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Investigating students' worldviews of complex multiplication and derivatives

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While there has been extensive research on introductory and proof-based undergraduate courses, complex analysis has been relatively understudied. Most of the research on complex variables has focused on specific interventions to teach particular concepts in complex arithmetic and derivatives or mathematicians' reasoning about complex variables. The purpose of this case study was to investigate how students in a complex analysis course interpret multiplication and its relationship to the complex derivative and the amplitwist concept using Tall's three worlds of mathematics. Overall, participants were able to describe multiplication from a proceptual and embodied perspective, but struggled to conceptualize the complex derivative from an embodied perspective.

Keywords: Amplitwist, complex derivatives, complex multiplication, undergraduate education.

Introduction

Complex analysis, being neither an introductory undergraduate course, nor an advanced proofs-based course, has received relatively little attention in research. Despite calls to incorporate complex variables into linear algebra and calculus courses (D'Angelo, 2017), the study of complex variables remains isolated in a course with wildly varying prerequisites in the undergraduate curriculum (Howell et al., 2017).

Research on complex analysis has initially focused on the reasoning of mathematicians and non-experts, typically undergraduate and high school students. Hanke (2020) investigated how mathematicians viewed the complex path integral. There has also been work on experts' geometric interpretations and mathematical meanings of complex derivatives and integrals, which suggested that more work is needed on the links between multiplication and the amplitwist concept of the complex derivative (Oehrtman et al., 2019); amplitwist will be explained in more detail later in the paper.

There is also a body of research on how non-experts view complex analysis. Nemirovsky et al. (2012) studied the use of physical activities and embodied cognition to promote an understanding of complex arithmetic in prospective teachers. High school students, when questioned about their understanding of complex arithmetic, are not flexible in their choice of representation and often have difficulty with problems when a geometric interpretation is called for (Panaoura et al., 2006). Soto-Johnson (2014) showed some success at developing representational flexibility in undergraduates using labs constructed using dynamical geometric software (DGS). Troup et al. (2017) expanded this work on DGS labs with undergraduates and the amplitwist concept of the complex derivative. Beginning with the slope of a tangent line, participants developed an understanding of amplitwist for linear and non-linear functions through a series of DGS labs. Soto-Johnson and Hancock (2019) piloted a refined version of these activities with some further success in developing the amplitwist concept. Other studies have investigated undergraduates' understanding of complex integration with varying levels

of success (Hancock, 2018; Soto & Oehrtman, 2022). Of particular note in these studies is a phenomenon found in most studies of non-experts and their understandings of complex variables: that of ‘thinking real, doing complex’, where participants attempt to reason about the complex plane and the structure of the complex plane and complex calculus from their understanding of the real variable calculus concepts. Hancock (2018) used Tall’s (2013) three worlds of mathematics to parse three ways in which students think real and do complex analysis. Thinking real, doing complex has even been documented in the studies of mathematicians (Oehrtman et al., 2019). One study looked at the understanding of complex arithmetic pre-service teachers had in a flipped classroom environment; generally, students were more comfortable with algebraic representations of complex numbers and their arithmetic but struggled to interpret complex arithmetic geometrically (Setyawan & Rohmah, 2021). This study aimed to investigate student thinking about complex analysis under a flipped classroom model. Specifically, we were interested in students’ reasoning about complex numbers in the context of multiplication and the closely related complex derivative.

Before describing the methods, we briefly overview complex derivatives and the *amplitwist* concept. Writing two complex numbers in polar coordinates, $a = |a|e^{i\psi}$ and $z = |z|e^{i\theta}$, and multiplying them we get $a \cdot z = |a||z|e^{i(\psi+\theta)}$. Multiplying two non-zero complex numbers adds one’s argument to the other’s, $(\psi + \theta)$, and the multiplying number’s magnitude $|a|$ scales the other’, $|a||z|$. A complex linear function $w = a \cdot z$ is simply multiplication by a complex number generating a scaling and a rotation on the complex plane. A function f being (complex) differentiable at a point z means f is approximately complex linear near z : Using the limit definition of differentiation of f at z , one can observe that the infinitesimally small change in f , notated df , is equal to the multiplication of an infinitesimally small complex number dz by the derivative of f at the point z , $df = f'(z)dz$. If f is a complex analytic function at z (complex-differentiable near z) and its complex derivative does not vanish, the infinitesimally small change in f is given as a linear function $dz \rightarrow f'(z)dz$, (D’Angelo, 2017). Thus, the infinitesimally small change in f is a multiplication of the infinitesimally small complex number dz by the complex number $f'(z)$, producing a scaling (amplification) and a rotation (a twist) on dz by the magnitude $|f'(z)|$ and the argument of $f'(z)$, respectively. The combination of these two local behaviors of a complex analytic function f at z , amplifying and twisting, is called *amplitwist* (Needham 1997; Needham, 2021). Consider dz at z as a tiny vector in a disk with radius $|dz|$ and center z . The $f'(z)$ amplifies the tiny vector dz and then twists it, amplifying and twisting the tiny disk. That is, if f is complex analytic, f *amplitwists* every infinitesimally small disk in its domain, local geometric act encoded in $f'(z)$. Although this concept is not generally taught in undergraduate complex analysis courses, we hoped that introducing the *amplitwist* concept to students would help provide some geometric intuition to their understanding of the complex derivative.

