\|_{L^{q_1^1}\dot{H}^{v,r_1^1}}\left\||u|^{p-h}\right\|_{L^{q_1^2}L^{r_1^2}}\prod_i^h \left\|D^{\beta_i}u\right\|_{L^{q_i^0}L^{r_i^0}} \qquad (3.21) \\ &+ \left\|P_{\gtrsim N}u\right\|_{L^{q_2^1}L^{r_2^1}}\left\||u|^{p-h}\right\|_{L^{q_2^2}L^{r_2^2}}\sum_{k=1}^h \left\|D^{\beta_k}u\right\|_{L^{q_k^k}\dot{H}^{v,r_k^k}}\prod_{i=1,i\neq k}^h \left\|D^{\beta_i}u\right\|_{L^{q_i^k}L^{r_i^k}}, \qquad (3.22) \end{aligned}$$

where the above parameters satisfy, for each $k \in [1,h]$

$$\frac{1}{q_0'} = \frac{1}{q_1} + \sum_{i=1}^{h} \frac{1}{q_i^0}, \tag{3.23}$$

$$\frac{1}{r_0'} = \frac{1}{r_1} + \sum_{i=1}^{h} \frac{1}{r_i^0}, \tag{3.24}$$

$$\frac{1}{q_0'} = \frac{1}{q_2} + \sum_{i=1}^{h} \frac{1}{q_i^k}, \tag{3.25}$$

$$\frac{1}{r_0'} = \frac{1}{r_2} + \sum_{i=1}^{h} \frac{1}{r_i^k}. \tag{3.26}$$

In (3.21), the parameters are chosen such that, for each $i \in [1,h]$

$$\frac{2}{q_i^0} + \frac{d}{r_i^0} = \frac{d}{2} - (s - |\beta_i|), \tag{3.27}$$

$$\frac{2}{q_1^2} + \frac{d}{r_1^2} = (p-h)\left(\frac{d}{2} - s\right). \tag{3.28}$$

Let s_1 be such that $\frac{2}{q_1^1} + \frac{d}{r_1^1} = \frac{d}{2} - s_1$. From (3.23), (3.24), (3.27), (3.28), we see that $s_1 = \lfloor s \rfloor - \left(2 - p\left(\frac{d}{2} - s\right)\right)$. It is easy to check that $0 < s_1$ and $s_1 + v < s$ (using $s > s_p \geqslant 1$). Thus, by Sobolev embedding Lemma 3.1 and Lemma 3.14, we have

$$\begin{aligned}
(3.21) &\lesssim \left\|D^{s_1+v} P_{\gtrsim N} u\right\|_{L^{q_1^1} L^{m_1^1}} \|D^s u\|^{p-h}_{L^{(p-h)q_1^2} L^{m_1^2}} \prod_{i=1}^{h} \|D^s u\|_{L^{q_i^0} L^{m_i^0}} \\
&\lesssim N^{s_1+v-s} \|D^s u\|_{L^{q_1^1} L^{m_1^1}} \|D^s u\|^{p-h}_{L^{(p-h)q_1^2} L^{m_1^2}} \prod_{i=1}^{h} \|D^s u\|_{L^{q_i^0} L^{m_i^0}} \\
&\lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)},
\end{aligned}$$

where it is easy to check that m_1^1, m_1^2, m_1^0 are such that (q_1^1, m_1^1), $((p-h)q_1^2, m_1^2)$, (q_i^0, m_i^0) (for $i \in [1,h]$) are admissible pairs.
In (3.22), the parameters are chosen such that,

$$\frac{2}{q_2^2} + \frac{d}{r_2^2} = (p-h)\left(\frac{d}{2} - s\right), \tag{3.29}$$

$$\frac{2}{q_k^k} + \frac{d}{r_k^k} = \frac{d}{2} - (s - |\beta_k| - v), \tag{3.30}$$

$$\frac{2}{q_i^k} + \frac{d}{r_i^k} = \frac{d}{2} - (s - |\beta_i|), \quad \text{for each} i \neq k, k \in [1,h]. \tag{3.31}$$

Let s_2 be such that $\frac{2}{q_2^1} + \frac{d}{r_2^1} = \frac{d}{2} - s_2$. From (3.25), (3.26), (3.29), (3.30), (3.31), we see that $s_2 = s - \left(2 - p\left(\frac{d}{2} - s\right)\right)$. Moreover, by Sobolev embedding Lemma 3.1 and Lemma 3.14, we have

$$\begin{aligned}
(3.22) &\lesssim \left\|D^{s_2} P_{\gtrsim N} u\right\|_{L^{q_2^1} L^{m_2^1}} \|D^s u\|^{p-h}_{L^{(p-h)q_2^2} L^{m_2^2}} \sum_{k=1}^{h} \|D^s u\|_{L^{q_k^k} L^{m_k^k}} \prod_{i=1, i\neq k}^{h} \|D^s u\|_{L^{q_i^k} L^{m_i^k}} \\
&\lesssim N^{s_2-s} \|D^s u\|_{L^{q_2^1} L^{m_2^1}} \|D^s u\|^{p-h}_{L^{(p-h)q_2^2} L^{m_2^2}} \sum_{k=1}^{h} \|D^s u\|_{L^{q_k^k} L^{m_k^k}} \prod_{i=1, i\neq k}^{h} \|D^s u\|_{L^{q_i^k} L^{m_i^k}} \\
&\lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)},
\end{aligned}$$

where m_2^1, m_2^2, m_i^k are such that (q_2^1, m_2^1), (q_2^2, m_2^2), (q_i^k, m_i^k) (for $i \in [1,h]$) are admissible pairs.
Combining the above, we have

$$(3.19) \lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)}.$$

Similarly, we have

$$(3.20) \lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)}.$$

This implies that

$$\|D^s P_N f(u)\|_{L^{q_0'}\dot{H}^{v,r_0'}} \lesssim N^{-\left(2-p\left(\frac{d}{2}-s\right)\right)}.$$

Moreover, since $s > s_p$, we have $2 - p\left(\frac{d}{2} - s\right) > 0$.

Our second task is to show the following

$$\|D^s P_N(Vu)\|_{L^{q'}L^{r'}} \lesssim N^{-\eta_1}, \tag{3.32}$$

for all (q, r) admissible pair. To prove (3.32), we use the following results.

Lemma 3.16. *$\|P_{\geqslant N} V\|_{L^r} \lesssim_{V,k} N^{-k}$ for all $k \geqslant 0$.*

Lemma 3.17. *(See [8][Corollary 1.1]) Let p, p_1, p_2 be such that $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The the following holds for all $f, g \in \mathcal{S}$*

$$\|D^s(fg) - D^s fg - D^s gf\|_{L^p} \leqslant C \|Df\|_{L^{p_1}} \left\|D^{s-1} g\right\|_{L^{p_2}}.$$

Denote $P_{<N^{\eta_0}} u = u_{\mathrm{lo}}$ and $u_{\mathrm{hi}} = u - u_{\mathrm{lo}}$. We have

$$\begin{aligned}\|D^s P_N(Vu)\|_{L^{r'}} &= \|D^s P_N(V_{\mathrm{lo}} u)\|_{L^{r'}} + \|D^s P_N(V_{\mathrm{hi}} u)\|_{L^{r'}} \\ &= \left\|D^s P_N(V_{lo} P_{>N/10} u)\right\|_{L^{r'}} + \|D^s P_N(V_{\mathrm{hi}} u)\|_{L^{r'}}\end{aligned}$$

Consider the second term. Using Lemma 3.16, Lemma 3.17 and Hölder inequality, we have

$$\begin{aligned}\|D^s P_N(V_{\mathrm{hi}} u)\|_{L^{q'}L^{r'}} &= \|D^s P_N(V_{\mathrm{hi}} u)\|_{L^{q'}L^{r'}} \\ &\leqslant \|D^s(V_{\mathrm{hi}} u)\|_{L^{q'}L^{r'}} \\ &\leqslant \|D^s(V_{\mathrm{hi}})u\|_{L^{q'}L^{r'}} + \|V_{\mathrm{hi}} D^s u\|_{L^{q'}L^{r'}} + \|DV_{\mathrm{hi}}\|_{L^{k_1}L^{h_1}} \left\|D^{s-1} u\right\|_{L^{k_2}L^{h_2}} \\ &\lesssim \|D^s V_{\mathrm{hi}}\|_{L^{q_1}L^{r_1}} \|u\|_{L^q L^r} + \|V_{\mathrm{hi}}\|_{L^{q_1}L^{r_1}} \|D^s u\|_{L^q L^r} \\ &\quad + \|DV_{\mathrm{hi}}\|_{L^{q_1}L^{r_1}} \left\|D^{s-1} u\right\|_{L^q L^r} \\ &\lesssim_V N^{-\eta_0}.\end{aligned}$$

where (q_1, r_1) is such that $\frac{1}{q_1} = \frac{1}{q'} - \frac{1}{q}$ and $\frac{1}{r_1} = \frac{1}{r'} - \frac{1}{r}$. Thus, this term is acceptable. Consider the first term. Using Lemma 3.17 and Hölder inequality, we have

