

Coupled system involving eddy coefficients and a right hand side in L^1 : review chapter

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1 Introduction

Let Ω be an open bounded subset of \mathbb{R}^N , this chapter focuses on the following problem. We search $u, k : \Omega \rightarrow \mathbb{R}$ satisfying

$$(1) \quad \begin{cases} -\operatorname{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -\operatorname{div}(B(k)\nabla k) = A(k)|\nabla u|^2 & \text{in } \Omega, \\ -A(k)\frac{\partial u}{\partial n} = \alpha u & \text{on } \partial\Omega, \\ k = 0 & \text{on } \partial\Omega. \end{cases}$$

In (1), $f \in L^2(\Omega)$ is a given source term, $\alpha > 0$ and there exists two constants $\nu, \mu > 0$ such that both viscosities A and B satisfy

$$(2) \quad \forall x \in \mathbb{R}, \quad \nu \leq A(x), B(x) \leq \mu.$$

The aim of this chapter is proving the existence of a couple (u, k) of distributional solutions of (1). This is not the only kind of solutions commonly searched. Indeed, some articles deal with renormalized solutions, which are defined and studied in [3]. This notion is interesting for the study of a system having a source term in $L^1(\Omega)$, as shown in [6].

The source term of the equation in k is in $L^1(\Omega)$. Without any better hypothesis, it remains possible to find distributional solutions of (1).

This figures in the works of L. Boccardo and T. Gallouët in [1], on which is based this chapter. The associated result, proved in section 4, is a consequence of the washer Lemma, stated and proved in section 2. This Lemma brings a bound for a solution in $W_0^{1,q}(\Omega)$, where $q < \frac{N}{N-1}$. Consequently, some particular function sets are needed.

Then, in section 4, we search weak solutions in $V = \bigcap_{1 < q < N'} W_0^{1,q}(\Omega)$, where N' refers to $\frac{N}{N-1}$; with test functions in $W = \bigcup_{q > N} W_0^{1,q}(\Omega)$. This method was also used by T. Gallouët and R. Herbin in [4].

Along this chapter, the viscosities A and B are bounded (assumption (2)). This does not really correspond to the physical situation, but is a right way to establish some interesting mathematical results, as a priori estimates, such as in section 3 and 4. The assumption of

bounded viscosities appears in [4], but also in the works of S. Clain and R. Touzani (see [2]), in which an existence result is proved in the two-dimensional case. In the present work, the existence results are proved when $N = 3$.

Of course it is possible to deal with unbounded viscosities, such as in [6], where they are only bounded from below. Otherwise, if they are not, other assumptions are required, concerning the decreasing. In addition, other function spaces need to be developed, like in [5].

In order to find the existence of a solution of (1), this work will be divided into two steps, that correspond to sections 3 and 4. Section 3 is devoted to the equation in u , which amounts to a linear problem. But when the problem is obviously non-linear, like the equation in k considered in section 4, another method is required. This consists in regularizing the system, then solving it by a linearization and a fixed-point theorem. This method is already used in [2].

Given any borelian subset or \mathbb{R}^N named A , the Lebesgue measure is simply written $|A|$.

2 Preliminary results

2.1 The washer lemma

This section aims at proving the following result (already proved by Boccardo-Gallouët in [1]).

Lemma 1. *Let $q \in [1; N'[$ where $N' = \frac{N}{N-1}$, and $u \in W_0^{1,q}(\Omega)$. We define*

$$(3) \quad B_n = \{x \in \Omega / n \leq |u(x)| < n + 1\},$$

and assume that there is a constant $C > 0$ such that

$$(4) \quad \forall n \in \mathbb{N}, \quad \int_{B_n} |\nabla u|^2 \leq C.$$

Then we have the inequality

$$(5) \quad \|u\|_{W^{1,q}(\Omega)} \leq A,$$

where A depends on Ω , q and C .

Remark 2. *Both constants C and A do not depend on u . Actually, the result may be rephrased as: let $(u_j)_{j \in J}$ a family of functions in $W_0^{1,q}(\Omega)$ with $q \in [1; N'[$ satisfying*

$$(6) \quad \forall j \in J, \quad \forall n \in \mathbb{N}, \quad \int_{B_n^j} |\nabla u_j(x)|^2 dx \leq C,$$

where B_n^j refers to $B_n^j = \{x \in \Omega / n \leq |u_j(x)| < n + 1\}$.

Then there exists a constant $A > 0$ only depending on Ω , q and C such that

$$(7) \quad \forall j \in J, \quad \|u_j\|_{W^{1,q}(\Omega)} \leq A.$$

The proof of lemma 1 is based on a decomposition given by the following lemma.

Lemma 3. *Let u satisfy the assumptions of lemma 1. Let q^* be the critical exponent in Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$, and any index $n_0 \in \mathbb{N}$. There exist two constants λ_1 and λ_2 depending on n_0 , q and Ω satisfying*

$$(8) \quad \|\nabla u\|_{L^q(\Omega)^N}^q \leq C^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{L^{q^*}(\Omega)}^{\frac{q^*(2-q)}{2}} \right).$$

Moreover, we have: $\lim_{n_0 \rightarrow +\infty} \lambda_2(n_0) = 0$.

Proof of lemma 3. At first, we decompose $\int_{\Omega} |\nabla u|^q$ as a sum

$$(9) \quad \int_{\Omega} |\nabla u|^q = \sum_{n=0}^{+\infty} \int_{B_n} |\nabla u|^q.$$

Recalling that B_n is bounded, Hölder inequality and (4) yield

$$(10) \quad \int_{B_n} |\nabla u|^q \leq |B_n|^{\frac{2-q}{2}} \left(\int_{B_n} |\nabla u|^2 \right)^{q/2} \leq C^{q/2} |B_n|^{\frac{2-q}{2}}.$$

We consider a fixed index $n_0 \in \mathbb{N}$.

If $n \leq n_0$, (10) directly yields

$$(11) \quad \sum_{n=0}^{n_0} \int_{B_n} |\nabla u|^q \leq C^{q/2} \sum_{n=0}^{n_0} |B_n|^{\frac{2-q}{2}} \leq (n_0 + 1) C^{q/2} |\Omega|^{\frac{2-q}{2}}.$$

If $n > n_0$, the definition of B_n yields

$$(12) \quad \int_{B_n} |u|^{q^*} \geq \int_{B_n} n^{q^*} = n^{q^*} |B_n|, \quad \text{hence } |B_n| \leq \frac{1}{n^{q^*}} \int_{B_n} |u|^{q^*}.$$

Inequalities (10) and (12) yield

$$(13) \quad \forall n > n_0, \quad \int_{B_n} |\nabla u|^q \leq C^{q/2} \left(\frac{1}{n^{q^*}} \right)^{\frac{2-q}{2}} \left(\int_{B_n} |u|^{q^*} \right)^{\frac{2-q}{2}},$$

which yields by addition

$$(14) \quad \sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u|^q \leq C^{q/2} \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left(\int_{B_n} |u|^{q^*} \right)^{\frac{2-q}{2}}.$$

The discrete Hölder inequality applied on the right hand side of (14) is written

$$(15) \quad \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left(\int_{B_n} |u|^{q^*} \right)^{\frac{2-q}{2}} \leq \left[\sum_{n=n_0+1}^{+\infty} \int_{B_n} |u|^{q^*} \right]^{\frac{2-q}{2}} \left[\sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}} \right]^{q/2}.$$

The inequality may be useful in the following if and only if the series $\sum_{n \geq n_0+1} n^{\frac{N(q-2)}{N-q}}$ converges, which is equivalent to $N(2-q) > N-q$. Some calculations show it is equivalent to $q < N'$.

Then we set $\lambda_2(n_0, q) = \left[\sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}} \right]^{q/2}$, so that we directly obtain $\lambda_2(n_0, q) \xrightarrow{n_0 \rightarrow +\infty} 0$. With this notation, (14) and (15) become

$$(16) \quad \sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u|^q \leq C^{q/2} \lambda_2(n_0, q) \left[\sum_{n=n_0+1}^{+\infty} \int_{B_n} |u|^{q^*} \right]^{\frac{2-q}{2}} \leq C^{q/2} \lambda_2(n_0, q) \left(\int_{\Omega} |u|^{q^*} \right)^{\frac{2-q}{2}} \\ = C^{q/2} \lambda_2(n_0, q) \|u\|_{L^{q^*}(\Omega)}^{\frac{q^*(2-q)}{2}}.$$

By setting $\lambda_1(n_0, q) = (n_0 + 1)|\Omega|^{\frac{2-q}{2}}$, (11) and (16) yield

$$(17) \quad \int_{\Omega} |\nabla u|^q \leq C^{q/2} \left(\lambda_1(n_0, q) + \lambda_2(n_0, q) \|u\|_{L^{q^*}(\Omega)}^{\frac{q^*(2-q)}{2}} \right).$$

That corresponds to (8), which concludes the proof. \square

This result can now be used to prove Lemma 1.