The theoretical perspective for this study was Tall’s (2013) three worlds of mathematics; in this paradigm, learners have three interdependent views of mathematics. The first is the embodied world, which includes mental perceptions of real-world objects and visuospatial imagery such as Euclidean geometry; for complex numbers we considered the embodied world a geometric interpretation of operations on the complex plane, such as *amplitwist*. The second world is that of symbols used for manipulation, called the *proceptual world*. This world does not develop like the embodied world,

rather the proceptual world develops by expanding numbers and counting to different contexts, such as the complex plane. For this study we considered the proceptual world to begin with computational fluency of complex multiplication and derivatives, and that advanced proceptual knowledge would begin to see the operation as a transformation of the complex plane. The third world is based on properties and is known as the formal world; this world was of less relevance to our inquiry as the course we studied was not proof-based. Finally, Tall (2013) discusses the idea of a ‘met-before’, a previously known concept that can interfere in the development of new concepts; the ‘met-before’ most commonly documented in complex analysis is that of ‘thinking real, doing complex’, where students’ intuitions from the real plane can lead to difficulties in the complex plane (e.g., Hancock, 2018). This theoretical perspective was chosen as it allowed us to consider algebraic and geometric reasoning about complex analysis while acknowledging the well-documented issue of ‘thinking real, doing complex’ with novice learners of complex analysis. The research questions that guided this study were: (1) What are undergraduate students’ proceptual and embodied views about complex multiplication and complex derivatives? (2) To what extent do undergraduate students in a complex analysis class think real, and do complex?

Methods

This study took place at a rural research university in the southern United States in a flipped complex analysis course. The course met twice weekly for 75 minutes per session, and consisted of twelve students, one junior and eleven seniors. Seven students had mathematics as their first major, three as their second major, and two as their minor. The prerequisite for the course is multivariate calculus. During the course, we primarily used the following books: Beck et al. (2002-2018), Ponnusamy & Silverman (2006), and Spiegel et al. (2009). Additionally, the instructor created short videos (5-15 minutes in length) that explained key concepts using theorems, proofs, and examples from various sources.

Before each class meeting, students studied the material presented in the videos shared at least a week in advance. During most classes, students worked on instructor-provided problem sets that scaffolded the concept of the day through computational examples to generalized computations justifying the phenomenon. These class sessions were a mix of small group discussions and whole class presentations. Several classes were dedicated to embodied perspectives; the instructor removed the desks, created a complex plane on the floor, and had students stand at starting points on the plane. They then performed various operations and moved to their new point to illustrate the transformation of the complex plane (Barnes & Libertini, 2018). Other activities associated with the geometric meaning of complex multiplication and complex derivative were discussion-oriented, such as using Linear Algebra tools in complex multiplication and complex derivative; these activities were based on Needham (1997).

After inviting all students in the course to participate in the study, eight consented to be interviewed. Seven of the interview participants were male, seven were seniors, and seven were mathematics majors or second majors. Students were interviewed individually twice during the semester. The first interview took place around the time of the first exam, approximately five weeks into a 15-week semester, which covered complex arithmetic and derivatives; this interview lasted approximately

thirty minutes. The second interview functioned as a member check; students were presented with the transcripts of the first interview in the last week of the semester and asked if any of their understandings had changed throughout the semester. No participant had substantive changes, though two noted that some of their mathematics was incorrect in their initial interview. For this paper, we analyzed only the questions about complex multiplication and the complex derivative.

