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# Can one integrate by adding up lines?

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*We present part of a project whose aim it is to investigate high school students' meanings for fundamental concepts of calculus, including rate of change and accumulation. We focus on students' reactions to a peer's statement that an integral or accumulated quantity can be found by adding up vertical lines under the graph of the rate of change function being integrated. We present results of 24 interviews with advanced level high school students after they studied integration. Our findings show that most of the interviewees consider the peer's thinking as correct in principle. On the other hand, more than half of the interviewees state that the peer's way of thinking is not close to their own. A tentative conclusion is that most of these advanced-level high school students proficiently carry out computations of integrals but fail to relate to the essence of a definite integral as a sum of products.*

*Keywords: Accumulation, adding up lines, integration, ways of thinking.*

## Introduction and background

Learning calculus is an important part of high school and university mathematics curricula worldwide (e.g., Bressoud et al., 2016). However, researchers have argued that even high ability students rarely acquire comprehension regarding the central concepts of calculus but, in the best case, formal techniques allowing them to answer exercises (e.g., Thompson & Harel, 2021).

According to Thompson and Silverman (2008) many processes in the real world are processes of change and when something changes, something accumulates; hence, the idea of accumulation is central to integration. In their recent review of research on integration, Jones and Ely (2022) have distinguished two approaches to accumulation, adding up pieces and accumulation from rate. A crucial aspect of the pieces that accumulate is that they usually products.

Israel is in the process of introducing a new high school curriculum, whose section on integration is based on the idea of accumulation (Dreyfus et al., 2021). In this connection, students' meanings for accumulation and for the bits that accumulate are of central interest. In the framework of a larger research project, we empirically investigate students' personal meanings for fundamental concepts of calculus including accumulation and the bits that accumulate. The study presented here contributes to this line of research. Specifically, we investigate whether high school students, who have studied integration as part of their advanced level mathematics studies, (i) consider as correct and (ii) identify with ways of finding a definite integral as a sum of vertical lines under the graph of the integrand.

## Methodology: questionnaire and interviews

In the framework of the larger research project, we built a questionnaire and an interview with the aim of examining how students relate to different ways of thinking about accumulation and integration. We named these different ways 'Antiderivative', 'Area', 'Lines', 'Sum of products',

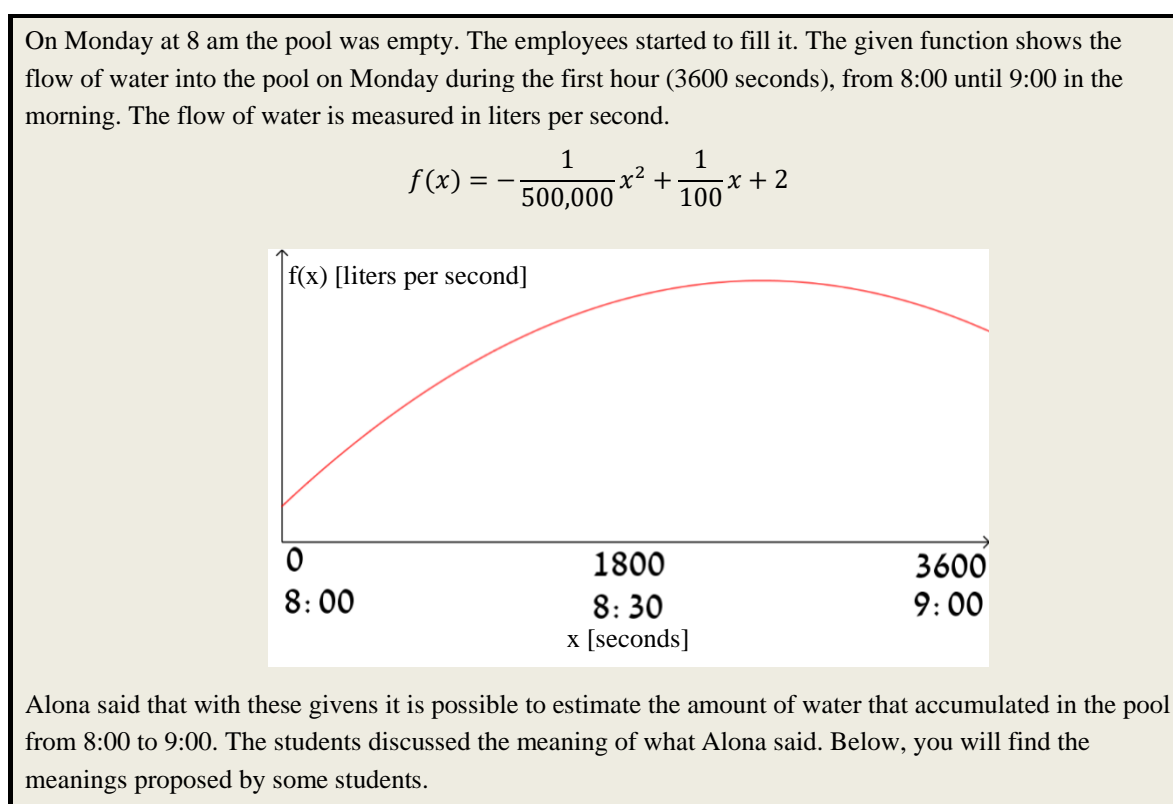
‘Rectangle’, and so on. In this paper, we focus on the way of thinking called ‘Lines’; ‘Lines’ is presented in the questionnaire as having been proposed by a hypothetical student called Lina.