Proof of lemma 1. We start with the case $N = 2$, which is easier. In that case, we have $\frac{q^*(2-q)}{2} = \frac{Nq(2-q)}{2(N-q)} = q$. Then (8) becomes

$$(18) \quad \|\nabla u\|_{L^q(\Omega)^N}^q \leq C^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{L^{q^*}(\Omega)}^q \right).$$

Gagliardo-Nirenberg inequality and (18) yield

$$(19) \quad \|u\|_{L^{q^*}(\Omega)}^q \leq K \|\nabla u\|_{L^q(\Omega)^N}^q \leq KC^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{L^{q^*}(\Omega)}^q \right),$$

where $K > 0$ depends on Ω and q . We recall that if $b < 1$, the following equivalence stands

$$(20) \quad x \leq a + bx \Leftrightarrow x \leq \frac{a}{1-b}.$$

Since $\lambda_2(n_0, q) \xrightarrow{n_0 \rightarrow +\infty} 0$, it is possible to choose $n_0 \in \mathbb{N}$ large enough to have $KC^{q/2}\lambda_2 < 1$. Then by (20), we obtain

$$(21) \quad \|u\|_{L^{q^*}(\Omega)}^q \leq \frac{KC^{q/2}\lambda_1}{1 - KC^{q/2}\lambda_2}.$$

Thus, from (18):

$$(22) \quad \|\nabla u\|_{L^q(\Omega)^N}^q \leq C^{q/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right).$$

And Poincaré-Friedrichs inequality yields

$$(23) \quad \|u\|_{W^{1,q}(\Omega)}^q \leq LC^{q/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right),$$

where L depends on q and Ω . This yields the result with

$$(24) \quad A = L^{1/q}C^{1/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right)^{1/q}.$$

We now focus on the case $N > 2$. Since $q < N' < N$, the Sobolev embedding used before yields

$$(25) \quad \|u\|_{L^{q^*}(\Omega)}^q \leq K\|\nabla u\|_{L^q(\Omega)^N}^q \leq KC^{q/2} \left(\lambda_1 + \lambda_2\|u\|_{L^{q^*}(\Omega)}^{\frac{q^*(2-q)}{2}} \right).$$

Inequality (25) is of the form $x \leq a + bx^\gamma$, with

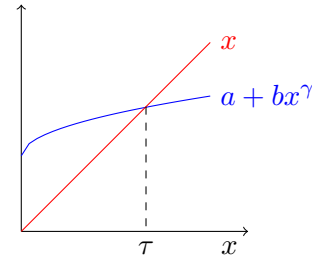
$$(26) \quad \gamma = \frac{q^*(2-q)}{2q} = \frac{N(2-q)}{2(N-q)}.$$

We have already shown that if $q \in [1; N'[,$

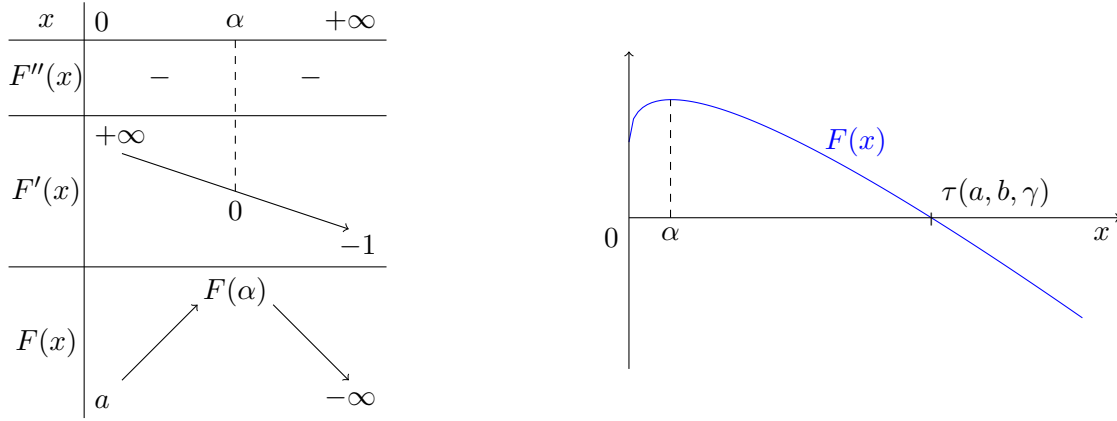
$$(27) \quad 1 < \frac{N(2-q)}{N-q} < N' \text{ hence } \frac{1}{2} < \gamma < \frac{N'}{2} < 1.$$

Thus, the inequation we need to solve is of the form: $x \leq a + bx^\gamma$, with $\gamma \in]0, 1[$ and $a, b > 0$.

The graph on the side shows the existence of $\tau \in]0; +\infty[$ depending on a, b and γ such that $x \leq a + bx^\gamma \Leftrightarrow x \leq \tau$.



Let us show it rigorously by studying the variations of $F : x > 0 \mapsto a + bx^\gamma - x$. This function is \mathcal{C}^∞ on $]0; +\infty[$ and $\begin{cases} F'(x) = b\gamma x^{\gamma-1} - 1 \\ F''(x) = b\gamma(\gamma-1)x^{\gamma-2} \end{cases}$, therefore $F''(x) < 0$. We obtain the following variations



Inequality (25) finally ensures the existence of $\tau = \tau(\lambda_1, \lambda_2, C) > 0$ such that

$$(28) \quad \|u\|_{L^{q^*}(\Omega)}^q \leq \tau.$$

Consequently, (8) and Poincaré-Friedrichs inequality yield

$$(29) \quad \|\nabla u\|_{L^q(\Omega)}^q \leq C^{q/2} (\lambda_1 + \lambda_2 \tau) \quad \text{hence} \quad \|u\|_{W^{1,q}(\Omega)}^q \leq LC^{q/2} (\lambda_1 + \lambda_2 \tau),$$

where L only depends on Ω and q . With $A = L^{1/q} C^{1/2} (\lambda_1 + \lambda_2 \tau)^{1/q}$, lemma 1 is proved in case $N > 2$. \square

The first hypothesis of lemma 1 is $q \in [1; N'[,$ where $N' = \frac{N}{N-1}$. It has been used many times, and we are able to show that is impossible to obtain similar results with higher values of q .

Indeed, considering the case $N > 2$, we recall that α is the only point where F' vanishes, which means

$$(30) \quad b\gamma\alpha^{\gamma-1} = 1 \quad \text{thus} \quad \alpha = (b\gamma)^{\frac{1}{1-\gamma}}.$$

Replacing b by the constants of lemma 3 yields $b(q) = M(q)\lambda_2(q)$, where $M(q)$ is bounded if $q \in [1; N'[,$. Consequently, $b(q) \xrightarrow{q \rightarrow N'} +\infty$ and then

$$(31) \quad \gamma(q) = \frac{N(2-q)}{2(N-q)} \xrightarrow{q \rightarrow N'} 2 \quad \text{hence} \quad \frac{1}{1-\gamma} \xrightarrow{q \rightarrow N'} \frac{1}{2}.$$

Therefore, (30) yields $\alpha \xrightarrow{q \rightarrow N'} +\infty$. And since $\tau \geq \alpha$, we obtain $\tau \xrightarrow{q \rightarrow N'} +\infty$, which means $A \xrightarrow{q \rightarrow N'} +\infty$.

There is a similar result in the case $N = 2$. The proof of lemma 1 requires to choose an index n_0 great enough to have $b < 1$. But as $\lambda_2(n_0, q) \xrightarrow{q \rightarrow N'} +\infty$, an increase of q implies an increase of n_0 .

Thus, as $\frac{2-q}{2} = 1 - \frac{q}{2} \xrightarrow{q \rightarrow N'} 1 - \frac{N'}{2} = 0$, it follows

$$(32) \quad \lambda_1(n_0, q) = (n_0 + 1)|\Omega|^{\frac{2-q}{2}} \xrightarrow[\substack{q \rightarrow N' \\ n_0 \rightarrow +\infty}]{} +\infty \text{ hence } A \xrightarrow{q \rightarrow N'} +\infty.$$

We have shown that q cannot be larger than the assumptions of lemma 1.

The result shown in this first section will be useful in the following. Generally, solving a PDE system is divided into two steps:

- simplifying the problem (for example by moving to a finite-dimension space, or by using a truncation function),
- coming back to the initial problem with the help of compacity results.