$$\begin{aligned}\left\|D^s P_N(V_{\mathrm{lo}} P_{>N/10} u)\right\|_{L^{q'}L^{r'}} &\leqslant \left\|D^s(V_{\mathrm{lo}} P_{>N/10} u)\right\|_{L^{q'}L^{r'}} \\ &\leqslant \left\|D^s V_{\mathrm{lo}} P_{>N/10} u\right\|_{L^{q'}L^{r'}} + \left\|V_{\mathrm{lo}} D^s(P_{>N/10} u)\right\|_{L^{q'}L^{r'}} \\ &\quad + \|DV_{\mathrm{lo}}\|_{L^{q_1}L^{r_1}} \left\|D^{s-1}(P_{>N/10} u)\right\|_{L^q L^r} \\ &\leqslant \|D^s V_{\mathrm{lo}}\|_{L^{q_1}L^{r_1}} \left\|P_{>N/10} u\right\|_{L^q L^r} + \left\|V_{\mathrm{lo}} D^s(P_{>N/10} u)\right\|_{L^{q'}L^{r'}} \\ &\quad + \|DV_{\mathrm{lo}}\|_{L^{q_1}L^{r_1}} \left\|D^{s-1}(P_{>N/10} u)\right\|_{L^q L^r}.\end{aligned}$$

The first term and the third term are acceptable since $\left\|P_{>N/10} u\right\|_{L^q L^r} \lesssim N^{-s} \left\|D^s(P_{>N/10} u)\right\|_{L^q L^r} \lesssim N^{-s}$ and $\left\|D^{s-1}(P_{>N/10} u)\right\|_{L^q L^r} \lesssim N^{-1} \left\|D^s(P_{>N/10} u)\right\|_{L^q L^r} \lesssim N^{-1}$. Consider now the second term. First, we have

$$\left\|V_{\mathrm{lo}} D^s(P_{>N/10} u)\right\|_{L^{\tilde{q}'}L^{\tilde{r}'}(I\times\mathbb{R}^d)} \lesssim 1,$$

for all $\tilde{q}, \tilde{r}$ is sufficiently close to q, r.
Moreover, using Lemma 2.5 for $0 < \delta < \frac{1}{2}$, we have

$$\begin{aligned}\left\|V_{\mathrm{lo}} D^s (P_{>N/10} u)\right\|_{L^2_{t,x}(I\times\mathbb{R}^d)} &\leqslant \sum_{\substack{M<N^{\eta_0} \text{ dyadic},\\ \tilde{M}>N/10 \text{ dyadic}}} \|V_M D^s u_{\tilde{M}}\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \\ &\lesssim \sum_{\substack{M<N^{\eta_0} \text{ dyadic},\\ \tilde{M}>N/10 \text{ dyadic}}} \frac{M^{(d-1)/2-\delta}\tilde{M}^{-1/2+\delta}}{\langle M\rangle^s} \\ &\lesssim \sum_{M<N^{\eta_0} \text{ dyadic}} M^{(d-1)/2-\delta} N^{-1/2+\delta} \\ &\lesssim N^{\eta_0((d-1)/2-\delta)} N^{-1/2+\delta},\end{aligned}$$

this is an acceptable term if we choose $\eta_0, \delta \ll 1$. By interpolation, this implies that the term $\left\|V_{\mathrm{lo}} D^s (P_{>N/10} u)\right\|_{L^{q'}L^{r'}}$ is also acceptable. Thus, (3.32) holds. Hence, $\|D^s P_N f(u)\|_{L^{q_0'}(I)L^{r_0'}(\mathbb{R}^d)} \lesssim N^{-\eta_1}$ and the proof of Proposition 3.15 is completed. □

Remark 3.18. In the above proof, the condition $s > s_p$ is used since we should have $2 > p\left(\frac{d}{2} - s\right)$. In the critical case $s = s_p$, the situation is more complex and local smoothing effect can not hold.

Lemma 3.19 (Schur's test)**.** *If*

$$\sup_{x\in\mathbb{R}^d}\left(\int_{y\in\mathbb{R}^d} |K(x,y)|^r dy\right)^{1/r} + \sup_{y\in\mathbb{R}^d}\left(\int_{x\in\mathbb{R}^d} |K(x,y)|^r dx\right)^{1/r} < C,$$

where r satisfies $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$, for some $1 \leqslant p \leqslant q \leqslant \infty$, then the operator $Tf(x) = \int_{y\in\mathbb{R}^d} K(x,y) f(y) dy$ extends to a continuous operator $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ with $\|T\|_{L^p\to L^q} \leqslant C$.

Lemma 3.20 ([12][Lemma 5.1])**.** *Let u be a global solution of energy at most E. Then, $e^{-it\Delta}u(t)$ is weakly convergent to $u_+ \in H$ as $t \to +\infty$.*

Proposition 3.21 ([12][Proposition 5.2])**.** *Let u be a global solution of energy at most $E > 0$. Then there exists a unique decomposition*

$$u(t) = e^{it\Delta}u_+ + v(t), \tag{3.33}$$

where $u_+ \in H$ with

$$\|u_+\|_H^2 \leqslant E \tag{3.34}$$

and

$$\|v(t)\|_H^2 \leqslant 2E, \tag{3.35}$$

for all $t \geqslant 0$. We also have the following identities

$$v(t) = e^{it\Delta}(u(0) - u_+) - i\int_0^t e^{i(t-t')\Delta} f(u(t'))dt' \tag{3.36}$$

and

$$v(t) = \underset{T\to+\infty}{iw\text{-}lim} \int_t^T e^{i(t-t')\Delta} f(u(t'))dt'. \tag{3.37}$$

Remark 3.22. Assume that the solution u in Proposition 3.21 is almost periodic in sense that $\{u(t) : t \in \mathbb{R}^+\}$ is pre-compact in H. Then $u_+ = 0$. Indeed, since $e^{-it\Delta}u(t)$ weakly converges to u_+ in H, $e^{-it\Delta}u(t)$ also weakly converges to u_+ in L^2. We only need to show that for all $\phi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\lim_{t\to\infty} \left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2} = 0.$$

Let $\varepsilon > 0$. By assumption, there exists $C_\varepsilon > 0$ such that

$$\|u(t,x)\|^2_{L^2(|x|\geqslant C_\varepsilon)} \leqslant \varepsilon, \quad \forall t \in \mathbb{R}^+.$$

We have

$$\begin{aligned}\left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2}\right|^2 &\lesssim \left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2(|x|\geqslant C_\varepsilon)}\right|^2 + \left|\left\langle u(t), e^{it\Delta}\phi\right\rangle_{L^2(|x|\leqslant C_\varepsilon)}\right|^2 \\ &\lesssim \|u(t)\|^2_{L^2(|x|\geqslant C_\varepsilon)}\|\phi\|^2_{L^2} + \|u(t)\|_{L^\infty_t L^2_x}\int_{|x|\leqslant C_\varepsilon}|e^{it\Delta}\phi|^2 dx \\ &\lesssim \varepsilon\|\phi\|^2_{L^2} + t^{-d}C_\varepsilon^d\|\phi\|^2_{L^1} \\ &\lesssim \varepsilon, \quad \text{if } t \text{ large enough.}\end{aligned}$$

This implies that $e^{-it\Delta}u(t)$ also weakly converges to 0 in L^2, hence, $u_+ = 0$. This completes the proof.

Lemma 3.23 (Weakly bound component are approximate solutions, [12][Lemma 5.6]). *Let $T \in \mathbb{R}$ and u, v be as in Proposition 3.21. Then for all t_0 large enough (depending on T, u), we have*

$$S(T)v(t_0) = v(T + t_0) + o_H(1)$$

where $S(t)$ is the flow of (1.1) and $o_H(1)$ goes to zero in H norm as $t_0 \to +\infty$.