After transcription, each author coded the interview questions pertaining to multiplication and derivatives separately using the coding dictionary derived from Tall (2013) and Hancock (2018), given in Table 1. The operationalization definitions that appear in the table were finalized after the initial coding; these definitions are largely the same as the initial operationalization, with the exception of the last two codes. For formal and thinking real, doing complex, the portion of the operationalization dealing with complex multiplication was deleted as we had no codes related to complex multiplication for these codes. After the initial coding, the authors discussed their coding until an agreement was reached on each participant's answers. Axial coding was then employed to develop a categorization system for participants' responses. Trustworthiness was maintained throughout the coding process using member checks and an audit trail was maintained through the use of researcher journals.

Table 1: Coding dictionary

Code	Definition	Operationalization
Embodied	Participant takes an embodied perception of complex multiplication or the derivative	Participant is able to interpret concepts as physical manifestations on the complex plane
Proceptual	Participant takes a proceptual view of complex multiplication or the derivative	Participant sees complex operations as procedures or transformations of the complex plane
Formal	Participant cites theorems or definitions about complex numbers to draw conclusions	Participant interprets complex derivative question by citing the definition of analytic rather than performing a computation
Thinking real, doing complex	Erroneously extending real intuition onto the complex plane	Participant misapplies calculus concepts to complex derivatives

Findings

For multiplication, we asked participants how complex numbers were multiplied and how multiplication could be interpreted. All eight participants initially gave an action-oriented answer indicating an operational proceptual view. Edward provides a typical description:

Edward: So you multiply two complex numbers uh the way I like to think is about it is you're multiplying um ah, what is it called. You're basically FOILING out your numbers. So you're taking your real terms multiplying them by each other and you're

multiplying your imaginary terms by each other and then you're crisscrossing between the real and imaginary components.

Eugene was the lone participant who described multiplication without the rectangular form of complex numbers, though his answer was also action-based rather than a process:

Eugene: So multiplication of complex numbers is easiest if you have them in the polar form when is when you have the number is $re^{i\theta}$, where r is the length of the vector and theta is the rotation counterclockwise from the real axis. Then to multiply two complex numbers, you multiply the r 's together and add the thetas. I think that's from a guy called DeMoivre.

Seven of the participants could describe the physical action of multiplication as a stretch and rotation. As Paul explained:

Paul: So basically when you multiply one complex number by another, you are simply adjusting the direction and magnitude of the resultant.

Mike, the lone participant who struggled to interpret the meaning of multiplication of complex numbers in a physical context, could recall that there was a change in magnitude, but did not mention a rotation.

Mike: Um, well my understanding, which is about 30% confident is that it expands or shrinks the magnitude of the complex number. So you had a circle, and multiplied it, no say you had a point, and a line from the origin to the point, and then multiplied it by a complex number. The length of the line would expand by the magnitude of the complex number you multiplied by.

For derivatives, since participants had only been doing derivatives for two weeks, we asked that participants think about a particular complex function, specifically $f(z) = z^2$. Participants were asked to first find the derivative of this function at $z = i$ algebraically, and then interpret the meaning of their answer in a geometric sense. Six of the participants were able to correctly find the derivative without assistance, with four participants using the definition of the derivative and the other two noting that the function was analytic because it was a polynomial and used the power rule to evaluate the derivative. Five of the six students saw the derivative as something to compute, though the specific example likely influenced their view. Stuart noted that the answer was $2z$, then used the definition of the derivative:

Interviewer: So we're going to work with the function $f(z) = z^2$. Can you calculate the derivative for me at $z = i$?

Stuart: Like I can already tell you that the power rule works and this is $2z$ but the derivative represents something totally different. But I can show you this as well. So $(z+h)^2 - z^2$ over h and $z^2 + 2zh + h^2 - z^2$, these cancel, and then we cancel the h and then we get $2z$. Oh and then you asked at i , so the derivative is $2i$.

Both participants who struggled to find the derivative treated the derivative as a computational action but had trouble with algebraic operation. Jordan began by substituting $z = x + iy$ into the equation, squared the new equation, and then acknowledged that her work was wrong because she did not know how to proceed from there. After a hint, she noted that the function was analytic and was able to provide a derivative.

Interviewer: So we're going to work with the function $f(z) = z^2$. Can you calculate the derivative for me at $z = i$?

Jordan: $f(z) = z^2$ and the derivative at i . So we know we can take $z = x + iy$. <writes without talking, substituting $x + iy$ into the function>.

Interviewer: I think you forgot to cross out some terms there.

Jordan: Oh yeah, sorry. <Goes back to writing.> This is wrong. I'm wrong.

Euclid also treated the derivative as a computational action. He began with a definition of the derivative that had not been presented in class, then did not recognize that he needed to factor the difference of two squares. Given a hint, he was eventually able to factor the numerator and find the derivative.