2.2 A general result on norms

Theorem 4 (Trace norm on H^1). *Let the application N_0 be defined for any $u \in H^1(\Omega)$ by*

$$(33) \quad N_0(u) = \left(\|\nabla u\|_{L^2(\Omega)^N}^2 + \|u\|_{L^2(\partial\Omega)}^2 \right)^{1/2},$$

where " $u|_{\partial\Omega}$ " refers to the trace of u on $\partial\Omega$.

Then N_0 is a norm, which is equivalent to the classical norm on $H^1(\Omega)$.

Proof. N_0 is non negative, homogeneous and satisfies triangular inequality (direct consequence of the fact that the norms L^2 also do). Let us consider $u \in H^1(\Omega)$ such that $N_0(u) = 0$. It comes $\|\nabla u\|_{L^2(\Omega)^N} = 0$ and $\|u\|_{L^2(\partial\Omega)} = 0$.

First, this yields that $\nabla u = 0$ almost everywhere on Ω . We write $u \stackrel{a.e.}{=} \lambda$, where λ is a real constant. We directly obtain that $\text{tr } u = \lambda$ a.e. on $\partial\Omega$, then $\lambda = 0$. We deduce that N_0 is a norm.

Equivalence between both norms :

At first, the trace operator $\text{tr} : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is surjective (see [7]), and Sobolev embeddings yield in particular $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$. Hence

$$(34) \quad \exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|_{L^2(\partial\Omega)}^2 \leq C \|u\|_{H^1(\Omega)}^2.$$

We deduce

$$(35) \quad \forall u \in H^1(\Omega), \quad N_0(u)^2 \leq (1 + C) \|u\|_{H^1(\Omega)}^2.$$

Therefore N_0 is dominated by $\|\cdot\|_{H^1(\Omega)}$.

Conversely, we intend to show

$$(36) \quad \exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|_{H^1(\Omega)}^2 \leq C N_0(u)^2.$$

By contradiction, let us suppose the negation of (36), which means in particular

$$(37) \quad \forall n \in \mathbb{N}, \exists u_n \in H^1(\Omega), \quad \|u_n\|_{H^1(\Omega)}^2 > n N_0(u_n)^2.$$

Inequality (37) yields that $\forall n \in \mathbb{N}^*$, $\|u_n\|_{L^2(\Omega)} > 0$. Then, both sides of (37) can be divided by $\|u_n\|_{L^2(\Omega)}$. Which means, keeping the same notation

$$(38) \quad \forall n \in \mathbb{N}, \exists u_n \in S, \quad \|u_n\|_{H^1(\Omega)}^2 > nN_0(u_n)^2,$$

where S refers to the " L^2 -sphere" defined by $S = \{v \in H^1(\Omega) / \|v\|_{L^2(\Omega)} = 1\}$. Then (38) can be rewritten as

$$(39) \quad \forall n \geq 2, \quad \frac{1}{n} + \frac{1}{n} \|\nabla u_n\|_{L^2(\Omega)^N}^2 > \|u_n\|_{L^2(\partial\Omega)}^2 + \|\nabla u_n\|_{L^2(\Omega)^N}^2.$$

Since $\|u_n\|_{L^2(\partial\Omega)}^2 \geq 0$, (39) yields in particular

$$(40) \quad \left(1 - \frac{1}{n}\right) \|\nabla u_n\|_{L^2(\Omega)^N}^2 \leq \frac{1}{n}, \quad \text{then } \|\nabla u_n\|_{L^2(\Omega)^N}^2 \leq \frac{1}{n-1}.$$

Therefore, the sequence $(\nabla u_n)_{n \in \mathbb{N}^*}$ is bounded in $L^2(\Omega)$; and as $(u_n)_{n \in \mathbb{N}^*}$ is a sequence of points of S , $(u_n)_{n \in \mathbb{N}^*}$ is also bounded in $L^2(\Omega)$. We deduce $(u_n)_{n \in \mathbb{N}^*}$ is bounded in $H^1(\Omega)$, and there exists a subsequence also denoted $(u_n)_{n \in \mathbb{N}^*}$ satisfying $u_n \xrightarrow{n \rightarrow +\infty} u$ in $H^1(\Omega)$.

However, (39) yields

$$(41) \quad \forall n \geq 2, \quad \frac{1}{n} > \|u_n\|_{L^2(\partial\Omega)}^2 + \left(1 - \frac{1}{n}\right) \|\nabla u_n\|_{L^2(\Omega)^N}^2 \geq \frac{1}{2}N_0(u_n)^2,$$

hence $N_0(u_n) \xrightarrow{n \rightarrow +\infty} 0$, then $u = 0$.

Rellich theorem yields that $u_n \xrightarrow{n \rightarrow +\infty} 0$ in $L^2(\Omega)$, meaning $\|u_n\|_{L^2(\Omega)} \xrightarrow{n \rightarrow +\infty} 0$. But for all $n \in \mathbb{N}^*$, $\|u_n\|_{L^2(\Omega)} = 1$, which is a contradiction. \square

3 Analysis of the first equation

In this section, given $p \in]1; +\infty[$ and $k \in W_0^{1,p}(\Omega)$, we consider the system

$$(42) \quad \begin{cases} -\operatorname{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -A(k)\frac{\partial u}{\partial n} = \alpha u & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^2(\Omega)$, $\alpha > 0$ and $A : \mathbb{R} \rightarrow \mathbb{R}_+$ satisfying

$$(43) \quad \forall k \in \mathbb{R}, \quad 0 < \nu \leq A(k) \leq \mu < +\infty.$$

3.1 Variational formulation and a priori estimates

We remind the Green formula, which is likely to be used all along this thesis.

Theorem 5 (Green formula). *Let $f \in H^1(\Omega)$ and $\mathbf{g} \in H^1(\Omega)^N$. Assume that \mathbf{n} is the normal vector of Ω , then the following formula stands*

$$(44) \quad \int_{\Omega} \operatorname{div}(\mathbf{g})f = - \int_{\Omega} \mathbf{g} \cdot \nabla f + \int_{\partial\Omega} (\mathbf{g} \cdot \mathbf{n})f \, dS.$$

Multiplying the first equation of (42) by $w \in H^1(\Omega)$ and then integrating yields

$$(45) \quad - \int_{\Omega} \operatorname{div}(A(k)\nabla u)w = \int_{\Omega} fw.$$

Then (44) and the second equation of (42) yield

$$(46) \quad - \int_{\Omega} \operatorname{div}(A(k)\nabla u)w = \int_{\Omega} A(k)\nabla u \cdot \nabla w + \alpha \int_{\partial\Omega} uw \, dS.$$

We obtain the variational formulation of (42)

$$(47) \quad \begin{cases} u \in H^1(\Omega) \\ \forall w \in H^1(\Omega), \quad \int_{\Omega} A(k)\nabla u \cdot \nabla w + \alpha \int_{\partial\Omega} uw \, dS = \int_{\Omega} fw. \end{cases}$$

All the terms of (47) are well defined. Moreover, the trace theorem (see [7]) yields that $u, w \in H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

The main result of this section is the following.

Theorem 6. *There exists a unique function $u \in H^1(\Omega)$ satisfying (47).*

Proof. This proof uses the Lax-Milgram theorem on the Hilbert space $H^1(\Omega)$. Let a_k be the bilinear form given by

$$(48) \quad \forall u, w \in H^1(\Omega), \quad a_k(u, w) = \int_{\Omega} A(k)\nabla u \cdot \nabla w + \alpha \int_{\partial\Omega} uw \, dS.$$

Let $u, w \in H^1(\Omega)$. First we have

$$(49) \quad \begin{aligned} |a_k(u, w)| &\leq \int_{\Omega} A(k)|\nabla u \cdot \nabla w| + \alpha \int_{\partial\Omega} |uw| \, dS \\ &\leq \mu \|\nabla u\|_{L^2(\Omega)^N} \|\nabla w\|_{L^2(\Omega)^N} + \alpha \|u\|_{L^2(\partial\Omega)} \|w\|_{L^2(\partial\Omega)} \\ &\leq (\mu + \alpha M) \|u\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}, \end{aligned}$$

where M does not depend on u . Therefore, a_k is continuous on $H^1(\Omega) \times H^1(\Omega)$.

Then we show the coercivity. Let $w \in H^1(\Omega)$, we have

$$(50) \quad \begin{aligned} a_k(w, w) &= \int_{\Omega} A(k)|\nabla w|^2 + \alpha \int_{\partial\Omega} |w|^2 \, dS \geq \nu \|\nabla w\|_{L^2(\Omega)^N}^2 + \alpha \|w\|_{L^2(\partial\Omega)}^2 \\ &\geq \inf(\nu, \alpha) N_0(w)^2, \end{aligned}$$

hence Theorem 4 yields the coercivity of a_k .