Proof. Fix T, and let t_0 large enough. Define $I = [t_0, t_0 + T]$. We see that v solves the following NLS:

$$Lv = f(v) + [f(v + e^{it\Delta}u_+) - f(v)].$$

As $t_0 \to +\infty$, we have

$$\left\|e^{it\Delta}u_+\right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \to 0.$$

Moreover,

$$\left\|e^{it\Delta}u_+\right\|_{C^0_t H(I\times\mathbb{R}^d)} \lesssim 1.$$

By Lemma 3.14 and triangle inequality, we have

$$\|v\|_{L^{q_0}H^{s,r_0}\cap C^0_t H(I\times\mathbb{R}^d)} \lesssim_T 1.$$

Let

$$X = \left\|f(v + e^{it\Delta}u_+) - f(v)\right\|_{L^{q_0'}H^{s,r_0'}(I\times\mathbb{R}^d)}.$$

By Lemma 3.11, we have

$$\begin{aligned}X \lesssim |T|^{1/q_0'-1/q_0}\left\|e^{it\Delta}u_+\right\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}\Big(1 + \left\|e^{it\Delta}u_+\right\|_{L^{q_0}\dot{H}^{s,r_0}(I\times\mathbb{R}^d)}\left\|(u, e^{it\Delta}u_+)\right\|^p_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \\ + \left\|e^{it\Delta}u_+\right\|_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\left\|(u, e^{it\Delta}u_+)\right\|^{p-1}_{L^\infty L^{Q_0}(I\times\mathbb{R}^d)}\left\|(u, e^{it\Delta}u_+)\right\|_{L^{q_0}H^{s,r_0}(I\times\mathbb{R}^d)}\Big),\end{aligned}$$

which go to zero as $t_0 \to +\infty$. Thus, by perturbation Lemma 3.13, the claim follows. □

4. Frequency localisation

In this section, we prove the following asymptotic localisation frequency of weakly bound component v of u:

Proposition 4.1. *Let u be a global solution of energy at most E and let v be the weakly bound component of u. Then, we have*

$$\limsup_{t\to+\infty}\sup_{N\leqslant 1} N^{-\eta_3}\|P_{\leqslant N}v(t)\|_H \lesssim 1 \tag{4.1}$$

and

$$\limsup_{t\to+\infty}\sup_{N\geqslant 1} N^{\eta_3}\|P_{\geqslant N}v(t)\|_H \lesssim 1. \tag{4.2}$$

Proof. See [12][Proposition 6.1]. We sketch the proof here. The main idea is to use the double Duhamel formula (3.36) and (3.37).
Fix $\varepsilon > 0$. Let $u_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ be such that

$$u(0) - u_+ = u_\varepsilon + O_H(\varepsilon^2).$$

Hence, from (3.36), we have

$$v(t) = e^{it\Delta}u_\varepsilon - i\int_0^t e^{i(t-t')\Delta}f(u(t'))dt' + O_H(\varepsilon^2). \tag{4.3}$$

To Prove (4.1), it suffices to prove that

$$\|P_{\leqslant N}v(t)\|_H \lesssim N^{\eta_3} + \varepsilon, \tag{4.4}$$

for t large enough (can depend on E, u, ε but is independent on N) and $N \leqslant 1$.

Using (4.3) and (3.37), we have

$$\begin{aligned}
\|P_{\leqslant N}v(t)\|_H^2 &= \left|\langle P_{\leqslant N}v(t), P_{\leqslant N}v(t)\rangle_H\right| \\
&\leqslant \Bigg\langle \operatorname*{iw-lim}_{T\to+\infty} \int_t^T e^{i(t-t')\Delta}P_{\leqslant N}f(u(t'))dt', \\
&\quad P_{\leqslant N}e^{it\Delta}u_\varepsilon - i\int_0^t e^{i(t-t'')\Delta}P_{\leqslant N}f(u(t''))dt''\Bigg\rangle_H + O(\varepsilon^2) \\
&\leqslant \int_t^T \left|\left\langle e^{i(t-t')\Delta}P_{\leqslant N}f(u(t')), P_{\leqslant N}e^{it\Delta}u_\varepsilon\right\rangle_H\right| dt' \\
&\quad + \left|\int_t^T\int_0^t Y_N(t',t'')dt''dt'\right| + O(\varepsilon^2),
\end{aligned}$$

for T large enough and Y_N is defined by

$$Y_N(t',t'') = \left\langle e^{i(t-t')\Delta}P_{\leqslant N}f(u(t')), e^{i(t-t'')\Delta}P_{\leqslant N}f(u(t''))\right\rangle_H.$$

For the first integral, we have

$$\left\|P_{\leqslant N}e^{it'\Delta}u_\varepsilon\right\|_{H^{s,R'}} \lesssim_{u_\varepsilon} \frac{1}{|t'|^{d(1/R-1/2)}}.$$

Since $R < \frac{2d}{d+4}$, the exponent of $\langle t'\rangle$ is larger than 2, hence, the first integral is bounded by $O(\varepsilon^2)$ if t, T $(t < T)$ large enough (not depend on N).

The second integral is bounded by

$$\int_t^\infty\int_0^t |Y_N(t',t'')|\,dt'dt''$$

.

Using Lemma 2.2, we have

$$\begin{aligned}
|Y_N(t',t'')| &= \left|\left\langle P_{\leqslant N}f(u(t'))), e^{i(t'-t'')\Delta}P_{\leqslant N}f(u(t''))\right\rangle_H\right| \\
&\lesssim \|P_{\leqslant N}f(u(t'))\|_{H^{s,R}} \left\|e^{i(t'-t'')\Delta}P_{\leqslant N}f(u(t'))\right\|_{H^{s,R'}} \\
&\lesssim \|P_{\leqslant N}f(u(t'))\|_{L^R} \|P_{\leqslant N}f(u(t''))\|_{L^R} \frac{1}{|t'-t''|^{d(1/R-1/2)}} \\
&\lesssim \frac{1}{|t'-t''|^{d(1/R-1/2)}}.
\end{aligned}$$

Moreover, using Lemma 2.2, Bernstein's inequality, we have

$$\begin{aligned}
|Y_N(t',t'')| &\lesssim \|P_{\leqslant N}f(u(t'))\|_{H^s} \|P_{\leqslant N}f(u(t''))\|_{H^s} \\
&\lesssim \|P_{\leqslant N}f(u(t'))\|_{L^2} \|P_{\leqslant N}f(u(t''))\|_{L^2} \\
&\lesssim N^{d(2/R-1)} \|f(u(t'))\|_{L^R} \|f(u(t''))\|_{L^R} \\
&\lesssim N^{d(2/R-1)}.
\end{aligned}$$

Combining all the above, we have

$$\int_t^\infty\int_0^t |Y_N(t',t'')|\,dt'dt'' \lesssim \int_t^\infty\int_0^t \min\left(\frac{1}{t'-t''}, N^2\right)^{d(1/R-1/2)} dt''dt'.$$

Since $R < \frac{2d}{d+4}$, the exponent $d(1/R-1/2) > 2$, hence, the claim (4.4) follows.

We now prove (4.2). As before, we have

$$\|P_{\geqslant N}v(t)\|_H^2 \lesssim \varepsilon^2 + \left|\int_t^T \int_0^t Z_N(t',t'')dt''dt'\right|,$$

for T large enough, where

$$Z_N(t',t'') = \left\langle e^{i(t-t')\Delta}P_{\geqslant N}f(u(t')), e^{i(t-t'')\Delta}P_{\geqslant N}f(u(t''))\right\rangle_H.$$

Thus, it suffices to show that

$$\left|\int_t^T \int_0^t Z_N(t',t'')dt''dt'\right| \lesssim N^{-\eta_3}.$$

Consider the region where $t' \geqslant t + N^{\eta_2}$. We have

$$|Z_N(t',t'')| \lesssim |t'-t''|^{-d(1/R-1/2)}.$$

Since $d(1/R-1/2) > 2$, the contribution of this term is

$$\lesssim \int_{t'>t+N^{\eta_2}} \int_{t''<t} |t'-t''|^{-d(1/R-1/2)}dt''dt' = O(N^{-\eta_3}),$$

where we use $N^{-\eta_2} \ll N^{-\eta_3}$. Similar argument for the region $t'' \leqslant t - N^{\eta_2}$. It remains to prove that

$$\left|\int_t^{t+N^{\eta_2}} \int_{\max(0,t-N^{\eta_2})}^t Z_N(t',t'')dt''dt'\right| \lesssim N^{-\eta_3}.$$

By Hölder inequality, the left hand side is bounded by

$$\|P_{\geqslant N}f(u(t'))\|_{L_{t'}^{q_0'}H_x^{s,r_0'}((t,t+N^{\eta_2})\times\mathbb{R}^d)} \times \left\|\int_{\max(0,t-N^{\eta_2})}^t e^{i(t'-t'')\Delta}P_{\geqslant N}f(u(t''))dt''\right\|_{L_{t'}^{q_0}H_x^{s,r_0}((t,t+N^{\eta_2})\times\mathbb{R}^d)}.$$

By Strichartz, the left hand side is bounded by

$$\|P_{\geqslant N}f(u)\|^2_{L^{q_0'}H^{s,r_0'}((\max(t-N^{\eta_2},0),t+N^{\eta_2})\times\mathbb{R}^d)},$$

which by Proposition 3.15 and dyadic decomposition can be bounded by

$$O(N^{2\eta_2/q_0'}N^{-2\eta_1}),$$

this is acceptable. The proof is completed.

