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Guillaume Leloup

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# Coupled system involving eddy coefficients and a right hand side in $L^1$ : Review chapter

Guillaume LELOUP

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## 1 Introduction

This chapter is focused on the following system:

$$\begin{cases} -\operatorname{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -\operatorname{div}(B(k)\nabla k) = A(k)|\nabla u|^2 & \text{in } \Omega, \\ -A(k)\frac{\partial u}{\partial n} = \alpha u & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma. \end{cases} \quad (1.1)$$

Where  $\Omega$  is an open bounded set in  $\mathbb{R}^N$  ( $N \geq 2$ ), and  $\Gamma = \partial\Omega$  its boundary. The viscosity functions are bounded and satisfy:

$$\exists \nu, \mu > 0, \forall x \geq 0, \nu \leq A(x), B(x) \leq \mu. \quad (1.2)$$

We also assume that  $f \in L^2(\Omega)$  and  $\alpha > 0$ .

The aim of this chapter is proving the existence of a couple  $(u, k)$  of distributional solutions of (1.1). This is not the only kind of solutions commonly searched. Indeed, some articles deal with renormalized solutions, which are defined and studied in [3]. This notion is interesting for the study of a system having one source term in  $L^1(\Omega)$ , as shown in [6].

The source term of the equation in  $k$  in (1.1) is in  $L^1(\Omega)$ . Without any better hypothesis, it remains possible to find distributional solutions of (1.1).

This figures in the works of L. Boccardo and T. Gallouët in [1], on which is based this chapter. The associated result, proved in section 4, is a consequence of the washer Lemma, stated and proved in section 2. This Lemma brings a bound for a solution in  $W_0^{1,q}(\Omega)$ , where  $q < \frac{N}{N-1}$ . Consequently, some particular function sets are needed.

Then, in section 4, we search weak solutions in  $V = \bigcap_{1 < q < N'} W_0^{1,q}(\Omega)$ , where  $N'$  refers to  $\frac{N}{N-1}$ ; with test functions in  $W = \bigcup_{q > N} W_0^{1,q}(\Omega)$ . This method was also used by T. Gallouët and R. Herbin in [4].

Along this chapter, the viscosities  $A$  and  $B$  are bounded (assumption (1.2)). This does not really correspond to the physical situation, but is a right way to establish some interesting mathematical results, as a priori estimates, such as in section 3 and 4. The assumption of bounded viscosities appears in [4], but also in the works of S. Clain and R. Touzani (see [2]), in which an existence result is proved in the two-dimensional case. In the present work, the existence results are proved when  $N = 3$ .

Of course it is possible to deal with unbounded viscosities, such as in [6], where they are only bounded from below. Otherwise, if they are not, other assumptions are required, concerning the decreasing. In addition, other function spaces need to be developed, like in [5].

In order to find the existence of a solution of (1.1), this work will be divided into two steps, that correspond to sections 3 and 4. Section 3 is devoted to the equation in  $u$ , which amounts to a linear problem. But when the problem is obviously non-linear, like the equation in  $k$  considered in section 4, another method is required. This consists in regularizing the system, then solving it by a linearization and a fixed-point theorem. This method is already used in [2].

Given  $m \in \mathbb{N}$  and  $p \in [1, +\infty]$ , the norm of  $W^{m,p}(\Omega)$  will be simply written:  $\|\cdot\|_{m,p,\Omega}$ . In particular,  $\|\cdot\|_{0,p,\Omega}$  is the  $L^p(\Omega)$  norm.

## 2 Preliminary results

### 2.1 The washer lemma

This section aims at proving the following result (already proved by Boccardo-Gallouët in [1]).

**Lemma 1** (Washer lemma). *Let  $q \in ]1, N'[$  where  $N' = \frac{N}{N-1}$ , and  $u \in W_0^{1,q}(\Omega)$ . Let us write:*

$$B_n = \{x \in \Omega / n \leq |u(x)| < n + 1\}. \quad (2.1)$$

We assume that there is a constant  $C > 0$  such that:

$$\forall n \in \mathbb{N}, \quad \int_{B_n} |\nabla u(x)|^2 dx \leq C. \quad (2.2)$$

Then we have the inequality:

$$\|u\|_{1,q,\Omega} \leq A. \quad (2.3)$$

Where  $A$  depends on  $\Omega$ ,  $q$  and  $C$  that is introduced by (2.2).