Interviewer: So we're going to work with the function $f(z) = z^2$. Can you calculate the derivative for me at $z = i$?

Euclid: <Writes $\lim(z^2 - i^2)/(z - i)$ > I think this is wrong. Am I doing this correctly?

Interviewer: You're not wrong so far.

Euclid: <Continues writing.> Then I'm gonna multiply by the conjugate. And then I'm not sure.

Participants struggled when asked to interpret the meaning of the derivative from an embodied perspective. Six participants could not interpret the derivative as a physical action on the complex plane, either saying that they were unable to answer the question (2/8), or by providing an interpretation that was not mathematically correct (4/8), as Paul did in this case:

Paul: The geometric meaning of a derivative would be a curve, given a function which is continuous and smooth.

Eugene and Matt, the two highest achieving students that consented to be interviewed, were able to provide at least a partially correct interpretation of the complex derivative that related to amplitude. Eugene's interpretation was correct, while Matt made some arithmetic errors in his interpretation.

Matt: I um believe it is...it's the same kind of feel as when you multiply two complex numbers. You take the derivative at a specific point, and that results in a magnification and a rotation at that point. So like on the derivative example, the derivative at i is $2i$. So the magnitude is 2, so we multiply by the magnitude of the derivative and rotate by the argument of the derivative. So the original point was at -1 , and the derivative is $2i$, which is $2e^{\pi/2}$ and then -1 is $e^{3\pi/2}$ and you multiply those together and we get $2e^{3\pi/2}$.

Although we attempted to code for all three types of 'thinking real, doing complex' described in Hancock (2018), only one code was found, which was students erroneously extending intuition about real numbers to the complex plane. Only two participants had any instances of 'thinking real, doing complex'; Edward conflated the definition of the complex derivative of the fundamental theorem of calculus while Paul, the lone math minor interviewed, struggled to interpret the derivative, as we can see in his quote above.

Interviewer: So we're going to work with the function $f(z) = z^2$. Can you calculate the derivative for me at $z = i$?

Edward: The derivative at i . Ok, so we know with complex functions the definition of the derivative is very similar to the definition of the derivative given by um I guess I guess calculus. I forget is the fundamental theorem of calculus the limit definition of the derivative?

Discussion

Overall, the participants in this study displayed both an operational proceptual and a mathematically robust embodied worldview of complex multiplication. Some participants were even able to talk about rectangular and polar forms, such as Eugene. However, most participants' first response to multiplication was the action of using the algebraic algorithm in rectangular form. This strong

preference for rectangular complex numbers for students beginning a study of complex analysis is typical, and has been documented in other studies (e.g., Panaoura et al., 2006).

For the complex derivative, six participants showed a mathematically standard action-oriented proceptual worldview and could correctly find the derivative. Of those six, two also noted that the algebra was unnecessary because the function was analytic, as Stuart did in his interview; this is arguably evidence of the beginnings of a formal understanding of the complex derivative. Although Jordan and Euclid were not successful at finding a complex derivative without help, Euclid appeared to have algebra problems, while Jordan's difficulty stemmed from wanting to work exclusively with complex numbers in rectangular form; arguably both saw the derivative as an action to take but struggled with prerequisite knowledge. Lack of representational flexibility is a well-documented challenge for novice learners of complex analysis (Nemirovsky et al., 2012; Panaoura et al., 2006). Participants had a developing understanding of the embodied worldview of the complex derivative, with only Eugene having a complete understanding of amplitwist. However, amplitwist has been a difficult concept for students to articulate, especially in the case of non-linear functions (Troup et al., 2017).

Surprisingly, there were few instances where students engaged in 'thinking real, doing complex'. Only two participants had codes documenting this phenomenon, and two of the three codes came from Paul, the only math minor and lowest achieving student we interviewed. Since it has been documented that even experts struggle with thinking real, doing complex (Oehrtman et al., 2019), this relatively low frequency of thinking real, doing complex may be an artifact of the heavy emphasis on sense-making and embodied activities related to multiplication and the complex derivative in class; primarily based on Needham (1997) and Barnes & Libertini (2018). It is also possible that the low rate of 'thinking real, doing complex' is a function of the relatively small number of participants we interviewed. However, other research on complex analysis has also been small-scale interview studies (e.g., Setyawan & Rohmah, 2021; Troup et al., 2017), and two-thirds of the students enrolled in the course completed an interview for this study, which is a fairly high response rate. Either way, further exploration of complex analysis courses is warranted.

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