□

5. Spatial localisation

In this section, we prove localisation in spatial of v and then the main result Theorem 1.3. First, we have the following localisation result:

Theorem 5.1. *Let $E > 0$, $0 < \mu_0 < 1$. There exist J, μ_4 (depending only on E, μ_0) with the following property: For each u is a forward-global solution of energy at most E, then there exists functions $x_1, x_2, \cdots, x_J : R^+ \to \mathbb{R}$, such that*

$$\limsup_{t\to+\infty} \int_{\inf_{1\leqslant j\leqslant J}|x-x_j(t)|\geqslant 1/\mu_4} |v(t,x)|^2dx \lesssim \mu_0^2. \tag{5.1}$$

5.1. **First step: L^∞_x spatial localisation at fixed times.** Define $v_{\rm med} = P_{\mu_2<\cdot<1/\mu_2}v$. We exploit the frequency localisation of $v_{\rm med}$. From Proposition 4.1, we have

$$\|v(t) - v_{\rm med}(t)\|_H \lesssim \mu_2^{\eta_3} \tag{5.2}$$

and thus,

$$u(t) = e^{it\Delta}u_+ + v_{\rm med}(t) + O_H(\mu_2^{\eta_3}). \tag{5.3}$$

For each t, let $x_1(t), x_2(t), \cdots, x_{J(t)}(t)$ be a set of points such that $|x_j(t) - x_k(t)| > \mu_3^{-1}$ for each $j \neq k$ and

$$|v_{\rm med}(t, x_j(t))| \geqslant \mu_3^{1/\eta_1} \quad \text{for all } 1 \leqslant j \leqslant J(t). \tag{5.4}$$

By the rapid decay of the convolution kernel of $P_{\mu_2<\cdot<\mu_2^{-1}}$ and Hölder, we have

$$|v_{\rm med}(t, x_j(t))|^2 \lesssim \mu_2^{-d} \int_{|x-x_j(t)|\leqslant 1/2\mu_3^{-1}} |v_{\rm med}(t,x)|^2 dx + O(\mu_3^{3/\eta_1})$$

and thus

$$\int_{|x-x_j(t)|\leqslant 1/2\mu_3^{-1}} |v_{\rm med}(t,x)|^2 dx \gtrsim \mu_3^{2/\eta_1}\mu_2^d \gtrsim \mu_3^{2/\eta_1+d}$$

for all j. Thus, by uniformly bounded of H-norm of v, there exists J depending on E, μ_3 such that $J(t) \leqslant J$ for all t. If $J(t) < J$ then for each $J(t) < j \leqslant J$, we define $x_j(t) = x_1(t)$. By the above construction, for all t, there exist $x_1(t), x_2(t), \cdots, x_J(t)$ such that

$$|v_{\rm med}(t,x)| < \mu_3^{1/\eta_1} \quad \text{whenever} \quad \inf_{1\leqslant j\leqslant J} |x - x_j(t)| \geqslant \mu_3^{-1}. \tag{5.5}$$

5.2. **Second step: L^∞_x spatial localisation on a time interval.** Fix t_0 a sufficient late time and $I = [t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$. Let $D : \mathbb{R}^d \to \mathbb{R}^+$ be the distance function $D(x) = \inf_{1\leqslant j\leqslant J} |x - x_j(t_0)|$; thus $|v_{\rm med}(t_0, x)| \leqslant \mu_3^{1/\mu_1}$ whenever $D \geqslant \mu_3^{-1}$.

Let $\chi : \mathbb{R}^d \to \mathbb{R}^+$ be a smooth cutoff function which equals to one when $D(x) \leqslant 2\mu_3^{-1}$, vanishes when $D(x) \geqslant 3\mu_3^{-1}$ and obeys the bounds $\nabla^k\chi = O_k(\mu_3^k)$ for $k \geqslant 0$. We can let $\chi(x) = \chi_1(\mu_3 x)$ and chose suitable function χ_1.

We have the following result:

Lemma 5.2. *We have*

$$\limsup_{t\to+\infty} \left\| e^{i(t-t_0)\Delta}((1-\chi)u(t_0)) \right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \lesssim \mu_2^{\eta_3}.$$

Proof. Let $\tilde{u}_+ \in \mathcal{D}(\mathbb{R}^d)$ be such that $\tilde{u}_+ = u_+ + O_H(\mu_2^{\eta_3})$, this gives

$$(1-\chi)u(t_0) = e^{it_0\Delta}\tilde{u}_+ + (1-\chi)v_{\rm med}(t_0) + O_H(\mu_2^{\eta_3}).$$

The contribution of the first term on the right hand side is acceptable if we choose t_0 large enough. For the second term on the right hand side, we have

$$\begin{aligned}(1-\chi)v_{\rm med}(t,x) = {}& (1-\chi)P_{<100\mu_2^{-1}}(1_{D\geqslant\mu_3^{-1}}v_{\rm med}(t_0)) \\ & + (1-\chi)P_{<100\mu_2^{-1}}(1_{D<\mu_3^{-1}}v_{\rm med}(t_0))\end{aligned}$$

Using Lemma 6.3, the second term in the right hand side is acceptable. It remains to consider the first term. We only need to prove that

$$\left\| e^{i(t-t_0)\Delta}\left((1-\chi)P_{100\mu_2^{-1}}(1_{D\geqslant\mu_3^{-1}}v_{\rm med}(t_0))\right)\right\|_{L^{q_0}H^{s,r_0}\cap L^\infty L^{Q_0}(I\times\mathbb{R}^d)} \lesssim \mu_2^{\eta_3}$$

for t_0 large enough.

Using Lemma 6.2, the left hand side is bounded by

$$O_{\mu_1}(\mu_2^{-1/\eta_0}) \left\| 1_{D\geqslant\mu_3^{-1}}v_{\rm med}(t_0)\right\|_{L^{r_0}\cap L^{Q_0}}.$$

From (5.5),

$$\left\| 1_{D\geqslant\mu_3^{-1}}v_{\rm med}(t_0)\right\|_{L^\infty} \leqslant \mu_3^{1/\eta_1}.$$

Interpolating this with bounded L^2 norm of v, we obtain the desired result.

□

We have the following spatial decay:

Lemma 5.3. *We have*

$$\limsup_{t_0\to+\infty} \left\|1_{D>\mu_3^{-2}}u\right\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} \mu_2^{\eta_3}.$$

Proof. Let $\tilde{u}$ be the solution of (1.1) with initial data $\tilde{u}(t_0) = \chi u(t_0)$. From Lemma 3.13 and Lemma 5.2, we see that

$$\|\tilde{u}\|_{L^{q_0}H^{s,r_0}\cap C^0H(I\times\mathbb{R}^d)} = O_H(\mu_1^{-1})$$

and

$$\|u-\tilde{u}\|_{L^{q_0}H^{s,r_0}} \lesssim_{\mu_1} \mu_2^{\eta_3}. \tag{5.6}$$

Let $W = 1+\mu_3\tilde{D}$ be such that $\tilde{D}\in C^2(\mathbb{R}^d)$, $\nabla\tilde{D}\lesssim 1$, $\nabla^2\tilde{D}\lesssim 1$ and $\left\|\tilde{D}-D\right\|_{L^\infty}\leqslant 1$ (see Lemma 6.1 for the choice of such $\tilde{D}$). We have $\nabla W, \nabla^2 W = O(\mu_3)$ and $W\chi = \chi+\mu_3 D\chi+\mu_3(\tilde{D}-D)\chi = O(1)$. Thus,

$$\|W\tilde{u}(t_0)\|_{L^2} \lesssim \|u(t_0)\|_{L^2} \lesssim 1. \tag{5.7}$$

We have

$$(i\partial_t+\Delta)(W\tilde{u}) = O(W|\tilde{u}|^p\tilde{u}) + O(WV\tilde{u}) + O(\mu_3|\tilde{u}|) + O(\mu_3\nabla\tilde{u}).$$

By Strichartz and $\left\|e^{it\Delta}f\right\|_{L^{q_0}L^{r_0}} \lesssim \|f\|_{L^2}$, we have

$$\|W\tilde{u}\|_{C^0L^2\cap L^{q_0}L^{r_0}(I'\times\mathbb{R}^d)} \lesssim \|W\tilde{u}(t')\|_{L^2} + \|W|\tilde{u}|^p\tilde{u}\|_{L^{q_0'}L^{r_0'}(I'\times\mathbb{R}^d)} + \mu_3|I'|\,\|\tilde{u}\|_H$$

for any $I'\subset I$, $t'\in I'$. Denote the left hand side by $X(I')$, by Hölder, we have

$$X(I') \lesssim \|W\tilde{u}(t')\|_{L^2} + |I'|^{1/q_0'-1/q_0}X(I') + |I'|\mu_3$$

and hence (for I' small enough), we have

$$X(I') \lesssim \|W\tilde{u}(t')\|_{L^2} + \mu_1^{-1}\mu_3.$$

By iterating this using (5.7), we have

$$X(I) \lesssim_{\mu_1} 1.$$

Thus,

$$\left\|1_{\tilde{D}>\frac{1}{2}\mu_3^{-2}}W\tilde{u}\right\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \leqslant \|W\tilde{u}\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} 1$$

and hence,

$$\left\|1_{\tilde{D}>\frac{1}{2}\mu_3^{-2}}\tilde{u}\right\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} \mu_3.$$