We did not invent the different ways of thinking but distilled them from ways of thinking actual students expressed during a collection of close to 30 informal and progressively developing preliminary interviews on integration and accumulation. The idea for Lina was distilled from an interview with Nathan (Noah-Sella et al., 2022) who explained:

Nathan: My logic is that when  $\Delta x$  approaches zero, or is even equal to zero, the size of the – I don’t want to say rectangle, it’s a line - it has no width. It’s just a line, and since its width is zero, when we add up all of these lines, we will get the area under the curve.

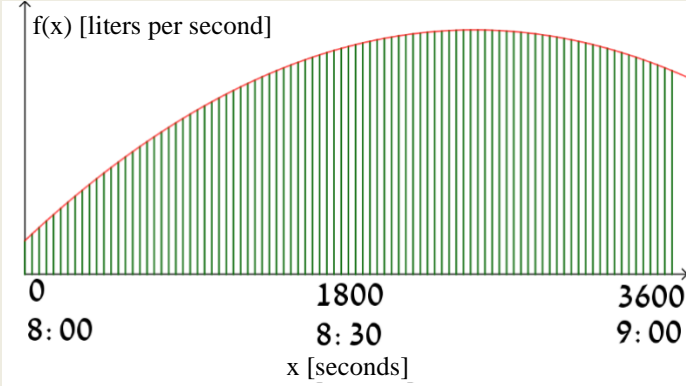
Nathan also mentioned adding up the lengths of all lines and adding up the y-coordinates of the infinite number of points on the graph of the given rate function. When constructing Lina, we chose not to specify this but to leave open what precisely is being added up. Similarly, Nathan collapsed the x-dimension (Oehrtman, 2009); Lina, however, exhibits only the result of the collapse. The intention behind these choices was to examine how participants considered the lines, for example whether they attributed a width to them, and how they interpreted adding them up.

The questionnaire including Lina’s and other ways of thinking presents a situation in eight versions: Two settings and four contexts in each setting (details below). One of these versions is presented in Figure 1.



**Figure 1: The situation - context pool (P) and setting quantity (Q)**

The situation was followed by several ways of thinking proposed by hypothetical students. One of these was Lina. Lina's statement and the questions about it are presented in Figure 2.

|      |   |                         |                     |                   |                     |                         |
|------|---|-------------------------|---------------------|-------------------|---------------------|-------------------------|
| Lina | <p>To me, if we draw vertical lines between the function <math>f(x)</math> and the <math>x</math>-axis at each and every point and add them up we'll get the amount of water that accumulated.</p>  |                         |                     |                   |                     |                         |
|      | To what extent do you think Lina's claim is correct?  | 1<br>Not at all correct | 2<br>Not so correct | 3<br>I don't know | 4<br>Pretty correct | 5<br>Completely correct |
|      | How close is Lina's claim to your way of thinking?  | 1<br>Not at all close   |                     | 2<br>Not so close | 3<br>Pretty close   | 4<br>Very close         |

**Figure 2: Lina - context pool (P) and setting quantity (Q)**

We note that in terms of Jones and Ely (2022), Lina makes no reference to accumulation from rate, but she does add up pieces. We were interested in the properties that research participants attributed to the lines, how they thought about adding them up, and what they thought they were adding up.

The two settings we used were I (for integral) and Q (for quantity); the situation in Figure 1 shows setting Q. In setting I, Alona's statement about estimating the quantity of water was replaced by Alona asking what the meaning of the integral  $\int_0^{3600} f(x)dx$  is.