*Remark 1.* The most important in this result is the fact that the two constants  $C$  and  $A$  are independent of  $u$ . Actually, the result that will be used is: let  $(u_j)_{j \in J}$  a family of functions in  $W_0^{1,q}(\Omega)$  with  $q \in ]1, N[$  satisfying:

$$\forall j \in J, \forall n \in \mathbb{N}, \quad \int_{B_n^j} |\nabla u_j(x)|^2 dx \leq C, \quad (2.4)$$

where  $B_n^j$  refers to:  $B_n^j = \{x \in \Omega / n \leq |u_j(x)| < n + 1\}$ .

Then there exists a constant  $A > 0$  only depending on  $\Omega$ ,  $q$  and  $C$  such that:

$$\forall j \in J, \quad \|u_j\|_{1,q,\Omega} \leq A. \quad (2.5)$$

The proof of lemma 1 is based on a decomposition given by the following lemma.

**Lemma 2.** *Let  $u$  satisfying the assumptions of lemma 1. Let  $q^*$  be the critical exponent in Sobolev embedding  $W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ , and any index  $n_0 \in \mathbb{N}$ . There exist two constants  $\lambda_1$  and  $\lambda_2$  depending on  $n_0$ ,  $q$  and  $\Omega$  satisfying:*

$$\|\nabla u\|_{0,q,\Omega}^q \leq C^{q/2} \left( \lambda_1 + \lambda_2 \|u\|_{0,q^*,\Omega}^{\frac{q^*(2-q)}{2}} \right). \quad (2.6)$$

Moreover, we have:  $\lim_{n_0 \rightarrow +\infty} \lambda_2(n_0) = 0$ .

*Proof of lemma 2.* At first, we decompose  $\int_{\Omega} |\nabla u(x)|^q dx$  as a sum:

$$\int_{\Omega} |\nabla u(x)|^q dx = \sum_{n=0}^{+\infty} \int_{B_n} |\nabla u(x)|^q dx. \quad (2.7)$$

Let us apply Hölder inequality, recalling that  $\Omega$  is bounded, thus for  $n \in \mathbb{N}$ ,  $B_n$  is bounded.

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \int_{B_n} |\nabla u(x)|^q dx &\leq \left( \int_{B_n} |\nabla u(x)|^2 dx \right)^{q/2} \left( \int_{B_n} dx \right)^{1-\frac{q}{2}} \\ &= \text{Meas}(B_n)^{\frac{2-q}{2}} \left( \int_{B_n} |\nabla u(x)|^2 dx \right)^{q/2} \\ &\leq \text{Meas}(B_n)^{\frac{2-q}{2}} C^{q/2}. \end{aligned} \quad (2.8)$$

Where "Meas" refers to the Lebesgue measure of any borelian subset of  $\mathbb{R}^N$ . The last inequality comes from (2.2).

We set  $n_0 \in \mathbb{N}$ , and we focus on the case  $n \leq n_0$ , then  $n > n_0$ .

On one hand, let  $n \leq n_0$ , we have

$$\int_{B_n} |\nabla u(x)|^q dx \leq \text{Meas}(B_n)^{\frac{2-q}{2}} C^{q/2} \leq \text{Meas}(\Omega)^{\frac{2-q}{2}} C^{q/2}, \quad (2.9)$$

then by addition

$$\sum_{n=0}^{n_0} \int_{B_n} |\nabla u(x)|^q dx \leq (n_0 + 1) \text{Meas}(\Omega)^{\frac{2-q}{2}} C^{q/2}. \quad (2.10)$$

On the other hand, let  $n > n_0$ , we have

$$\int_{B_n} |u(x)|^{q^*} dx \geq \int_{B_n} n^{q^*} dx = n^{q^*} \text{Meas}(B_n), \quad \text{then } \text{Meas}(B_n) \leq \frac{1}{n^{q^*}} \int_{B_n} |u(x)|^{q^*} dx. \quad (2.11)$$

Inequalities (2.8) and (2.11) yield

$$\forall n > n_0, \quad \int_{B_n} |\nabla u(x)|^q dx \leq C^{q/2} \left( \frac{1}{n^{q^*}} \right)^{\frac{2-q}{2}} \left( \int_{B_n} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}}, \quad (2.12)$$

which yields by addition

$$\sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u(x)|^q dx \leq C^{q/2} \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left( \int_{B_n} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}}. \quad (2.13)$$