Moreover, $1_{D>\mu_3^{-2}} \leqslant 1_{\tilde{D}>\frac{1}{2}\mu_3^{-2}}$ by the choice of $\tilde{D}$. Thus,

$$\left\|1_{D>\mu_3^{-2}}\tilde{u}\right\|_{L^{q_0}L^{r_0}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} \mu_3.$$

Combining with (5.6), we obtain the desired result. □

As a consequence, we have local in time of Strichartz Duhamel:

Corollary 5.4. *For t_0 large enough and any $I'\subset I$, we have*

$$\left\|1_{D>\mu_3^{-3}}\int_{I'} e^{i(t_0-t')\Delta}f(u(t'))dt'\right\|_{L^2_x(\mathbb{R}^d)} \lesssim_{\mu_1} \mu_2^{\eta_3}.$$

Proof. Let $\chi_2:\mathbb{R}^d\to\mathbb{R}^+$ be a smooth cutoff such that $\chi_2 = 1$ when $D\leqslant\mu_3^{-2}$, $\chi_2 = 0$ when $D>2\mu_3^{-2}$ and χ_2 has bounded derivatives.
We have

$$f(u(t')) = P_{\geqslant\mu_2^{-1}}f(u(t')\chi_2) + P_{\leqslant\mu_2^{-1}}(1_{D\leqslant 2\mu_3^{-2}}f(u(t')\chi_2)) + O(1_{D>\mu_3^{-2}}(|u|^{p+1}(t')+|Vu(t')|)).$$

By Lemma 5.3, Hölder, we have

$$\left\|1_{D>\mu_3^{-2}}(|u|^{p+1}+|Vu|)\right\|_{L^{q_0'}L^{r_0'}(I\times\mathbb{R}^d)} \lesssim_{\mu_1} \mu_2^{\eta_3}$$

and hence this term is acceptable.
Consider the term $P_{\leqslant\mu_2^{-1}}(1_{D\leqslant 2\mu_3^{-2}}f(u(t')\chi_2))$. Using Lemma 6.4, and $\|f(u(t'))\|_{L^R}\lesssim 1$, we obtain

the contribution of $P_{\leqslant \mu_2^{-1}}(1_{D\leqslant 2\mu_3^{-2}} f(u(t')\chi_2))$ is acceptable.

It remains to consider the contribution of the term $P_{\geqslant \mu_2^{-1}} f(u(t')\chi_2)$. We have

$$\begin{aligned}&\left\|1_{D>\mu_3^{-3}}\int_{I'} e^{i(t_0-t')\Delta}P_{\geqslant \mu_2^{-1}} f(u(t')\chi_2)dt'\right\|_{L^2}\\ &\lesssim \left\|P_{\geqslant \mu_2^{-1}} f(u\chi_2)\right\|_{L^{q_0'}L^{r_0'}(I\times\mathbb{R}^d)} \lesssim \mu_2^s \left\|f(u\chi_2)\right\|_{L^{q_0'}\dot H^{s,r_0'}(I\times\mathbb{R}^d)} \lesssim \mu_2^{\eta_3}.\end{aligned}$$

this term is acceptable. This completes the proof. □

5.3. **Third step: L^2_x localisation at fixed time.** We now prove the following result:

Proposition 5.5. *Let $\chi_3 : \mathbb{R}^d \to \mathbb{R}^+$ be a smooth cutoff function which equals 1 when $D \geqslant 2\mu_3^{-3}$, equals 0 when $D \leqslant \mu_3^{-3}$ and χ_3 has bounded derivatives. Then for t_0 large enough, we have*

$$\chi_3 v(t_0) = O_{L^2}(\mu_1^c),$$

for some $c > 0$.

Proof. From corollary 5.4, we have

$$\chi_3 \int_{I'} e^{i(t_0-t')\Delta} f(u(t'))dt' = O_{L^2}(\mu_2^{\eta_3/2}).$$

for all $I' \subset I$. Thus, by Duhamel's formula, we have

$$\chi_3 v(t_0) = \chi_3 e^{-i\mu_1^{-1}\Delta} v(t_0 + 1/\mu_1) + O_{L^2}(\mu_2^{\eta_3/2})$$

and

$$\chi_3 v(t_0) = \chi_3 e^{+i\mu_1^{-1}\Delta} v(t_0 - 1/\mu_1) + O_{L^2}(\mu_2^{\eta_3/2})$$

This implies that

$$\|\chi_3 v(t_0)\|_{L^2}^2 = \left\langle \chi_3 e^{-i\mu_1^{-1}\Delta} v(t_0 + 1/\mu_1), \chi_3 e^{+i\mu_1^{-1}\Delta} v(t_0 - 1/\mu_1)\right\rangle_{L^2} + O(\mu_2^{\eta_3/2}).$$

Thus, it suffices to show that

$$\left\langle e^{-i\mu_1^{-1}\Delta} v(t_0 + 1/\mu_1), \chi_3^2 e^{+i\mu_1^{-1}\Delta} v(t_0 - 1/\mu_1)\right\rangle_{L^2} \lesssim \mu_1^c.$$

Let ϕ be a Schwartz function be such that $u(0) - u_+ = \phi + O_{L^2}(\mu_2)$. Thus, by (3.36), we have

$$v(t_0 - 1/\mu_1) = e^{i(t_0-1/\mu_1)\Delta}\phi - i\int_0^{t_0-1/\mu_1} e^{i(t_0-\mu_1-t')\Delta} f(u(t'))dt' + O_{L^2}(\mu_2)$$

and hence

$$\chi_3^2 e^{i\mu_1^{-1}\Delta} v(t_0 - 1/\mu_1) = e^{it_0\Delta}\phi - (1-\chi_3^2)e^{it_0\Delta}\phi - i\chi_3^2\int_0^{t_0-1/\mu_1} e^{i(t_0-t')\Delta} f(u(t'))dt' + O_{L^2}(\mu_2).$$

From dispersive estimate we see that $(1-\chi_3^2)e^{it_0\Delta}\phi = O_{L^2}(\mu_2)$ for t_0 large enough. Using (3.37), it suffices to show that

$$\int_{t_0+1/\mu_1}^{+\infty} \left|\left\langle e^{i(t_0-t')\Delta} f(u(t')), e^{it_0\Delta}\phi\right\rangle_{L^2}\right| dt' \lesssim \mu_1^{2c}, \tag{5.8}$$

and

$$\int_{t_0+1/\mu_1}^{+\infty}\int_0^{t_0-1/\mu_1} \left|\left\langle e^{i(t_0-t')\Delta} f(u(t')), \chi_3^2 e^{i(t_0-t'')\Delta} f(u(t''))\right\rangle_{L^2}\right| dt''dt' \lesssim \mu_1^{2c}. \tag{5.9}$$

To prove (5.8), using (2.2) and Hölder, we have

$$\left|\left\langle e^{i(t_0-t')\Delta} f(u(t')), e^{it_0\Delta}\phi\right\rangle_{L^2}\right| = \left|\left\langle f(u(t')), e^{it'\Delta}\phi\right\rangle_{L^2}\right| \lesssim \left\|e^{it'\Delta}\phi\right\|_{L^{R'}} \lesssim_\phi |t'|^{-d(1/R-1/2)}$$

. Since $d(1/R - 1/2) > 2$, we obtain (5.8) for t_0 large enough.

Now, we prove (5.9). Writing $\chi_3^2 = 1 - (1-\chi_3^2)$, we have

$$\left\langle e^{i(t_0-t')\Delta} f, \chi_3^2 e^{i(t_0-t'')\Delta} g\right\rangle_{L^2} = \left\langle e^{i(t''-t')\Delta} f, g\right\rangle_{L^2} - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\overline{g(z)}K_{t',t''}(x,z)dxdz, \tag{5.10}$$

for each test functions f, g and $K_{t',t''}$ is defined by

$$K_{t',t''}(x,z) = c(t_0-t')^{-d/2}(t_0-t'')^{-d/2}\int_{\mathbb{R}^d} e^{i|x-y|^2/4(t_0-t')-i|x-z|^2/4(t_0-t'')}(1-\chi_3^2)(y)dy,$$

for some constant c.

We prove that

$$K_{t',t''}(x,z) \lesssim |t'-t''|^{-d/2}. \tag{5.11}$$

Let $\varphi = 1-\chi_3^2$. Then $\varphi = 0$ on $D \geqslant 2\mu_3^{-3}$ and $\varphi = 1$ on $D \leqslant \mu_3^{-3}$. Without loss of generality, we assume $\operatorname{supp}(\varphi) \subset B(x_0, 2\mu_3^{-3})$ for some $x_0 \in \mathbb{R}^d$. Rewriting $\varphi(x) = \tilde{\varphi}(\mu_3^3(x-x_0))$, we have $\operatorname{supp}(\tilde{\varphi}) \subset B(0,2)$.