Finally, it is clear that the linear form $[w \mapsto \int_{\Omega} fw]$ is continuous. Problem (47) then admits a unique solution $u \in H^1(\Omega)$, by Lax-Milgram theorem. \square

Now the existence and uniqueness of u is proved, it is relevant to state some a priori estimates on it.

Proposition 7. *Let $k \in W_0^{1,p}(\Omega)$ with $p \in [1; +\infty[$. The solution $u = u(k)$ of (47) satisfies the inequalities*

$$(51) \quad \|u\|_{H^1(\Omega)} \leq C_1(\Omega, \alpha, \nu, f),$$

and

$$(52) \quad \|A(k)^{1/2} \nabla u(k)\|_{L^2(\Omega)^N} \leq C_2(\Omega, \alpha, \mu, \nu, f).$$

Proof. Taking $w = u$ as a test in (47) leads to

$$(53) \quad \int_{\Omega} A(k) |\nabla u|^2 + \alpha \int_{\partial\Omega} |u|^2 dS = \int_{\Omega} f u.$$

Cauchy-Schwarz inequality then yields

$$(54) \quad \nu \|\nabla u\|_{L^2(\Omega)^N}^2 + \alpha \|u\|_{L^2(\partial\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)},$$

hence

$$(55) \quad \min(\nu, \alpha) N_0(u)^2 \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}.$$

Theorem 4 directly yields

$$(56) \quad \frac{\min(\nu, \alpha)}{C} \|u\|_{H^1(\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)},$$

hence (51).

To obtain (52), the higher bound on A and Poincaré inequality yield

$$(57) \quad \|A(k)^{1/2} \nabla u\|_{L^2(\Omega)^N} \leq \sqrt{\mu} \|\nabla u\|_{L^2(\Omega)^N} \leq c \sqrt{\mu} \|u\|_{H^1(\Omega)}.$$

Finally, (51) yields (52). □

3.2 An energy method

This subsection is devoted to show the H^1 convergence of solutions of (42). More precisely, if $(k_n)_{n \in \mathbb{N}}$ is a sequence converging to a certain k in a sense precised after, a subsequence of $(u(k_n))_{n \in \mathbb{N}}$ converges in $H^1(\Omega)$ to $u(k)$. An intermediary result is first needed.

Theorem 8 (L^2 convergence). *Let $p \in [1; +\infty[$ and $(k_n)_{n \in \mathbb{N}}$ a sequence of functions converging weakly to k in $W_0^{1,p}(\Omega)$. Let $u_n = u(k_n)$ and $u = u(k)$ the associated solutions of problem (42).*

There exists a subsequence still denoted $(u_n)_{n \in \mathbb{N}}$ satisfying

$$(58) \quad A(k_n)^{1/2} \nabla u_n \xrightarrow{n \rightarrow +\infty} A(k)^{1/2} \nabla u \text{ in } L^2(\Omega).$$

Proof. The weak convergence of $(k_n)_{n \in \mathbb{N}}$ and compact Sobolev embeddings yield that $k_n \xrightarrow[n \rightarrow +\infty]{} k$ in $L^1(\Omega)$. By Lebesgue inverse theorem, a subsequence named the same way converges a.e. in Ω to k . The continuity of A yields

$$(59) \quad A(k_n) \xrightarrow[n \rightarrow +\infty]{} A(k) \quad \text{a.e. in } \Omega.$$

We recall that u_n and u satisfy

$$(60) \quad \forall v \in H^1(\Omega), \forall n \in \mathbb{N}, \quad \int_{\Omega} A(k_n) \nabla u_n \cdot \nabla v + \alpha \int_{\partial\Omega} u_n v \, dS = \int_{\Omega} f v.$$

$$(61) \quad \forall v \in H^1(\Omega), \quad \int_{\Omega} A(k) \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} u v \, dS = \int_{\Omega} f v.$$

Inequality (51) yields that $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega)$, which ensures the existence of a same-named subsequence converging weakly to a certain $u^* \in H^1(\Omega)$. The continuity of the trace operator yields in particular

$$(62) \quad \forall v \in H^1(\Omega), \quad \int_{\partial\Omega} u_n v \, dS \xrightarrow[n \rightarrow +\infty]{} \int_{\partial\Omega} u^* v \, dS.$$

Then, in order to show the convergence of the first term in (60), an integration lemma is needed.

Lemma 9 (Convergence of integrals). *Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be two sequences of $L^2(\Omega)$ functions, and $f, g \in L^2(\Omega)$ such that*

$$(63) \quad f_n \xrightarrow[n \rightarrow +\infty]{} f \quad \text{and} \quad g_n \xrightarrow[n \rightarrow +\infty]{} g \quad \text{in } L^2(\Omega).$$

Then we have

$$(64) \quad \int_{\Omega} f_n g_n \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} f g.$$

This result will be proved after finishing the proof of Theorem 8.

The assumptions of Lemma 9 have to be checked.

- Since the sequence $(A(k_n))_{n \in \mathbb{N}}$ converges almost everywhere to $A(k)$, the domination $A(k_n)|\nabla v| \leq \mu|\nabla v|$ and the Lebesgue theorem yield $A(k_n)\nabla v \xrightarrow[n \rightarrow +\infty]{} A(k)\nabla v$ in $L^2(\Omega)$.
- As a consequence of Rellich theorem, $\nabla u_n \xrightarrow[n \rightarrow +\infty]{} \nabla u^*$ in $L^2(\Omega)$.

Therefore, Lemma 9 yields

$$(65) \quad \forall v \in H^1(\Omega), \quad \int_{\Omega} A(k_n) \nabla u_n \cdot \nabla v \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} A(k) \nabla u^* \cdot \nabla v.$$

Both convergences (62) and (65) ensure that the limit u^* satisfies (61). And since the solution of (61) is unique (by Theorem 6), $u^* = u = u(k)$.

Hence, $\nabla u_n \xrightarrow{n \rightarrow +\infty} \nabla u$ in $L^2(\Omega)$, and (52) ensures that there exists a subsequence of $(A(k_n)^{1/2} \nabla u_n)_{n \in \mathbb{N}}$ converging weakly in L^2 . Therefore, Lemma 9 yields

$$(66) \quad A(k_n)^{1/2} \nabla u_n \xrightarrow{n \rightarrow +\infty} A(k)^{1/2} \nabla u \text{ in } L^2(\Omega).$$

In order to obtain the strong convergence of this sequence, the convergence of the norms remains to be proved. Let us write (60) with $v = u_n$ and (61) with $v = u$.

$$(67) \quad \forall n \in \mathbb{N}, \quad \int_{\Omega} A(k_n) |\nabla u_n|^2 + \alpha \int_{\partial\Omega} |u_n|^2 dS = \int_{\Omega} f u_n.$$

$$(68) \quad \int_{\Omega} A(k) |\nabla u|^2 + \alpha \int_{\partial\Omega} |u|^2 dS = \int_{\Omega} f u.$$

The strong convergence of $(u_n)_{n \in \mathbb{N}}$ in L^2 directly yields

$$(69) \quad \int_{\Omega} f u_n \xrightarrow{n \rightarrow +\infty} \int_{\Omega} f u.$$

The compact Sobolev embedding $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ yields in particular

$$(70) \quad \int_{\partial\Omega} |u_n|^2 dS \xrightarrow{n \rightarrow +\infty} \int_{\partial\Omega} |u|^2 dS.$$

Convergences (69) and (70) yield

$$(71) \quad \int_{\Omega} A(k_n) |\nabla u_n|^2 \xrightarrow{n \rightarrow +\infty} \int_{\Omega} A(k) |\nabla u|^2.$$

The result of Theorem 8 is proved. □

It remains to prove Lemma 9, so that the proof is complete.