We then apply the discrete Hölder inequality on the right hand side of (2.13) to obtain

$$\sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left( \int_{B_n} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}} \leq \left[ \sum_{n=n_0+1}^{+\infty} \int_{B_n} |u(x)|^{q^*} dx \right]^{\frac{2-q}{2}} \left[ \sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}} \right]^{q/2}. \quad (2.14)$$

We need to prove that the series  $\sum_{n \geq n_0+1} n^{\frac{N(q-2)}{N-q}}$  converges, which is equivalent to  $\frac{N(2-q)}{N-q} > 1$ .

By simple calculations, this inequality is equivalent to  $q < N'$ .

Then we set  $\lambda_2(n_0, q) = \left[ \sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}} \right]^{q/2}$ . And since the series converges, we directly have:  $\lambda_2(n_0, q) \xrightarrow{n_0 \rightarrow +\infty} 0$ . With this notation, (2.14) becomes

$$\begin{aligned} \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left( \int_{B_n} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}} &\leq \lambda_2(n_0, q) \left[ \sum_{n=n_0+1}^{+\infty} \int_{B_n} |u(x)|^{q^*} dx \right]^{\frac{2-q}{2}} \\ &\leq \lambda_2(n_0, q) \left( \int_{\Omega} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}}. \end{aligned} \quad (2.15)$$

Hence, (2.15) combined with (2.13) yield

$$\sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u(x)|^q dx \leq C^{q/2} \lambda_2(n_0, q) \left( \int_{\Omega} |u(x)|^{q^*} dx \right)^{\frac{2-q}{2}} = C^{q/2} \lambda_2(n_0, q) \|u\|_{0, q^*, \Omega}^{\frac{q^*(2-q)}{2}}. \quad (2.16)$$

We set  $\lambda_1(n_0, q) = (n_0 + 1) \text{Meas}(\Omega)^{\frac{2-q}{2}}$ , and we add (2.10) to (2.16) to obtain, by (2.7)

$$\int_{\Omega} |\nabla u(x)|^q dx \leq C^{q/2} \left( \lambda_1(n_0, q) + \lambda_2(n_0, q) \|u\|_{0, q^*, \Omega}^{\frac{q^*(2-q)}{2}} \right). \quad (2.17)$$

That corresponds to (2.6), which concludes the proof.  $\square$

Then we use this result to prove the washer lemma.

*Proof of lemma 1.* We start with the case  $N = 2$ , which is easier. In that case, we have  $\frac{q^*(2-q)}{2} = \frac{Nq(2-q)}{2(N-q)} = q$ . Then (2.6) becomes

$$\|\nabla u\|_{0, q, \Omega}^q \leq C^{q/2} \left( \lambda_1 + \lambda_2 \|u\|_{0, q^*, \Omega}^q \right). \quad (2.18)$$

As  $q < 2$ , we can apply Sobolev embedding  $W^{1, q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$  on the right hand side of (2.18), and then the Poincaré-Friedrichs inequality

$$\|u\|_{0, q^*, \Omega}^q \leq K \|\nabla u\|_{0, q, \Omega}^q \leq KC^{q/2} \left( \lambda_1 + \lambda_2 \|u\|_{0, q^*, \Omega}^q \right), \quad (2.19)$$

where  $K > 0$  depends on  $\Omega$  and  $q$ . We recall that if  $b < 1$ , the following equivalence stands

$$x \leq a + bx \Leftrightarrow x \leq \frac{a}{1-b}. \quad (2.20)$$

Since  $\lambda_2(n_0, q) \xrightarrow{n_0 \rightarrow +\infty} 0$ , it is possible to choose  $n_0 \in \mathbb{N}$  large enough to have  $KC^{q/2} \lambda_2 < 1$ . Then by (2.20), we obtain

$$\|u\|_{0, q^*, \Omega}^q \leq \frac{KC^{q/2} \lambda_1}{1 - KC^{q/2} \lambda_2}. \quad (2.21)$$