We have:

$$\begin{aligned}
K_{t',t''}(x,z) &= c(t_0-t')^{-d/2}(t_0-t'')^{-d/2}\int_{\mathbb{R}^d} e^{i\left(\frac{1}{4(t_0-t')}-\frac{1}{4(t_0-t'')}\right)|y-\tilde{c}|^2}\tilde{\varphi}(\mu_3^3(y-x_0))dy, \quad \text{(for some constant } c)\\
&= c(t_0-t')^{-d/2}(t_0-t'')^{-d/2}\int_{\mathbb{R}^d} e^{i\frac{t'-t''}{4(t_0-t')(t_0-t'')}|y|^2}\tilde{\varphi}(\mu_3^3(y-x_1))dy, \quad \text{(where } x_1 = x_0+\tilde{c})\\
&= c(t_0-t')^{-d/2}(t_0-t'')^{-d/2}\int_{\mathbb{R}^d} e^{-i\alpha|y|^2}\tilde{\varphi}(\mu_3^3(y-x_1))dy, \quad \left(\text{where } \alpha = \frac{|t'-t''|}{4|t_0-t'||t_0-t''|}\right)\\
&= c(t_0-t')^{-d/2}(t_0-t'')^{-d/2}\alpha^{-d/2}\int_{\mathbb{R}^d} e^{-i|\tilde{y}|^2}\tilde{\varphi}(\mu_3^3/\sqrt{\alpha}(\tilde{y}-\sqrt{\alpha}x_1))d\tilde{y}\\
&= c|t'-t''|^{-d/2}\int_{\mathbb{R}^d} e^{-i|y|^2}\tilde{\varphi}(\mu_3^3/\sqrt{\alpha}(y-x_2))dy, \quad (\text{ where } x_2 = \sqrt{\alpha}x_1).
\end{aligned}$$

Define $L = (1+|y|^2)^{-1}(1+iy/2\partial_y)$. We have $L^*(f) = (1+iy/2\partial_y)((1+|y|^2)^{-1}f$ and

$$L(e^{-i|y|^2}) = e^{-i|y|^2}.$$

Thus, for some $M \in \mathbb{N}$ large enough chosen later,

$$\begin{aligned}
K_{t',t''}(x,z) &= c|t'-t''|^{-d/2}\int_{\mathbb{R}^d} L^M(e^{-i|y|^2})\tilde{\varphi}(\mu_3^3/\sqrt{\alpha}(y-x_2))dy\\
&= c|t'-t''|^{-d/2}\int_{\mathbb{R}^d} e^{-i|y|^2}(L^*)^M(\tilde{\varphi}(\mu_3^3/\sqrt{\alpha}(y-x_2)))dy.
\end{aligned}$$

To prove (5.11), we only need to prove the following lemma

Lemma 5.6. *The following estimate holds for $M \in \mathbb{N}$ large enough:*

$$\left|\int_{\mathbb{R}^d} e^{-i|y|^2}(L^*)^M(\tilde{\varphi}(\mu_3^3/\sqrt{\alpha}(y-x_2)))dy\right| \lesssim 1.$$

Proof. For convenience, define $M_3 = \mu_3^3/\sqrt{\alpha}$. By induction, we see that

$$(L^*)^M = \sum_{k=0}^{M}(yM_3)^k\partial_y^k\tilde{\varphi}(M_3(y-x_2))(1+|y|^2)^{-2M+k}(c_{k0}+c_{k1}|y|^2+\cdots+c_{k(M-k)}|y|^{2M-2k}).$$

Moreover,

$$y^k = (y-x_2+x_2)^k = \sum_{h=0}^{k} c_h(y-x_2)^{k-h}x_2^h.$$

Combing to uniform bounded of $y^k\varphi(y)$, using $c_{k0}+c_{k1}|y|^2+\cdots+c_{k(M-k)}|y|^{2M-2k} \lesssim (1+|y|^2)^{M-k}$ it suffices to estimate

$$Q_0 := \int_{M_3|y-x_2|\leqslant 2}(M_3x_2)^h(1+|y|^2)^{-M}dy, \quad (\text{for each } 0 \leqslant h \leqslant M).$$

Since $|M_3x_2|^h \lesssim 1+|M_3x_2|^M$ $\forall 0 \leqslant h \leqslant M$, we only need to check the case $h=0$ and $h=M$. Choosing $M = 2d$. The first case is easy to estimate Q_0. Consider the second case. If $|M_3x_2| \leqslant 10$ then we see that $Q_0 \lesssim 1$. Consider the case $|M_3x_2| \geqslant 10$. If $M_3 \geqslant 1$ then $|y-x_2| \leqslant \frac{2}{M_3} \leqslant 2$. Thus, $1+|y|^2 \gtrsim (1+|y|^2)(1+|y-x_2|^2) \gtrsim 1+|x_2|^2 \gtrsim |x_2|$. This implies that $Q_0 \lesssim M_3^M(2/M_3)^d = 2^d$.

Consider the case $M_3 \leqslant 1$. Since $|M_3 x_2| \geqslant 10$ and $M_3|y-x_2| \leqslant 2$ and $|y-x_2|+|y| \geqslant |x_2|$, we have $|y| \gtrsim |x_2|$. Thus, $Q_0 \lesssim |x_2|^M \int_{M_3|y-x_2|\leqslant 2}(1+|y|^2)^{-M/2}|y|^{-M}dy \lesssim \int_{M_3|y-x_2|\leqslant 2}(1+|y|^2)^{-M/2}dy \lesssim 1$. This completes the proof of Lemma 5.6.

Thus, $K_{t',t''}(x,z) \lesssim |t'-t''|^{-d/2}$. This implies that (5.10) is estimated by $|t'-t''|^{-d/2}\|f\|_{L^1}\|g\|_{L^1}$. Moreover, (5.10) is also estimated by $\|f\|_{L^2}\|g\|_{L^2}$. Hence, by bilinear interpolation, (5.10) is estimated by $|t'-t''|^{-d(1/R-1/2)}\|f\|_{L^R}\|g\|_{L^R}$. By the choice of R, $d(1/R-1/2)>2$, hence (5.9) is proved. This completes the proof of Proposition 5.5.

□

Theorem 5.1 is followed from Proposition 5.5. As a consequence of Theorem 5.1, we have the following spatial localisation result:

Proposition 5.7. *Let $E>0$ and $0<\mu_0<1$. Then there exists $\mu_5=\mu_5(E,\mu_0)$ depending on E,μ_0 (and on the other fixed parameters) with the following properties: for any u radial global solution of energy at most E, we have*

$$\limsup_{t\to+\infty}\int_{|x|\geqslant 1/\mu_5}|v(t,x)|^2dx \lesssim \mu_0^2, \tag{5.12}$$

where v is the weak bound component of u.

Proof. Apply Theorem 5.1, for all t sufficiently large, there exist $x_1(t),\cdots,x_J(t)$ such that

$$\int_{\mathbb{R}^d} 1_{\inf_{1\leqslant j\leqslant J}|x-x_j(t)|\geqslant 1/\mu_4}|v(t,x)|^2dx \lesssim \mu_0^2.$$

Since u is radial, u_+ and v are also radial. Thus, we may average the above estimate over rotation and obtain that

$$\int_{\mathbb{R}^d}\left(\int_{S^{d-1}} 1_{\inf_{1\leqslant j\leqslant J}||x|\omega-x_j(t)|\geqslant 1/\mu_4}d\omega\right)|v(t,x)|^2dx \lesssim \mu_0^2.$$

In particular, we have

$$\int_{|x|\geqslant 1/\mu_5}\left(\int_{S^{d-1}}\left(1-\sum_{j=1}^J 1_{||x|\omega-x_j(t)|<1/\mu_4}\right)d\omega\right)|v(t,x)|^2dx \lesssim \mu_0^2.$$

But if μ_5 is sufficiently small depending on μ_4, by using $|x|\geqslant 1/\mu_5$, we have

$$\begin{aligned}
&\int_{S^{d-1}} 1_{||x|\omega-x_j(t)|<1/\mu_4}d\omega\\
&=\int_{S^{d-1}} 1_{|\omega-x_j(t)/|x||<\frac{1/\mu_4}{|x|}}d\omega\\
&\leqslant \int_{S^{d-1}} 1_{|\omega-x_0|<\mu_5/\mu_4}d\omega \quad (\text{where } x_0=x_j(t)/|x|)\\
&=\mathrm{Vol}(\{|\omega-x_0|<\mu_5/\mu_4|\}\cap S^{d-1})\\
&\leqslant (\mu_5/\mu_4)^{d-1}.
\end{aligned}$$

Thus, if μ_5 is small enough depending on both J and μ_4, we have

$$\int_{S^{d-1}}\left(1-\sum_{j=1}^J 1_{||x|\omega-x_j(t)|<1/\mu_4}\right)d\omega \geqslant \frac{1}{2}.$$

for all $|x|\geqslant 1/\mu_5$ and the desired result is proved.

□

Combining the above analysis, we give the proof of the main result:

Proof of Theorem 1.3. By Proposition 1.7, Proposition 4.1 and Theorem 5.1, there exists a compact set $K\subset H$ such that

$$\lim_{t\to+\infty}\mathrm{dist}_H(v(t),K)=0$$

for all radial global bounded solution u of energy at most E.