Proof of Lemma 9. The difference between the two integrals is written

$$(72) \quad \int_{\Omega} (f_n g_n - f g) = \int_{\Omega} f (g_n - g) + \int_{\Omega} (f_n - f) g_n.$$

The weak convergence of $(g_n)_{n \in \mathbb{N}}$ yields the convergence of the first integral of (72), and the fact that $(g_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$ yields

$$(73) \quad \left| \int_{\Omega} (f_n - f) g_n \right| \leq \|f_n - f\|_{L^2(\Omega)} \|g_n\|_{L^2(\Omega)} \leq M \|f_n - f\|_{L^2(\Omega)} \xrightarrow{n \rightarrow +\infty} 0.$$

□

Theorem 10 (H^1 convergence). *With the same assumptions as in Theorem 8, there is a subsequence of $(u_n)_{n \in \mathbb{N}}$ still named the same such that*

$$(74) \quad u_n \xrightarrow{n \rightarrow +\infty} u \text{ in } H^1(\Omega).$$

Proof of Theorem 10. The weak convergence and Rellich Theorem ensure that $u_n \xrightarrow[n \rightarrow +\infty]{} u$ in $L^2(\Omega)$. It remains to prove that $\nabla u_n \xrightarrow[n \rightarrow +\infty]{} \nabla u$ in $L^2(\Omega)$.

$$(75) \quad \begin{aligned} \sqrt{\nu} \|\nabla u_n - \nabla u\|_{L^2(\Omega)^N} &\leq \left\| A(k_n)^{1/2} \nabla u_n - A(k_n)^{1/2} \nabla u \right\|_{L^2(\Omega)^N} \\ &\leq \left\| A(k_n)^{1/2} \nabla u_n - A(k)^{1/2} \nabla u \right\|_{L^2(\Omega)^N} + \left\| \left(A(k_n)^{1/2} - A(k)^{1/2} \right) \nabla u \right\|_{L^2(\Omega)^N} \xrightarrow[n \rightarrow +\infty]{} 0, \end{aligned}$$

by Theorem 8 and Lebesgue Theorem. \square

4 Analysis of the second equation

The system considered in this section is the following

$$(76) \quad \begin{cases} -\operatorname{div}(B(k)\nabla k) = A(k)|\nabla u(k)|^2 & \text{in } \Omega, \\ k = 0 & \text{on } \partial\Omega. \end{cases}$$

The notations are lightened by defining

$$(77) \quad D(k) = A(k)|\nabla u(k)|^2.$$

4.1 Variational formulation and description of the method

In order to write the variational formulation of (76), we start by defining two function sets, and stating a regularity result.

Definition 11. Let the sets V and W be defined by

$$(78) \quad V = \bigcap_{1 \leq r < N'} W_0^{1,r}(\Omega) \quad \text{and} \quad W = \bigcup_{N < r \leq +\infty} W_0^{1,r}(\Omega).$$

Note that V is a vectorial space, but not W .

We aim at writing a variational formulation searching solutions in V with test functions in W .

Proposition 12. *Let $k \in V$ and $\psi \in W$. Then $B(k)\nabla k \cdot \nabla \psi \in L^1(\Omega)$ and $D(k)\psi \in L^1(\Omega)$.*

Proof. Let $k \in V$ and $\psi \in W$. By applying Hölder inequality with $r > N$ such that $\psi \in W_0^{1,r}(\Omega)$, we obtain

$$(79) \quad \int_{\Omega} B(k)|\nabla k \cdot \nabla \psi| \leq \mu \int_{\Omega} |\nabla k \cdot \nabla \psi| \leq \mu \|\nabla k\|_{L^{r'}(\Omega)} \|\nabla \psi\|_{L^r(\Omega)}.$$

Moreover, (52) yields $\|D(k)\|_{L^1(\Omega)} \leq C$, then $D(k) \in L^1(\Omega)$ and $\psi \in L^\infty(\Omega)$. Hence $D(k)\psi \in L^1(\Omega)$. \square

Remark 13. *The compact Sobolev embedding*

$$(80) \quad V \hookrightarrow L^2(\Omega)$$

stands if $N \in \{2; 3\}$.

Definition 14 (Variational formulation). A function k is a weak solution of (76) if

$$(81) \quad \begin{cases} k \in V \\ \forall \psi \in W, \int_{\Omega} B(k) \nabla k \cdot \nabla \psi = \int_{\Omega} D(k) \psi. \end{cases}$$

As proved before, $D(k) \in L^1(\Omega)$, but it seems difficult to obtain a better regularity. In addition, (81) is not a linear problem, and could not be solved using only Lax-Milgram theorem, even if $D(k)$ was in $L^2(\Omega)$. These problems lead to use another method, which is divided into three steps:

- Regularizing the equation, by changing the source term.
- Linearizing it to solve the regularized problem with a fixed-point theorem.
- Taking the limit when the regularizing parameter goes to zero.

4.2 Two first steps: solving the regularized problem

Before linearizing Problem (81), the source term needs to be approached by a sequence of regular functions.

Definition 15. If $w \in L^r(\Omega)$ with $r \in [1, +\infty]$, $\tilde{w} \in L^r(\mathbb{R}^N)$ refers to the extension by zero

$$\tilde{w} : x \mapsto \begin{cases} w(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Definition 16 (Mollifier). Let $\rho : \mathbb{R}^N \rightarrow \mathbb{R}$ be a C^∞ function satisfying

- $\text{Supp}(\rho) = \bar{B}(0, 1)$,
- $\int_{\mathbb{R}^N} \rho(y) dy = 1$.
- $\rho \geq 0$,

Let the family $(\rho_\varepsilon)_{\varepsilon>0}$ be defined by

$$(82) \quad \forall x \in \mathbb{R}^N, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right).$$

Then $(\rho_\varepsilon)_{\varepsilon>0}$ is called a mollifier.

Remark 17. *From now on, the term "mollifier" always refers to Definition 16.*

Definition 18 (Regularized problem). Let the approximated source term $D_\varepsilon(k)$ be defined for any $\varepsilon > 0$ by

$$(83) \quad D_\varepsilon(k) = \left| \widetilde{U(k)} * \rho_\varepsilon \right|^2,$$

where $U(k) = A(k)^{1/2} \nabla u(k)$.

A function k is called a weak solution of the regularized problem if it satisfies

$$(84) \quad \begin{cases} k \in H_0^1(\Omega) \\ \forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} B(k) \nabla k \cdot \nabla \psi = \int_{\Omega} D_\varepsilon(k) \psi. \end{cases}$$

Linearizing Problem (84) is useful if and only if Lax-Milgram theorem can be applied in the Hilbert space $H_0^1(\Omega)$. To do so, the linear form $[\psi \mapsto \int_{\Omega} D_\varepsilon(k) \psi]$ must be continuous on $H_0^1(\Omega)$.

Theorem 19. *Let $\varepsilon > 0$. The function $D_\varepsilon(k)$ defined just above is in $L^2(\Omega)$ and there exists a constant $C > 0$ independent of k and ε such that*

$$(85) \quad \|D_\varepsilon(k)\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^2}.$$

The proof of Theorem 19 is based on a general result.

Lemma 20. *Let $v \in H^m(\mathbb{R}^N)$, $\ell \in \mathbb{N}$ and $(\rho_\varepsilon)_{\varepsilon>0}$ a mollifier. For $\varepsilon > 0$ sufficiently close to 0, we have*

$$(86) \quad \|v * \rho_\varepsilon\|_{H^{m+\ell}(\mathbb{R}^N)} \leq \frac{C(m, \ell)}{\varepsilon^\ell} \|v\|_{H^m(\mathbb{R}^N)}.$$

Proof of Lemma 20. The $H^{m+\ell}$ norm can be defined by

$$(87) \quad \|v * \rho_\varepsilon\|_{H^{m+\ell}(\mathbb{R}^N)}^2 = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq \ell}} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2.$$

Let $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| \leq m$ and $|\beta| \leq \ell$. Plancherel theorem yields

$$(88) \quad \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2 = \frac{1}{(2\pi)^N} \|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{L^2(\mathbb{R}^N)}^2,$$

where \mathcal{F} refers to the Fourier transform operator.

Since $v \in H^m(\mathbb{R}^N)$, the derivation formula $\partial^{\alpha+\beta}(v * \rho_\varepsilon) = (\partial^\alpha v) * (\partial^\beta \rho_\varepsilon)$ and the Fourier transform yield

$$(89) \quad \mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon)) = \mathcal{F}(\partial^\alpha v) \mathcal{F}(\partial^\beta \rho_\varepsilon).$$

Then with norms

$$(90) \quad \|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\mathcal{F}(\partial^\alpha v)(\xi)|^2 \cdot |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 d\xi.$$

A linear change of variables yields

$$(91) \quad \mathcal{F}(\rho_\varepsilon)(\xi) = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} e^{-ix\xi} \rho\left(\frac{x}{\varepsilon}\right) dx = \int_{\mathbb{R}^N} e^{-i\varepsilon y\xi} \rho(y) dy = \mathcal{F}(\rho)(\varepsilon\xi).$$

Then we obtain

$$(92) \quad \begin{aligned} |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 &= |\xi^\beta \mathcal{F}(\rho_\varepsilon)(\xi)|^2 = |\xi^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 \\ &= \frac{1}{\varepsilon^{2|\beta|}} |(\varepsilon\xi)^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 = \frac{1}{\varepsilon^{2|\beta|}} |\mathcal{F}(\partial^\beta \rho)(\varepsilon\xi)|^2. \end{aligned}$$

Yet, $\partial^\beta \rho \in \mathcal{D}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, which yields $\forall \xi \in \mathbb{R}^N$, $|\mathcal{F}(\partial^\beta \rho)(\xi)| \leq \|\partial^\beta \rho\|_{L^1(\mathbb{R}^N)}$. Then (90) ensures the existence of a constant $c_0 > 0$ independent of v such that

$$(93) \quad \|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\mathcal{F}(\partial^\alpha v)\|_{L^2(\mathbb{R}^N)}^2.$$

Plancherel Theorem then yields

$$(94) \quad \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\partial^\alpha v\|_{L^2(\mathbb{R}^N)}^2.$$

Then, by addition

$$(95) \quad \sum_{|\alpha| \leq m} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|v\|_{H^m(\mathbb{R}^N)}^2.$$

Taking $\varepsilon \in]0, 1[$ yields $\frac{1}{\varepsilon^{2|\beta|}} \leq \frac{1}{\varepsilon^{2\ell}}$. Thus, adding for $|\beta| \leq \ell$ yields

$$(96) \quad \|v * \rho_\varepsilon\|_{H^{m+\ell}(\mathbb{R}^N)}^2 \leq \frac{c(\ell)}{\varepsilon^{2\ell}} \|v\|_{H^m(\mathbb{R}^N)}^2,$$

hence (86). □

We can now apply this result to prove Theorem 19, recalling that

$$(97) \quad \|U(k)\|_{L^2(\Omega)^N} \leq C,$$

where $C > 0$ depends on f , Ω , α , μ and ν . This is a rewriting of (52).