Thus, from (2.18):

$$\|\nabla u\|_{0, q, \Omega}^q \leq C^{q/2} \left( \lambda_1 + \frac{KC^{q/2} \lambda_1 \lambda_2}{1 - KC^{q/2} \lambda_2} \right). \quad (2.22)$$

And the Poincaré-Friedrichs inequality yields

$$\|u\|_{1, q, \Omega}^q \leq LC^{q/2} \left( \lambda_1 + \frac{KC^{q/2} \lambda_1 \lambda_2}{1 - KC^{q/2} \lambda_2} \right), \quad (2.23)$$

where  $L$  depends on  $q$  and  $\Omega$ . This yields the result with

$$A = L^{1/q} C^{1/2} \left( \lambda_1 + \frac{KC^{q/2} \lambda_1 \lambda_2}{1 - KC^{q/2} \lambda_2} \right)^{1/q}. \quad (2.24)$$

Let us focus now on the case  $N > 2$ . Since  $q < N' < N$ , the Sobolev embedding used before yields

$$\|u\|_{0,q^*,\Omega}^q \leq K \|\nabla u\|_{0,q,\Omega}^q \leq KC^{q/2} \left( \lambda_1 + \lambda_2 \|u\|_{0,q^*,\Omega}^{\frac{q^*(2-q)}{2}} \right). \quad (2.25)$$

Inequality (2.25) is of the form  $x \leq a + bx^\gamma$ , with

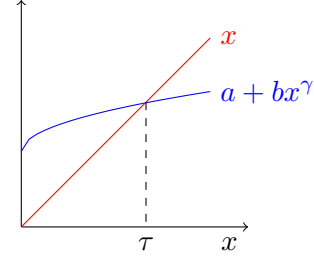
$$\gamma = \frac{q^*(2-q)}{2q} = \frac{N(2-q)}{2(N-q)}. \quad (2.26)$$

We have already shown that if  $q \in ]1, N'[$ , then:

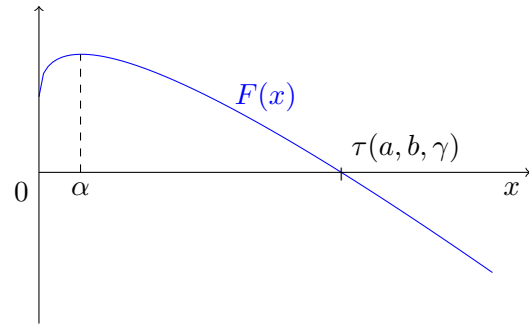
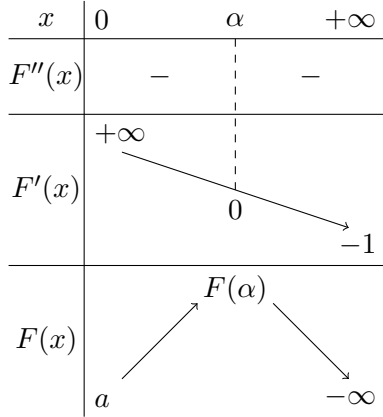
$$1 < \frac{N(2-q)}{N-q} < N' \text{ hence } \frac{1}{2} < \gamma < \frac{N'}{2} < 1. \quad (2.27)$$

Thus, the inequation we need to solve is of the form:  $x \leq a + bx^\gamma$ , with  $\gamma \in ]0, 1[$  and  $a, b > 0$ .

The graph on the side shows the existence of  $\tau \in ]0, +\infty[$  depending on  $a, b$  and  $\gamma$  such that  $x \leq a + bx^\gamma \Leftrightarrow x \leq \tau$ .



Let us show it rigorously by studying the variations of  $F : x > 0 \mapsto a + bx^\gamma - x$ . This function is  $C^\infty$  on  $]0, +\infty[$  and  $\begin{cases} F'(x) = b\gamma x^{\gamma-1} - 1 \\ F''(x) = b\gamma(\gamma-1)x^{\gamma-2} \end{cases}$ , therefore  $F''(x) < 0$ . We obtain the following variations