Let $\mathcal{K}_{E,\mathrm{rad}}$ be the closure of the set of all limit points $\lim_{t_n\to+\infty} v(t_n)$, where t_n ranges over sequences of times which converge to infinity. Then, $\mathcal{K}_{E,\mathrm{rad}}$ is close and hence is a compact subset of K. Let us prove that $\mathcal{K}_{E,\mathrm{rad}}$ is also a attractor. Indeed, let $t_n\to+\infty$. Since, $\mathrm{dist}_H(v(t_n),K)\to 0$ as $n\to+\infty$, there exists $k_n\in K$ such that $\|v(t_n)-k_n\|_K\to 0$. By compactness of K, for each sub-sequence of (t_n) there exists $\varphi\in K$ and a sub-sequence of (t_n) which for convenience still denotes by (t_n) such that $\|v(t_n)-\varphi\|_H\to 0$. Thus, $\varphi\in\mathcal{K}_{E,\mathrm{rad}}$ and hence, $\lim_{t\to+\infty}\mathrm{dist}_H(v(t),\mathcal{K}_{E,\mathrm{rad}})=0$. By local theory and Lemma 3.23, $\mathcal{K}_{E,\mathrm{rad}}$ is invariant under $S(t)$ for t small and hence for all t large also.

Finally, we show the uniqueness of u_+. If $\tilde{u}_+$ satisfies the same properties of u_+ then $\{e^{it\Delta}(u_+-\tilde{u}_+), t\geqslant 0\}$ is pre-compact in H. Moreover, for $2<p\leqslant\frac{2d}{d-2s}$,

$$\left\|e^{it\Delta}(u_+-\tilde{u}_+)\right\|_{L^p}\to 0,\qquad \text{as } t\to+\infty.$$

Combining the above

$$\left\|e^{it\Delta}(u_+-\tilde{u}_+)\right\|_{H}\to 0\qquad \text{as } t\to+\infty.$$

Thus, $u_+=\tilde{u}_+$, which completes the proof. □

6. Appendix

In this section, we prove some useful results which is used in the proof of the main result.

Lemma 6.1. *Let $x_1,x_2,\cdots,x_J\in\mathbb{R}^d$ and $D(x)=\inf_{1\leqslant j\leqslant J}|x-x_j|$. There exists a function $\tilde{D}\in C^2(\mathbb{R}^d)$ such that $\left\|\tilde{D}-D\right\|_{L^\infty}\leqslant 1$ and $\nabla\tilde{D},\nabla^2\tilde{D}\lesssim 1$.*

Proof. We see that $D\in C^0(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)\backslash\mathcal{A}$ for some zero measure set $\mathcal{A}$. Indeed, $\mathcal{A}=\cup_i\{x_i\}\cup_{j\neq k}\{x:|x-x_j|=|x-x_k|\}$. Moreover, D is small inside $\cup_{i=1}^J B(x_i,1)$ and $|\nabla D|,|\nabla^2 D|\lesssim 1$ outside $\cup_{i=1}^J B(x_i,1)\cup_{j\neq k}\{x:|x-x_j|=|x-x_k|\}$.

We first study the graph of D in $\mathbb{R}^{d+1}$ i.e the set of elements of form $(x,D(x))$, $x\in\mathbb{R}^d$. For each j, the graph of $|x-x_j|$ is $\{(x,|x-x_j|),x\in\mathbb{R}^d\}\subset\mathbb{R}^{d+1}$, which is a co-dimension one surface having a critical point at x_j. The graph of $|x-x_k|$ can be given by move the graph of $|x-x_j|$ by vector x_j-x_k. For each $x_j\neq x_k$, the intersection of the graph $(x,|x-x_j|)$ and $(x,|x-x_k|)$ is a co-dimension two surface: $\{(x,|x-x_j|):|x-x_j|=|x-x_k|\}$. Thus, the graph of D is a co-dimension one surface which is singular at each x_j and at each co-dimension two surface $\{(x,|x-x_j|):|x-x_j|=|x-x_k|\}$. We can easily take a regular surface $\mathcal{H}$ close to the graph of D, which is graph of a smooth function $\tilde{D}$. We see that $\tilde{D}\in C^\infty(\mathbb{R}^d)$ and $\tilde{D}$ satisfies the desired properties if we take $\mathcal{H}$ is close enough to the graph of D.

□

Let $t_0,t,t',\chi,\mu_2,\mu_3,v_{\mathrm{med}}$ be as in Subsection 5.2. We have the following results:

Lemma 6.2. *The operator $\nabla^k e^{i(t-t_0)\Delta}(1-\chi)P_{\lesssim\mu_2^{-1}}$ have an operator norm of $O_{\mu_1}(\mu_2^{-c(k)})$ on L^r for $2<r<\frac{2d}{d-2s}$, $c(k)$ depending on k and $\forall k\in\mathbb{N}$. Similar result holds for $D^j e^{i(t-t_0)\Delta}(1-\chi)P_{\lesssim\mu_2^{-1}}$ for $j=0,s$.*

Proof. By density, we need only work on Schwartz functional space. We consider the case $k=0$, the general case is obtained by bounded derivatives of χ and interpolation.
Let ψ be smooth function such that $\psi(\xi)=1$ if $|\xi|\leqslant 1$ and $\psi(\xi)=0$ if $|\xi|\geqslant 2$.
For each $t\in\mathbb{R}$ and N dyadic, we have

$$\begin{aligned}e^{it\Delta}P_{\lesssim N}f&=\mathcal{F}^{-1}\left(e^{-it|\xi|^2}\psi(\xi/N)\hat{f}(\xi)\right)\\&=\mathcal{F}^{-1}(e^{-it|\xi|^2}\psi(\xi/N))*f.\end{aligned}$$

Thus, by Young inequality, for $M > d$ an integer, the operator norm of $e^{it\Delta}P_{\lesssim N}f$ on L^r is bounded by

$$\begin{aligned}
\left\|\mathcal{F}^{-1}(e^{-it|\xi|^2}\psi(\xi/N))(x)\right\|_{L^1} &= \left\|N^d\mathcal{F}^{-1}(e^{-itN^2|\xi|^2}\psi(\xi))(Nx)\right\|_{L^1_x} \\
&= \left\|\mathcal{F}^{-1}(e^{-i(tN^2)|\xi|^2}\psi(\xi))(x)\right\|_{L^1_x} \\
&= \left\|\mathcal{F}(e^{-i(tN^2)|\cdot|^2}\psi(\cdot))(\xi)\right\|_{L^1_\xi} \\
&\lesssim \left\|(1+|\xi|)^M\mathcal{F}(e^{-i(tN^2)|\xi|^2}\psi(\xi))\right\|_{L^2}\left\|(1+|\xi|)^{-M}\right\|_{L^2} \\
&\lesssim \left\|e^{-i(tN^2)|\xi|^2}\psi(\xi)\right\|_{H^M_\xi} \\
&\lesssim \left\|(1+|\xi|)^M\psi(\xi)\right\|_{H^M_\xi}(1+|tN^2|)^M \\
&\lesssim (1+|tN^2|)^M.
\end{aligned}$$

Thus, the operator norm of $e^{i(t-t_0)\Delta}P_{\lesssim\mu_2^{-1}}$ on L^r is bounded by

$$(1+|t-t_0|^2\mu_2^{-2})^M = O_{\mu_1}(\mu_2^{-2M}).$$

It remains to prove that $e^{i(t-t_0)\Delta}(1-\chi)P_{\lesssim\mu_2^{-1}}$ has bounded operator on L^r by $O_{\mu_1}(\mu_2^{-c})$ for some constant $c > 0$. The above expression is of form $e^{it\Delta}(1-\chi)P_{\lesssim\mu_2^{-1}} = e^{i\tilde{t}\Delta}(1-\chi)\Psi(\mu_2 D)$ for $|t| \leqslant \mu_1^{-1}$ and Ψ is a cut off function which vanishes in $\{\xi : |\xi| \geqslant 2\}$. Recall that

$$e^{it\Delta}f(x) = \int \frac{1}{t^{d/2}}e^{\frac{-i|x-y|^2}{t}}f(y)dy \quad \text{(up to multiply by a constant)}.$$

Thus,

$$e^{it\Delta}(\chi f)(x) = \int \frac{1}{t^{d/2}}e^{\frac{-i|x-y|^2}{t}}\chi(y)f(y)dy,$$

which means that the Schwartz kernel of the operator $T_1 : f \mapsto e^{it\Delta}(\chi f)$ is

$$K_1(x,y) = \frac{1}{t^{d/2}}e^{\frac{-i|x-y|^2}{t}}\chi(y).$$

Similarly, using the inverse Fourier transform:

$$\Psi(\mu D)u = \int\int \Psi(\mu\xi)e^{i(x-y)\xi}u(y)dyd\xi$$

and thus the Schwartz kernel of the operator $T_2 : u \mapsto \Psi(\mu D)u$ is

$$K_2(x,y) = \int \Psi(\mu\xi)e^{i(x-y)\xi}d\xi = \frac{1}{\mu^d}\int \Psi(\xi)e^{\frac{i(x-y)\xi}{\mu}}d\xi = \frac{1}{\mu^d}\hat{\Psi}\left(\frac{y-x}{\mu}\right).$$