Proof of Theorem 19. The definition of D_ε yields

$$(98) \quad \|D_\varepsilon(k)\|_{L^2(\Omega)} = \left(\int_{\Omega} |\widetilde{U(k)} * \rho_\varepsilon|^4 \right)^{1/2} = \left\| \widetilde{U(k)} * \rho_\varepsilon \right\|_{L^4(\Omega)^N}^2.$$

Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and Lemma 20 then yield

$$(99) \quad \left\| \widetilde{U(k)} * \rho_\varepsilon \right\|_{L^4(\Omega)^N} \leq c_1 \left\| \widetilde{U(k)} * \rho_\varepsilon \right\|_{H^1(\Omega)^N} \leq \frac{c_2}{\varepsilon} \|U(k)\|_{L^2(\Omega)^N}.$$

Finally, (97) yields the result. □

In order to solve the regularized problem (84), we first linearize it, as said before.

Definition 21. A function $g = g(k)$ is said to be a weak solution of the linearized problem associated to (76) if

$$(100) \quad \begin{cases} g = g(k) \in H_0^1(\Omega) \\ \forall \psi \in H_0^1(\Omega), \int_{\Omega} B(k) \nabla g \cdot \nabla \psi = \int_{\Omega} D_{\varepsilon}(k) \psi. \end{cases}$$

Lemma 22. Let $k \in L^1(\Omega)$, there exists a unique solution to Problem (100).

Proof. This is a consequence of Lax-Milgram theorem. Theorem 19 ensures that $[\psi \mapsto \int_{\Omega} D_{\varepsilon}(k) \psi]$ is a continuous linear form on $H_0^1(\Omega)$. Let the bilinear form b be defined by

$$(101) \quad b : (g, \psi) \mapsto \int_{\Omega} B(k) \nabla g \cdot \nabla \psi.$$

First, let $g, \psi \in H_0^1(\Omega)$

$$(102) \quad |b(g, \psi)| \leq \mu \|\nabla g\|_{L^2(\Omega)^N} \|\nabla \psi\|_{L^2(\Omega)^N} \leq \|g\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}.$$

The continuity of b is proved.

Then, let $\psi \in H_0^1(\Omega)$, we have

$$(103) \quad b(\psi, \psi) = \int_{\Omega} B(k) |\nabla \psi|^2 \geq \nu \|\nabla \psi\|_{L^2(\Omega)^N}^2 \geq c(\Omega) \nu \|\psi\|_{H^1(\Omega)}^2,$$

as a consequence of Poincaré inequality. The coercivity of b is proved.

Lax-Milgram theorem yields the existence and unicity of the solution g . \square

Now, we aim at showing the existence of a solution to problem (84). To do so, we recall the Leray-Schauder fixed-point theorem (see [9]).

Theorem 23 (Leray-Schauder). Let X a Banach space, and B a convex closed bounded nonempty subset of X . Let $f : B \rightarrow B$ a continuous function with $f(B)$ compact.

Then f admits at least one fixed point in B .

This result is the main ingredient of the proof of the following statement.

Theorem 24. The regularized problem (84) admits a solution.

Proof. Let the operator T be defined by

$$(104) \quad T : k \mapsto g(k),$$

where $g(k)$ refers to the solution of Problem (100). In other words, $Tk \in H_0^1(\Omega)$ satisfies

$$(105) \quad \forall \psi \in H_0^1(\Omega), \int_{\Omega} B(k) \nabla Tk \cdot \nabla \psi = \int_{\Omega} D_{\varepsilon}(k) \psi.$$

The method consists in applying Theorem 23 to the operator T . First, taking $\psi = Tk$ in (105) yields

$$(106) \quad \int_{\Omega} B(k) |\nabla Tk|^2 = \int_{\Omega} D_{\varepsilon}(k) Tk, \text{ hence } \nu \|\nabla Tk\|_{L^2(\Omega)^N} \leq c_1 \|D_{\varepsilon}(k)\|_{L^2(\Omega)}.$$

Then Theorem 19 yields

$$(107) \quad \|\nabla T k\|_{L^2(\Omega)^N} \leq c_2 \|D_\varepsilon(k)\|_{L^2(\Omega)} \leq M,$$

where $M > 0$ does not depend on k .

Considering the notations of Theorem 23, in this problem we take $X = H_0^1(\Omega)$, equipped with the scalar product

$$(108) \quad (f, g)_{H_0^1(\Omega)} = \int_{\Omega} \nabla f \cdot \nabla g,$$

and its associated topology.

Moreover, we take $B = \bar{B}(0, M)$ which is a convex closed bounded nonempty subset of $H_0^1(\Omega)$. It has been shown that $T(B) \subset B$. Now it remains to prove that T is continuous and $T(B)$ is a compact set.

Compactness of $T(B)$:

Let $(k_n)_{n \in \mathbb{N}}$ be any sequence of B . We aim at proving the existence of a subsequence (named the same way) converging in $H_0^1(\Omega)$ to a certain Tk , with $k \in B$. Given that B is bounded, there is a subsequence satisfying

$$(109) \quad k_n \xrightarrow[n \rightarrow +\infty]{} k \text{ in } H_0^1(\Omega),$$

where $k \in B$.

The assumptions on B and (107) yield that $(B(k_n)^{1/2} \nabla T k_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)^N$, so there exists a subsequence converging weakly to a certain function. In the following, we state this limit is $B(k)^{1/2} \nabla T k$.

Let $(k_n)_{n \in \mathbb{N}}$ be a sequence of elements of B . Since it is bounded, there is a subsequence named the same satisfying $k_n \xrightarrow[n \rightarrow +\infty]{} k$ in $H_0^1(\Omega)$.

We first aim at showing that for any $\varepsilon > 0$ and $\psi \in H_0^1(\Omega)$

$$(110) \quad \int_{\Omega} D_\varepsilon(k_n) \psi \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} D_\varepsilon(k) \psi.$$

The difference is rewritten as follows

$$(111) \quad \begin{aligned} D_\varepsilon(k_n) - D_\varepsilon(k) &= |\widetilde{U(k_n)} * \rho_\varepsilon|^2 - |\widetilde{U(k)} * \rho_\varepsilon|^2 \\ &= \left[(\widetilde{U(k_n)} - \widetilde{U(k)}) * \rho_\varepsilon \right] \cdot \left[(\widetilde{U(k_n)} + \widetilde{U(k)}) * \rho_\varepsilon \right]. \end{aligned}$$

Hölder and then Gagliardo-Nirenberg inequality yield

$$(112) \quad \begin{aligned} & \left| \int_{\Omega} (D_\varepsilon(k_n) - D_\varepsilon(k)) \psi \right| \\ & \leq \left\| (\widetilde{U(k_n)} - \widetilde{U(k)}) * \rho_\varepsilon \right\|_{L^3(\mathbb{R}^N)^N} \left\| (\widetilde{U(k_n)} + \widetilde{U(k)}) * \rho_\varepsilon \right\|_{L^3(\mathbb{R}^N)^N} \|\psi\|_{L^3(\Omega)} \\ & \leq c_1 \left\| (\widetilde{U(k_n)} - \widetilde{U(k)}) * \rho_\varepsilon \right\|_{H^1(\mathbb{R}^N)^N} \left\| (\widetilde{U(k_n)} + \widetilde{U(k)}) * \rho_\varepsilon \right\|_{H^1(\mathbb{R}^N)^N} \|\psi\|_{L^3(\Omega)}. \end{aligned}$$

Lemma 20 and Young convolution inequality yield

$$(113) \quad \left| \int_{\Omega} (D_{\varepsilon}(k_n) - D_{\varepsilon}(k)) \psi \right| \leq \frac{c_2}{\varepsilon^2} \|U(k_n) - U(k)\|_{L^2(\Omega)^N} \|U(k_n) + U(k)\|_{L^2(\Omega)^N} \|\psi\|_{L^3(\Omega)}.$$

According to Theorem 8, the right hand side of (113) converges to 0 when n goes to $+\infty$. This yields (110).