Inequality (2.25) finally ensures the existence of  $\tau = \tau(\lambda_1, \lambda_2, C) > 0$  such that

$$\|u\|_{0,q^*,\Omega}^q \leq \tau. \quad (2.28)$$

Thus, (2.6) and Poincaré-Friedrichs inequality yield

$$\|\nabla u\|_{0,q,\Omega}^q \leq C^{q/2} (\lambda_1 + \lambda_2 \tau) \text{ hence } \|u\|_{1,q,\Omega}^q \leq LC^{q/2} (\lambda_1 + \lambda_2 \tau), \quad (2.29)$$

where  $L$  only depends on  $\Omega$  and  $q$ . With  $A = L^{1/q}C^{1/2}(\lambda_1 + \lambda_2\tau)^{1/q}$ , lemma 1 is proved in case  $N > 2$ .  $\square$

The first hypothesis of lemma 1 is  $q \in ]1, N'[$ , where  $N' = \frac{N}{N-1}$ . It has been used many times, and we are able to show that is impossible to obtain similar results with higher values of  $q$ .

Indeed, considering the case  $N > 2$ , we recall that  $\alpha$  is the only point where  $F'$  vanishes, which means

$$b\gamma\alpha^{\gamma-1} = 1 \quad \text{thus} \quad \alpha = (b\gamma)^{\frac{1}{1-\gamma}}. \quad (2.30)$$

Replacing  $b$  by the constants of lemma 2 yields  $b(q) = M(q)\lambda_2(q)$ , where  $M(q)$  is bounded if  $q \in ]1, N'[$ . Consequently,  $b(q) \xrightarrow{q \rightarrow N'} +\infty$  and then

$$\gamma(q) = \frac{N(2-q)}{2(N-q)} \xrightarrow{q \rightarrow N'} 2 \quad \text{hence} \quad \frac{1}{1-\gamma} \xrightarrow{q \rightarrow N'} \frac{1}{2}. \quad (2.31)$$

Therefore, (2.30) yields  $\alpha \xrightarrow{q \rightarrow N'} +\infty$ . And since  $\tau \geq \alpha$ , we obtain  $\tau \xrightarrow{q \rightarrow N'} +\infty$ , thus  $A \xrightarrow{q \rightarrow N'} +\infty$ .

There is a similar result in the case  $N = 2$ . The proof of lemma 1 requires to choose an index  $n_0$  great enough to have  $b < 1$ . But as  $\lambda_2(n_0, q) \xrightarrow{q \rightarrow N'} +\infty$ , an increase of  $q$  implies an increase of  $n_0$ .

Thus, as  $\frac{2-q}{2} = 1 - \frac{q}{2} \xrightarrow{q \rightarrow N'} 1 - \frac{N'}{2} = 0$ , it follows

$$\lambda_1(n_0, q) = (n_0 + 1)\text{Meas}(\Omega)^{\frac{2-q}{2}} \xrightarrow[\substack{q \rightarrow N' \\ n_0 \rightarrow +\infty}]{} +\infty \quad \text{hence} \quad A \xrightarrow{q \rightarrow N'} +\infty. \quad (2.32)$$

We have shown that  $q$  cannot be larger than the assumptions of lemma 1.

The result shown in this first section will be useful in the following. Generally, solving a PDE system is divided into two steps:

- simplifying the problem (for example by moving to a finite-dimension space, or by using a truncation function),
- coming back to the initial problem with the help of compacity results.

## 2.2 A general result on norms

**Prop./Def. 3** (Trace norm on  $H^1$ ). *Let us define the application:  $N_0(u) = \left( \|\nabla u\|_{0,2,\Omega}^2 + \|u\|_{0,2,\Gamma}^2 \right)^{1/2}$ , " $u|_\Gamma$ " referring to the trace of  $u$  on  $\Gamma$ . Then  $N_0$  is a norm on  $H^1(\Omega)$ .*



*Proof.*  $N_0$  is non negative, homogeneous and satisfies triangular inequality (direct consequence of the fact that the norms  $L^2$  also do). Let us consider  $u \in H^1(\Omega)$  such that  $N_0(u) = 0$ . It comes  $\|\nabla u\|_{0,2,\Omega} = 0$  and  $\|u\|_{0,2,\Gamma} = 0$ .

First, this yields that  $\nabla u = 0$  almost everywhere on  $\Omega$ . Thus, we write:  $u \stackrel{ae.}{=} \lambda$ , where  $\lambda$  is a real constant. And if  $u$  is constant almost everywhere, its trace has the same value almost everywhere on  $\Gamma$ . Yet,  $\text{tr}(u) = 0$ , therefore  $u \stackrel{ae.}{=} 0$ .