Thus, the Schwartz kernel of $f \mapsto e^{it\Delta}(\chi\Psi(\mu D)f)$ is

$$\begin{aligned}
K(x,z) &= \int K_1(x,y)K_2(y,z)dy = \int \chi(y)\frac{1}{t^{d/2}}e^{\frac{-i|x-y|^2}{t}}\frac{1}{\mu^d}\hat{\Psi}\left(\frac{z-y}{\mu}\right)dy \\
&= \int \chi(x-y)\frac{1}{t^{d/2}}e^{\frac{i|y|^2}{t}}\frac{1}{\mu^d}\hat{\Psi}\left(\frac{y+z-x}{\mu}\right)dy \\
&= \int \chi(x-\sqrt{t}y)e^{i|y|^2}\frac{1}{\mu^d}\hat{\Psi}\left(\frac{\sqrt{t}y+z-x}{\mu}\right)dy.
\end{aligned}$$

Let $\theta(y) = 1$ for $|y| \leqslant 1$, $\theta(y) = 0$ for $|y| \geqslant 2$.
Then write $K = K_i + K_e$, where

$$K_i = \int \theta(y)\chi(x-\sqrt{t}y)e^{i|y|^2}\frac{1}{\mu^d}\hat{\Psi}\left(\frac{\sqrt{t}y+z-x}{\mu}\right)dy,$$

$$K_e = \int (1-\theta(y))\chi(x-\sqrt{t}y)e^{i|y|^2}\frac{1}{\mu^d}\hat{\Psi}\left(\frac{\sqrt{t}y+z-x}{\mu}\right)dy.$$

We have

$$\begin{aligned}\int |K_i(x,z)|dz &\leqslant \int_{|y|\leqslant 2}\int_{\mathbb{R}^d}\left|\frac{1}{\mu^d}\hat{\Psi}\left(\frac{\sqrt{t}y+z-x}{\mu}\right)\right|dzdy\\ &\leqslant \int_{|y|\leqslant 2}\int_{\mathbb{R}^d}|\hat{\Psi}(z)|dzdy \leqslant C \quad (\text{independent of} x),\end{aligned}$$

Similarly, $\int |K_i(x,z)|dx \leqslant C$ (independent of z).

It remains to control K_e. We have

$$K_e(x,z) = \int \chi(x-\sqrt{t}y)(1-\theta(y))\left(\frac{-iy}{|y|^2}\frac{\partial}{\partial y}\right)(e^{i|y|^2})\frac{1}{\mu^d}\hat{\Psi}\left(\frac{\sqrt{t}y+z-x}{\mu}\right)dy$$

After integration by part, we get:

$$|K_e(x,z)| \lesssim C\left(\sqrt{t}+1+\frac{\sqrt{t}}{\mu}\right)^M \frac{1}{\mu^d}\int_{|y|\geqslant 1}\left|\Psi_d\left(\frac{\sqrt{t}y+z-x}{\mu}\right)\right|\frac{1}{|y|^M}dy,$$

where $\Psi_d \in \mathcal{S}(\mathbb{R}^d)$ (sum of derivatives of $\hat{\Psi}$). It is easy to check that

$$\sup_z \int |K_e(x,z)|dx + \sup_x \int |K_e(x,z)|dz \lesssim C\left(\sqrt{t}+1+\frac{\sqrt{t}}{\mu}\right)^M.$$

By Schur's test 3.19, we get the desired result. □

Lemma 6.3. *We have*

$$\left\|(1-\chi)P_{<100\mu_2^{-1}}(1_{D<\mu_3^{-1}}v_{med}(t_0))\right\|_H \lesssim \mu_2^{\eta_3}$$

Proof. Since the convolution kernel of $P_{<100\mu_2^{-1}}$ is a Schwartz function and the support of $1-\chi$ belongs to $\{|x| : D(x) > 2\mu_3^{-1}\}$, we have kernel of $(1-\chi)P_{<100\mu_2^{-1}}1_{D<\mu_3^{-1}}$ is bounded by

$$|(1-\chi)P_{<100\mu_2^{-1}}(1_{D<\mu_3^{-1}}(x,y)| \lesssim_l N^{d-l}|x-y|^{-l}1_{|x-y|>\mu_3^{-1}},$$

for any $l > 0$. Thus, from Young inequality and H^s-norm bounded of v_{med} we obtain the desired result. □

Lemma 6.4. *The operator $1_{D>\mu_3^{-3}}e^{i(t_0-t')\Delta}P_{\leqslant\mu_2^{-1}}1_{D\leqslant 2\mu_3^{-2}}$ has an $L^R \to L^2$ norm of $O(\mu_3^{100})$ for each $t' \in I$.*

Proof Lemma 6.4. Let $f \in L^R$. We have, for φ is a smooth cutoff function which equals 1 in $|\xi| \leqslant 1$ and equals 0 in $|\xi| \geqslant 2$:

$$\begin{aligned}A_0 :=& 1_{D>\mu_3^{-3}}e^{i(t_0-t')\Delta}P_{\leqslant\mu_2^{-1}}1_{D\leqslant 2\mu_3^{-2}}f(x)\\ &= 1_{D>\mu_3^{-3}}(x)\mathcal{F}^{-1}\left(e^{-i(t_0-t')|\xi|^2}\varphi(\xi/\mu_2^{-1})\mathcal{F}(1_{D\leqslant 2\mu_3^{-2}}f)\right)(x)\\ &= 1_{D>\mu_3^{-3}}(x)\mathcal{F}^{-1}\left(e^{-i(t_0-t')|\xi|^2}\varphi(\xi/\mu_2^{-1})\right) * (1_{D\leqslant 2\mu_3^{-2}}f)(x)\\ &= 1_{D>\mu_3^{-3}}(x)\left(1_{|\cdot|>\mu_3^{-3}/2}\mathcal{F}^{-1}(e^{-i(t_0-t')|\xi|^2}\varphi(\xi/\mu_2^{-1})) * (1_{D\leqslant 2\mu_3^{-2}}f)\right)(x).\end{aligned}$$

Let $\tilde{R}$ satisfies $\frac{1}{R}+\frac{1}{\tilde{R}}=1+1/2$.
Thus, by Young inequality and using $\|f\|_{H^s}=\left\|(1+|x|)^s\mathcal{F}^{-1}f(x)\right\|_{L^2}$, we have

$$\begin{aligned}
\|A_0\|_{L^2} &\lesssim \|f\|_{L^R}\left\|1_{|\cdot|>\mu_3^{-3}/2}\mathcal{F}^{-1}\left(e^{-i(t_0-t')|\xi|^2}\varphi(\xi/\mu_2^{-1})\right)\right\|_{L^{\tilde{R}}}\\
&\lesssim \|f\|_{L^R}\,\mu_2^{d(1/\tilde{R}-1)}\left\|1_{|\cdot|>\mu_3^{-3}\mu_2^{-1}/2}\mathcal{F}^{-1}\left(e^{-i(t_0-t')\mu_2^{-2}|\xi|^2}\varphi(\xi)\right)\right\|_{L^{\tilde{R}}}\\
&\lesssim \|f\|_{L^R}\,\mu_2^{d(1/\tilde{R}-1)}\left\|\mathcal{F}^{-1}\left(e^{-i(t_0-t')\mu_2^{-2}|\xi|^2}\varphi(\xi)\right)\right\|_{L^{\tilde{R}}(|x|\geqslant\mu_3^{-3}\mu_2^{-1}/2)}\\
&\lesssim \|f\|_{L^R}\,\mu_2^{d(1/\tilde{R}-1)}\left\|(1+|x|)^M(1+|x|)^{-M}\mathcal{F}^{-1}\left(e^{-i(t_0-t')\mu_2^{-2}|\xi|^2}\varphi(\xi)\right)(x)\right\|_{L^{\tilde{R}}(|x|\geqslant\mu_3^{-3}\mu_2^{-1}/2)}\\
&\lesssim \|f\|_{L^R}\,\mu_2^{d(1/\tilde{R}-1)}O((\mu_3^3\mu_2)^l)\left\|e^{-i(t_0-t')\mu_2^{-2}|\xi|^2}\varphi(\xi)\right\|_{H^M_\xi}\quad\text{for } M>d \text{ and any } l>0\\
&\lesssim_\varphi \|f\|_{L^R}\,O(\mu_3^l)(1+\mu_1^{-1}\mu_2^{-2})^M\\
&\lesssim \|f\|_{L^R}\,O(\mu_3^{100}),\quad (\text{for } l \text{ large enough})
\end{aligned}$$

This completes the proof. □

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