Both (110) and (105) imply

$$(114) \quad \forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} B(k_n) \nabla T k_n \cdot \nabla \psi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} B(k) \nabla T k \cdot \nabla \psi.$$

The uniqueness of the limit yields that

$$(115) \quad B(k_n)^{1/2} \nabla T k_n \xrightarrow{n \rightarrow +\infty} B(k)^{1/2} \nabla T k \text{ in } L^2(\Omega)^N.$$

In addition, Lebesgue inverse theorem and the continuity of $1/B$ yield $\frac{1}{B(k_n)} \xrightarrow{n \rightarrow +\infty} \frac{1}{B(k)}$ a.e. in Ω , and in addition $\frac{1}{B(k_n)} |\nabla \psi| \leq \frac{1}{\nu} |\nabla \psi|$. Then we obtain

$$(116) \quad \frac{1}{B(k_n)} \nabla \psi \xrightarrow{n \rightarrow +\infty} \frac{1}{B(k)} \nabla \psi \text{ in } L^2(\Omega)^N.$$

Convergences (115) and (116), and Lemma 9 then yield for any $\psi \in H_0^1(\Omega)$

$$(117) \quad \int_{\Omega} \nabla T k_n \cdot \nabla \psi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla T k \cdot \nabla \psi.$$

In other words, $T k_n \xrightarrow{n \rightarrow +\infty} T k$ in $H_0^1(\Omega)$.

The convergence of the norms has to be established. To do so, we have to prove

$$(118) \quad \int_{\Omega} D_{\varepsilon}(k_n) T k_n \xrightarrow{n \rightarrow +\infty} \int_{\Omega} D_{\varepsilon}(k) T k.$$

We start by writing the inequality

$$(119) \quad \begin{aligned} \|D_{\varepsilon}(k_n) T k_n - D_{\varepsilon}(k) T k\|_{L^1(\Omega)} &\leq \|(D_{\varepsilon}(k_n) - D_{\varepsilon}(k)) T k\|_{L^1(\Omega)} \\ &\quad + \|D_{\varepsilon}(k_n) (T k_n - T k)\|_{L^1(\Omega)}. \end{aligned}$$

On one hand, (110) yields $\|(D_{\varepsilon}(k_n) - D_{\varepsilon}(k)) T k\|_{L^1(\Omega)} \xrightarrow{n \rightarrow +\infty} 0$.

On the other hand, by compact Sobolev embedding, we have $T k_n \xrightarrow{n \rightarrow +\infty} T k$ in $L^2(\Omega)$. Hölder inequality then yields

$$(120) \quad \|D_{\varepsilon}(k_n) (T k_n - T k)\|_{L^1(\Omega)} \leq \|D_{\varepsilon}(k_n)\|_{L^2(\Omega)} \|T k_n - T k\|_{L^2(\Omega)} \xrightarrow{n \rightarrow +\infty} 0.$$

Hence, (118) is proved. Reporting in (105) yields

$$(121) \quad \int_{\Omega} B(k_n) |\nabla T k_n|^2 \xrightarrow{n \rightarrow +\infty} \int_{\Omega} B(k) |\nabla T k|^2.$$

Both (115) and (121) yield

$$(122) \quad B(k_n)^{1/2} \nabla T k_n \xrightarrow{n \rightarrow +\infty} B(k)^{1/2} \nabla T k \text{ in } L^2(\Omega)^N.$$

Lebesgue inverse theorem ensures that another subsequence of $(\nabla T k_n)_{n \in \mathbb{N}}$ is dominated by a $L^1(\Omega)^N$ function. Lebesgue theorem then yields

$$(123) \quad \int_{\Omega} (B(k_n) - B(k)) |\nabla T k_n|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Consequently, the following decomposition

$$(124) \quad \int_{\Omega} (B(k_n) |\nabla T k_n|^2 - B(k) |\nabla T k|^2) = \int_{\Omega} B(k) (|\nabla T k_n|^2 - |\nabla T k|^2) \\ + \int_{\Omega} (B(k_n) - B(k)) |\nabla T k_n|^2$$

implies that

$$(125) \quad \int_{\Omega} B(k) (|\nabla T k_n|^2 - |\nabla T k|^2) \xrightarrow{n \rightarrow +\infty} 0.$$

Let N_B be defined by

$$(126) \quad N_B : f \in H_0^1(\Omega) \mapsto \|B(k) \nabla f\|_{L^2(\Omega)^N}.$$

The norms $\|\cdot\|_{H_0^1(\Omega)}$ and N_B are equivalent, and (125) ensures that $(T k_n)_{n \in \mathbb{N}}$ converges strongly for the norm N_B , so as for the $H_0^1(\Omega)$ norm.

We now have proved that $T(B)$ is compact.

Continuity of T :

Let $(k_n)_{n \in \mathbb{N}}$ be a sequence of B such that $k_n \xrightarrow{n \rightarrow +\infty} k$ in $H_0^1(\Omega)$. The computations made before ensure that a subsequence of $(T k_n)_{n \in \mathbb{N}}$ converges to $T k$ in $H_0^1(\Omega)$. Moreover, $T k$ is the only accumulation point of this bounded sequence. As a result, the whole sequence satisfies

$$(127) \quad T k_n \xrightarrow{n \rightarrow +\infty} T k \text{ in } H_0^1(\Omega).$$

All the assumptions of Theorem 23 are satisfied. Consequently, T admits a fixed point in B , which corresponds to a solution to Problem (84). \square

4.3 Last step: Taking the limit when ε goes to zero

This subsection aims at proving the existence of a solution to the initial variational TKE problem, by analyzing the convergence of the approximated solutions found just before.

Theorem 25. *There exists a function $k \in V$ satisfying (81).*

Proof. Let $(k_\varepsilon)_{\varepsilon>0}$ be the sequence of approximation functions given in the previous subsection. Then for any $\varepsilon > 0$, k_ε satisfies

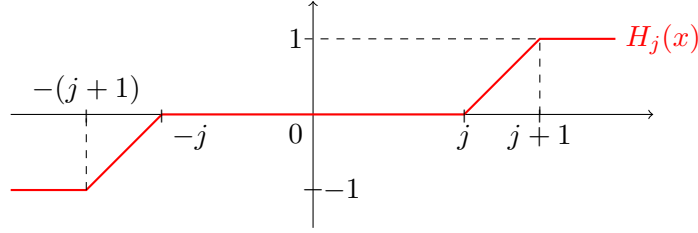
$$(128) \quad \forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} B(k_\varepsilon) \nabla k_\varepsilon \cdot \nabla \psi = \int_{\Omega} D_\varepsilon(k_\varepsilon) \psi.$$

We intend to show that $(k_\varepsilon)_{\varepsilon>0}$ is bounded for a certain topology. To do so, we use Lemma 1. A general theorem from Stampacchia (see [8]) is needed before.

Theorem 26 (Regularity and truncation function). *Let $H \in W^{1,\infty}(\mathbb{R})$ with a finite number of discontinuities, and such that $H(0) = 0$. Let $q \in [1; +\infty[$ and $v \in W_0^{1,q}(\Omega)$.*

Then $H(v) \in W_0^{1,q}(\Omega)$ and $\nabla H(v) = H'(v) \nabla v$.

Until the end of this chapter, for $j \in \mathbb{N}$, H_j refers to the truncation function defined by



This function satisfies the assumptions of Theorem 26, so that we can take $\psi = H_j(k_\varepsilon)$ in (128) to obtain

$$(129) \quad \forall j \in \mathbb{N}, \quad \int_{\Omega} B(k_\varepsilon) |\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) = \int_{\Omega} D_\varepsilon(k_\varepsilon) H_j(k_\varepsilon).$$

On one hand,

$$(130) \quad \int_{\Omega} B(k_\varepsilon) |\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) = \int_{B_j^\varepsilon} B(k_\varepsilon) |\nabla k_\varepsilon|^2 \geq \nu \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2,$$

where B_j^ε refers to the subset $\{x \in \Omega / j \leq |k_\varepsilon(x)| < j+1\}$.