The four contexts we used were water flowing into a Pool (P), the distance covered by a bicycle rider in Motion (M), Area under the graph (A) and a formal setting introducing a function  $g$  whose derivative is the function  $f$  given in the situation (F). Below we will often group the contexts P and M under the heading 'extra-mathematical context', and similarly group the contexts A and F under the heading 'mathematical context'.

We now turn to the interviews. In this paper, we consider 24 individual interviews, three on each of the eight versions of the questionnaire. All interviewees were volunteers learning advanced level mathematics and had studied integration of polynomial and trigonometric functions. They were asked to respond to the questionnaire and were interviewed immediately afterwards. Respondents were asked what additional subjects they took at the advanced level (physics, chemistry, English, literature, etc.). They were also asked whether they were familiar with the term accumulation and from where. The interview related to all ways of thinking presented in the questionnaire, but in this paper, we focus on the part pertaining to Lina's way of thinking.

The overall aim of the interview was to clarify the meanings interviewees hold about integration and accumulation by discussing with them their responses to the questions in Figure 2 and to corresponding questions about other ways of thinking. Interviewers were warned not to suggest ideas and to bring into the discussion as little of their own thinking as possible. Instead, they were asked to stimulate statements of opinion by the interviewees by asking about how the interviewee interprets the statements in the questionnaire, including Lina’s. Interviewers were familiar with the five criteria *Distinct language, Repetition, Reasoning, Unexpectedness, and Statement of opinion* (Noah-Sella et al., 2022) and instructed to question interviewees in depth, when one of these criteria applied.

Here we consider specifically how interviewees thought about Lina’s statement, that is about the possibility of finding an integral or accumulated quantity by summing up lines. Interviewers asked how Lina’s statement refers to what Alona said; they also asked about similarities and differences between how Lina thinks and how the interviewee thinks.

## Findings

In order to provide the reader with an overview, we begin by providing some numerical information on the questionnaire responses of the 24 interviewees, including to what extent their reaction to Lina depended on setting and context. We then use quotes from their interviews to answer the question how they interpret Lina, and why they consider her way of thinking correct or incorrect.

### Numerical overview of the results

Among the 24 interviewees, 12 also took physics at the advanced level, whereas 11 did not (one did not answer that question). Of the 24 interviewees, 14 stated that they were somewhat familiar with the term accumulation (eight from mathematics, one from mathematics and physics, and five from chemistry), nine that they were not, and one did not answer that question.

Table 1 shows how many interviewees answered positively or negatively to the question “To what extent do you think Lina’s claim is correct?” in the questionnaire (Figure 2). We remind the reader that interviewees were asked to answer on a scale of 1 to 5, with 3 meaning “I don’t know”. We group 1 and 2 together as ‘incorrect’; and we group 4 and 5 together as ‘correct’. We also group the mathematical contexts (F, A) together and the extra-mathematical contexts (M, P) together.

**Table 1: Number of interviewees judging Lina’s statement as correct or incorrect**

| Lina’s statement is | Total | Setting  |          | Context             |                           |
|---------------------|-------|----------|----------|---------------------|---------------------------|
|                     |       | Integral | Quantity | Mathematical (F, A) | Extra-mathematical (M, P) |
| Incorrect (1, 2)    | 10    | 5        | 5        | 6                   | 4                         |
| Don’t know (3)      | 2     | 2        | 0        | 0                   | 2                         |
| Correct (4, 5)      | 12    | 5        | 7        | 6                   | 6                         |

Out of the 24 interviewees, 22 took a definite position, 12 of them accepting Lina’s statement as correct. This could be because their ‘mathematical culture’ made them hesitate to declare that they don’t know, but it could also suggest that Lina’s way of thinking is quite common. On the other hand, the numbers in Table 1 suggest that neither the setting nor the context have an influence on the interviewees’ judgment whether Lina’s statement was correct or not.

Turning to the second questionnaire question, namely “How close is Lina’s statement to your own way of thinking?”, somewhat more than half of the participants answered ‘not close’, and somewhat less than half answered ‘close’. The corresponding results are presented in Table 2. Again, we group 1 and 2 together as ‘not close’ and 3 and 4 together as ‘close’.