Consequently,  $N_0$  is a norm on  $H^1(\Omega)$ .  $\square$

**Theorem 4** (Two equivalent norms). *Let us recall the definition of the classical Sobolev norm on  $H^1(\Omega)$ :*

$$\|u\|_{1,2,\Omega} = \left( \|u\|_{0,2,\Omega}^2 + \|\nabla u\|_{0,2,\Omega}^2 \right)^{1/2}. \quad (2.33)$$

*Then this norm and  $N_0$  previously defined are equivalent.*

*Proof.* At first, the trace operator  $\text{tr} : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$  is surjective (see [7]), and by Sobolev embedding,  $H^{1/2}(\Gamma) \hookrightarrow L^q(\Gamma)$ , with  $\frac{1}{q} = \frac{1}{2} - \frac{1}{2N}$ , thus  $q = \frac{2N}{N-1} \geq 2$ .

Then, since  $\Omega$  is bounded, we have:  $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ . Hence

$$\exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|_{0,2,\Gamma}^2 \leq C \|u\|_{1,2,\Omega}^2. \quad (2.34)$$

We deduce

$$\forall u \in H^1(\Omega), \quad N_0(u)^2 \leq (1 + C) \|u\|_{1,2,\Omega}^2. \quad (2.35)$$

Therefore  $N_0$  is dominated by  $\|\cdot\|_{1,2,\Omega}$ .

Conversely, we intend to show

$$\exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|_{1,2,\Omega}^2 \leq C N_0(u)^2. \quad (2.36)$$

By contradiction, let us suppose the negation of (2.36), which means

$$\forall C > 0, \exists u \in H^1(\Omega), \quad \|u\|_{1,2,\Omega}^2 > C N_0(u)^2. \quad (2.37)$$

Then in particular

$$\forall n \in \mathbb{N}, \exists u_n \in H^1(\Omega), \quad \|u_n\|_{1,2,\Omega}^2 > n N_0(u_n)^2. \quad (2.38)$$

Inequality (2.38) yields that  $\forall n \in \mathbb{N}^*$ ,  $\|u_n\|_{0,2,\Omega} > 0$ . Then, both sides of (2.38) can be divided by  $\|u_n\|_{0,2,\Omega}$ . Which means, keeping the same notation

$$\forall n \in \mathbb{N}, \exists u_n \in S, \quad \|u_n\|_{1,2,\Omega}^2 > n N_0(u_n)^2, \quad (2.39)$$

where  $S$  refers to the " $L^2$ -sphere" defined by:  $S = \{v \in H^1(\Omega) / \|v\|_{0,2,\Omega} = 1\}$ . Then (2.39) can be rewritten as

$$\forall n \geq 2, \quad \frac{1}{n} + \frac{1}{n} \|\nabla u_n\|_{0,2,\Omega}^2 > \|u_n\|_{0,2,\Gamma} + \|\nabla u_n\|_{0,2,\Omega}^2. \quad (2.40)$$

Since  $\|u_n\|_{0,2,\Gamma}^2 \geq 0$ , (2.40) yields in particular

$$\left(1 - \frac{1}{n}\right) \|\nabla u_n\|_{0,2,\Omega}^2 \leq \frac{1}{n}, \quad \text{then } \|\nabla u_n\|_{0,2,\Omega}^2 \leq \frac{1}{n-1}. \quad (2.41)$$

Therefore, the sequence  $(\nabla u_n)_{n \in \mathbb{N}^*}$  is bounded in  $L^2(\Omega)$ ; and as  $(u_n)_{n \in \mathbb{N}^*}$  is a sequence of points of  $S$ ,  $(u_n)_{n \in \mathbb{N}}$  is also bounded in  $L^2(\Omega)$ . We deduce  $(u_n)$  is bounded in  $H^1(\Omega)$ . Thus, Banach-Alaoglu theorem ensures the existence of a subsequence still denoted  $(u_n)_{n \in \mathbb{N}^*}$  satisfying  $u_n \xrightarrow[n \rightarrow +\infty]{H^1} u$ .