On the other hand, Young convolution inequality yields

$$(131) \quad \int_{\Omega} D_\varepsilon(k_\varepsilon) \underbrace{H_j(k_\varepsilon)}_{\leq 1} \leq \left\| \widetilde{U(k_\varepsilon)} * \rho_\varepsilon \right\|_{L^2(\Omega)^N}^2 \leq \|U(k_\varepsilon)\|_{L^2(\Omega)^N}^2.$$

Inequalities (130), (131) and (97) reported in (129) yield the existence of a constant $C > 0$ not depending on k and ε such that

$$(132) \quad \forall j \in \mathbb{N}, \quad \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2 \leq C.$$

Lemma 1 then yields $\|k_\varepsilon\|_{W^{1,q}(\Omega)} \leq A$, for $q \in [1; N'[,$ and given that C does not depend on ε , it is the same for A .

Thus, there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \xrightarrow{n \rightarrow +\infty} 0$ and $k_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} k$ in $W_0^{1,q}(\Omega)$. The function k is in V .

We aim at showing that k is a solution of (81), which means

$$(133) \quad \forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} B(k) \nabla k \cdot \nabla \psi = \int_{\Omega} D(k) \psi.$$

We recall that the intermediary solutions (k_{ε_n}) satisfy

$$(134) \quad \forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} B(k_{\varepsilon_n}) \nabla k_{\varepsilon_n} \cdot \nabla \psi = \int_{\Omega} D_{\varepsilon_n}(k_{\varepsilon_n}) \psi.$$

Convergence of the left hand side:

Given that $k_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} k$ in V , we have

$$(135) \quad \forall \psi \in W, \quad \int_{\Omega} \nabla k_{\varepsilon_n} \cdot \nabla \psi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla k \cdot \nabla \psi.$$

In order to add the term $B(k_n)$, it seems useful to state a variant of Lemma 9.

Lemma 27 (Double convergence in V and W). *Let Ω be an open subset of \mathbb{R}^N , $f \in \bigcap_{1 < r < N'} L^r(\Omega)$ and $g \in L^q(\Omega)$, where $q > N$. Let us consider two sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ such that*

$$(136) \quad \begin{cases} f_n \xrightarrow{n \rightarrow +\infty} f & \text{in } L^r(\Omega) \text{ for } r \in [1; N'[, \\ g_n \xrightarrow{n \rightarrow +\infty} g & \text{in } L^q(\Omega). \end{cases}$$

Then we have

$$(137) \quad \int_{\Omega} f_n g_n \xrightarrow{n \rightarrow +\infty} \int_{\Omega} f g.$$

Proof of Lemma 27. The proof of Lemma 27 is similar to the proof of Lemma 9. We write

$$(138) \quad \int_{\Omega} (f_n g_n - f g) = \int_{\Omega} (f_n - f) g_n + \int_{\Omega} f (g_n - g).$$

On one hand, the Hölder inequality yields

$$(139) \quad \left| \int_{\Omega} (f_n - f) g_n \right| \leq \|f_n - f\|_{L^{q'}(\Omega)} \|g_n\|_{L^q(\Omega)} \leq M \|f_n - f\|_{L^{q'}(\Omega)} \xrightarrow{n \rightarrow +\infty} 0,$$

where q' refers to the conjugate of q , which is in $[1; N'[,$

On the other hand, the weak convergence of $(g_n)_{n \in \mathbb{N}}$ yields

$$(140) \quad \int_{\Omega} (g_n - g) f \xrightarrow{n \rightarrow +\infty} 0.$$

Hence the result. □

We now check if the assumptions of Lemma 27 are satisfied.

By compact Sobolev embedding, $k_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} k$ in $L^1(\Omega)$. Thus, Lebesgue inverse theorem yields that a subsequence satisfies $k_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} k$ a.e. in Ω . The continuity of B yields $B(k_{\varepsilon_n}) \xrightarrow{n \rightarrow +\infty} B(k)$ a.e. in Ω . And the domination $B(k_{\varepsilon_n})|\nabla\psi| \leq \mu|\nabla\psi|$ stands. Consequently, Lebesgue theorem yields

$$(141) \quad \forall r \in [1; N'[, \forall \psi \in W, \quad B(k_{\varepsilon_n})\nabla\psi \xrightarrow{n \rightarrow +\infty} B(k)\nabla\psi \text{ in } L^r(\Omega).$$

The weak convergence of $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ being already established, Lemma 27 yields

$$(142) \quad \forall \psi \in W, \quad \int_{\Omega} B(k_{\varepsilon_n})\nabla k_{\varepsilon_n} \cdot \nabla\psi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} B(k)\nabla k \cdot \nabla\psi.$$

Convergence of the right hand side:

We intend to show that

$$(143) \quad \forall \psi \in W, \quad \int_{\Omega} D_{\varepsilon_n}(k_{\varepsilon_n})\psi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} D(k)\psi,$$

at least for a subsequence.

Given that $\psi \in L^\infty(\Omega)$, it is sufficient to prove that

$$(144) \quad D_{\varepsilon_n}(k_{\varepsilon_n}) \xrightarrow{n \rightarrow +\infty} D(k) \text{ in } L^1(\Omega).$$

The L^1 norm is decomposed as follows

$$(145) \quad \int_{\Omega} |D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)| \leq \int_{\Omega} |D_{\varepsilon_n}(k_{\varepsilon_n}) - D_{\varepsilon_n}(k)| + \int_{\Omega} |D_{\varepsilon_n}(k) - D(k)|.$$

Cauchy-Schwarz inequality first yields

$$(146) \quad \begin{aligned} \int_{\Omega} |D_{\varepsilon_n}(k) - D(k)| &= \int_{\Omega} \left| |\widetilde{U(k)} * \rho_{\varepsilon_n}|^2 - |U(k)|^2 \right| \\ &\leq \left\| \widetilde{U(k)} * \rho_{\varepsilon_n} - \widetilde{U(k)} \right\|_{L^2(\mathbb{R}^N)^N} \left\| \widetilde{U(k)} * \rho_{\varepsilon_n} + \widetilde{U(k)} \right\|_{L^2(\mathbb{R}^N)^N}. \end{aligned}$$

We recall the following classical result.

Theorem 28. *Let $p \in [1; +\infty[$, $f \in L^p(\mathbb{R}^N)$, and $(\rho_\varepsilon)_{\varepsilon > 0}$ a mollifier. Then we have*

$$(147) \quad \rho_\varepsilon * f \xrightarrow{\varepsilon \rightarrow 0} f \text{ in } L^p(\mathbb{R}^N).$$

Theorem 28 and (146) yield

$$(148) \quad \int_{\Omega} |D_{\varepsilon_n}(k) - D(k)| \xrightarrow{n \rightarrow +\infty} 0.$$

Similarly, we write

$$(149) \quad \int_{\Omega} |D_{\varepsilon_n}(k_{\varepsilon_n}) - D_{\varepsilon_n}(k)| \leq \left\| \left(\widetilde{U(k_{\varepsilon_n})} - \widetilde{U(k)} \right) * \rho_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)^N} \left\| \left(\widetilde{U(k_{\varepsilon_n})} + \widetilde{U(k)} \right) * \rho_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)^N}.$$

On one hand, Young convolution inequality and (97) yield

$$(150) \quad \left\| \left(\widetilde{U(k_{\varepsilon_n})} + \widetilde{U(k)} \right) * \rho_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)^N} \leq \|U(k_{\varepsilon_n})\|_{L^2(\Omega)^N} + \|U(k)\|_{L^2(\Omega)^N} \leq C.$$

On the other hand, Young convolution inequality and Theorem 8 yield for a subsequence

$$(151) \quad \left\| \left(\widetilde{U(k_{\varepsilon_n})} - \widetilde{U(k)} \right) * \rho_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)^N} \leq \|U(k_{\varepsilon_n}) - U(k)\|_{L^2(\Omega)^N} \xrightarrow{n \rightarrow +\infty} 0.$$

We have obtained

$$(152) \quad \int_{\Omega} |D_{\varepsilon_n}(k_{\varepsilon_n}) - D_{\varepsilon_n}(k)| \xrightarrow{n \rightarrow +\infty} 0,$$

hence (144).

As a conclusion, k satisfies (81). □

4.4 Conclusion: solution of the coupled problem

Let us consider the sequence (ε_n) mentioned previously, after all the extractions that were done. We have shown that (k_{ε_n}) strongly converges in $L^2(\Omega)$ to a function k .

Moreover, (k_{ε_n}) weakly converges to k in every space $W_0^{1,q}(\Omega)$ with $q \in [1; N']$. Therefore, theorem 10 yields that $(u(k_{\varepsilon_n}))$ strongly converges to $u(k)$ in $H^1(\Omega)$.

All of this ensures the existence of a couple $(u; k)$ of solutions of (1) at the distribution sense, with $u \in H^1(\Omega)$ and $k \in V$ (in particular $k \in L^2$).

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