**Table 2: Number of interviewees judging Lina’s statement as close to their own way of thinking**

| Lina’s statement is | Total | Setting  |          | Context             |                           |
|---------------------|-------|----------|----------|---------------------|---------------------------|
|                     |       | Integral | Quantity | Mathematical (F, A) | Extra-mathematical (M, P) |
| Not close (1, 2)    | 14    | 9        | 5        | 5                   | 9*                        |
| Close (3, 4)        | 10    | 3        | 7        | 7                   | 3                         |

\* Among the 9 interviewees in this cell, 6 answered the Integral setting and 3 the Quantity setting questionnaire.

The lack of influence of setting and context did not carry over to the question to what extent Lina’s statement was close to the interviewee’s own way of thinking. While the numbers are too small to allow for definite conclusions, in the more familiar Integral setting, more interviewees thought differently from Lina’s way than in the Quantity setting. Similarly, in the familiar Motion and Pool contexts, many interviewees felt that Lina did not represent how they would themselves think.

### **Interviewees’ interpretations of Lina’s way of thinking**

Out of ten interviewees marking Lina’s way of thinking as incorrect in the questionnaire, six gave explicit reasons for rejecting Lina’s way of thinking. We quote three of them (the other quotes were similar but less explicit); we mention the context of each interviewee’s questionnaire in parentheses.

- S1 (A): You can’t sum lines and define them as area, ‘cause in a given domain there can be lots of lines, in fact infinitely many.
- S14 (F): You can’t really do that because you don’t know the function values; and in mathematics, you shouldn’t use drawings and approximations but formulas.
- S16 (M): That’s wrong. You have to take rectangles; and even that gives you an approximation... You can’t take the values at each point... infinitely many points... you get infinity.

Five of the six interviewees, including S1 and S16, rejected Lina’s way of thinking because of reasons related to infinity, either that one cannot sum infinitely many terms or that the result would be infinite (presumably implying that the infinite result could not possibly represent the finite area or distance). Two of these five interviewees, including S1 also referred to area; both their questionnaires used the area context. S16 clearly stated that rectangles would have to be used rather than lines; he was almost alone in mentioning rectangles (see below). S14 interpreted Lina’s “summing lines” as summing function values (which some other interviewees did as well; see below) and was bothered by not knowing these values; moreover, Lina’s use of graphics and approximation was at variance with S14’s view that doing mathematics means using formulas.

Most of the interviewees who rejected Lina’s claim also stated that their own way of thinking differed from Lina’s. S1 is an exception to this; she explained that Lina’s statement is ‘pretty close’ to her own way of thinking because it helps her to relate to area in a certain way and split it into parts.

The other four interviewees who marked Lina's way of thinking as incorrect in the questionnaire, neither rejected nor accepted it in the interview. One of them only related to how complex Lina's way was. The three others were more explicit; we quote two of them (the third argued similarly):

- S7 (F): There will always be some area missing... so it will be really close but not exact.  
S13 (M): Depends how many lines she makes. Mmm... if there are infinitely many lines, I suppose. But you can't really get to infinity... it won't be completely correct.

The arguments of these three interviewees were close to those of another group of three interviewees, who also related to Lina's way of thinking as being an approximation but were among the 12 interviewees who considered Lina's way of thinking as 'pretty correct' or 'completely correct'.

- S17 (M): If you sum the length of the lines, then it depends on the number of lines how correct it is; if you assume infinitely many lines, then it's precise.  
S19 (P): That's correct in principle, but not exact... we can't indicate infinitely many lines... it's something like this that I did in my head... I interpret that in each second a certain quantity of water enters and the second symbolizes the point that we mark on the function... and to know the amount of water that accumulated, we sum, add all the values of these points.

All three of these interviewees indicated in the questionnaire that Lina's statement was close to their own way of thinking, although S17 changed his mind during the interview. Altogether, for seven interviewees, approximation was a salient aspect of Lina's way of thinking (the six discussed right now, and S16 above). Among these seven interviewees, only two referred to approximation as a process, S13 and S17; S17 even stated that this process is exact in the limit. S19, when discussing how close Lina's statement was to his own way of thinking, related to the question what exactly Lina adds up, namely the (y-)values of the points on the function (graph). We had intentionally left this point vague in the questionnaire; only five interviewees referred explicitly to what was being added up, among them S14 and S19 (quoted above), as well as

- S21 (P): At first, I didn't think of an integral, but I rather thought of substituting x's... That's a much more complicated method but... it is very correct... less efficient than an integral... If you draw a vertical line... that means if you substitute these x's... it's the same thing.

These five interviewees all added up function values. In addition to them, only three others seemed to ask themselves what was being added up. Two of them mentioned rectangles, S16 above and S22 who interpreted Lina's lines as rectangles and thought in terms of adding up their areas:

- S22 (F): She computes the area in the integral; she, like, simply makes such little rectangles but they are vertical.

The third one was the only one among the 24 interviewees who referred to units:

- S6 (M): The lines are between meter per second and time. So, each line is each meter that he passes per second, and if one adds them, that may give the distance... I don't know how to compute the sum, the size of the lines; maybe do an integral... The integral will give the area, but I don't think the area represents the distance; the lines represent the distance.

S6 seems to add speeds ("each meter that he passes per second") and get a distance as result. This justifies Lina's statement for S6; she characterizes it as completely correct but not at all close to her own way of thinking. Moreover, she associates the lines with the distance but not the area.

## Discussion

Although the setting and context of the questionnaires may have had some influence on interviewees' thinking, the number of participants in our study is too small to draw any conclusions about such influence. We therefore ignore setting and context in the discussion.

We remind the reader that the interviewers did not suggest ideas such as approximation or rectangles. Rather, only ideas suggested by the interviewees were discussed. For example, only two interviewees mentioned rectangles. Some interviewees related to Lina's way of thinking as an approximation whereas others did not; if an interviewee brought up approximation, the interviewer did not suggest increasing the number of lines; many interviewees considered a single static approximation, while only a few saw a process of approximation getting better as the number of lines increased.

Out of 24 interviewees, only six rejected Lina's statement consistently, six were undecided, and 12 accepted Lina's statement. On the other hand, only ten out of the 24 interviewees stated that Lina's statement was close to their own way of thinking. This raises the question on what basis the interviewees accepted Lina's way of thinking is, given that they stated it was not close to their own.

In considering this question, area is a keyword. Nine interviewees mentioned area explicitly, and several others like S13 and S17 seemed to refer to area implicitly. Most, if not all of them strongly connected to the area meaning of the integral; this is supported also by what they said about the 'Area' way of thinking in another part of their interviews. On the other hand, with three exceptions (S6, S16 and S22), they did not relate to the fact that lines are infinitely thin and that therefore even a countably infinite number of lines cannot add up to any area; put differently, they did not show awareness of the product nature of the bits that accumulate when integrating; differentials or  $\Delta x$ -intervals, whose importance for accumulation has been stressed by Jones and Ely (2022), were absent from their thinking. Lina's way of thinking can be seen as the result of a collapse without the process of collapse identified by Oehrtman (2009); in their attempt to live in a world with infinitesimal quantities, many interviewees readily accepted this novel aspect of collapse. They ignore that adding up lengths of lines or values of the integrand does not produce a quantity with the same units as the integral. Most of them related to integral as area but failed to relate to the essence of a definite integral as a sum of products. Many of them seem to view an integral as something you carry out rather than something that carries meaning. Since these are students from at least 8 different schools, who had studies integration recently, this result may be due to the way integration is taught in many schools in Israel.

On the other hand, many interviewees addressed technical aspects of Lina's way of thinking: They were worried about possible infinities. They saw Lina as providing an approximation to the value of the integral and related to the efficiency of this approximation. For most this approximation was static; only S13 and S17 mentioned a process of approximation, and only S17 referred to the limit.

The interviewees did not consider integration and accumulation as a process. Lina's way of thinking does not suggest accumulation from rate in the sense of Jones and Ely (2022), but it does refer to adding up pieces. The nature of these pieces is left vague by the majority of the interviewees. What is closer to their thinking is that adding up pieces provides an approximation. But even this approximation is static for almost all of them. The only process they see is computational: how to compute the approximation.



The participants in our research come from a population who learn mathematics at the advanced level. Students learning at the advanced level are being taught and often succeed to solve rather demanding computational exercises in calculus. In view of how they handled approximation and the relationship between lines and area, our findings suggest that despite their high achievements in mathematics, for most of them, the notion of integral remains linked to area but not to accumulation, not even in the sense of adding up pieces, and much less in the sense of accumulation from rate.

Three directions for further research emerge from our study: 1. More data are needed to draw conclusions about the influence of setting and context. 2. Our data were collected among Israeli high school students; further research is needed to examine whether a similar situation pertains in other countries and with college and university students. 3. We wonder whether it is possible to didactically exploit a combination of approximation and collapse to find an entry to an “adding up (multiplicative) pieces” view of the definite integral.

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