But (2.40) yields

$$\forall n \geq 2, \quad \frac{1}{n} > \|u_n\|_{0,2,\Gamma}^2 + \left(1 - \frac{1}{n}\right) \|\nabla u_n\|_{0,2,\Omega}^2 \geq \frac{1}{2} N_0(u_n)^2, \quad (2.42)$$

hence  $N_0(u_n) \xrightarrow[n \rightarrow +\infty]{} 0$ , then  $u = 0$ .

Last, by Rellich theorem:  $u_n \xrightarrow[n \rightarrow +\infty]{L^2} 0$ , which implies:  $\|u_n\|_{0,2,\Omega} \xrightarrow[n \rightarrow +\infty]{} 0$ . But we have assumed that  $\forall n \in \mathbb{N}$ ,  $\|u_n\|_{0,2,\Omega} = 1$ , which is a contradiction.  $\square$

### 3 Analysis of the first equation

In this subsection, given  $p \in ]1, +\infty[$  and  $k \in W_0^{1,p}(\Omega)$ , we consider the system:

$$\begin{cases} -\operatorname{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -A(k)\frac{\partial u}{\partial n} = \alpha u & \text{on } \Gamma. \end{cases} \quad (3.1)$$

Where  $f \in L^2(\Omega)$ , and  $\alpha > 0$ . We recall that  $A : \mathbb{R} \rightarrow \mathbb{R}_+$  is bounded from above and below:

$$\forall k \in \mathbb{R}, \quad 0 < \nu \leq A(k) \leq \mu < +\infty. \quad (3.2)$$

#### 3.1 Variational formulation

Let us write the variational form of this equation. To do so, we need the Green formula.

**Theorem 5** (Green formula). *Let  $f \in H^1(\Omega)$  and  $\mathbf{g} \in H^1(\Omega)^N$ . Assume that  $\mathbf{n}$  is the normal vector of  $\Omega$ , then the following formula stands*

$$\int_{\Omega} \operatorname{div}(\mathbf{g})f \, dx = - \int_{\Omega} \mathbf{g} \cdot \nabla f \, dx + \int_{\Gamma} (\mathbf{g} \cdot \mathbf{n})f \, dS(x). \quad (3.3)$$

Let us consider  $w \in H^1(\Omega)$  and multiply the first equation of (3.1) by  $w$ , then integrate.

$$- \int_{\Omega} \operatorname{div}(A(k)\nabla u)w \, dx = \int_{\Omega} fw \, dx. \quad (3.4)$$

The Green formula (3.3) yields

$$\begin{aligned} - \int_{\Omega} \operatorname{div}(A(k)\nabla u)w \, dx &= \int_{\Omega} A(k)\nabla u \cdot \nabla w \, dx - \int_{\Gamma} A(k)\frac{\partial u}{\partial n}w \, dS(x) \\ &= \int_{\Omega} A(k)\nabla u \cdot \nabla w \, dx + \alpha \int_{\Gamma} uw \, dS(x). \end{aligned} \quad (3.5)$$

That leads to the variational formulation of (3.1):

$$\begin{cases} u \in H^1(\Omega) \\ \forall w \in H^1(\Omega), \int_{\Omega} A(k) \nabla u \cdot \nabla w \, dx + \alpha \int_{\Gamma} uw \, dS(x) = \int_{\Omega} fw \, dx. \end{cases} \quad (3.6)$$

All the terms of (3.6) are well defined, since  $u, w \in H^1(\Omega)$ , hence  $\nabla u, \nabla w \in L^2(\Omega)$ . And  $A$  is bounded. Moreover, by the trace theorem (see [7]),  $u, w \in H^1/2(\Gamma) \hookrightarrow L^2(\Gamma)$ .

The main result of this section is the following.

**Theorem 6.** *There exists a unique function  $u \in H^1(\Omega)$  satisfying (3.6).*

*Proof.* This proof uses the Lax-Milgram theorem on the Hilbert space  $H^1(\Omega)$ . Let  $a(k, \cdot, \cdot)$  be the bilinear form given by

$$\forall u, w \in H^1(\Omega), \quad a(k, u, w) = \int_{\Omega} A(k) \nabla u \cdot \nabla w \, dx + \alpha \int_{\Gamma} uw \, dS(x). \quad (3.7)$$

Let  $u, w \in H^1(\Omega)$

$$\begin{aligned} |a(k, u, w)| &\leq \int_{\Omega} A(k) |\nabla u \cdot \nabla w| \, dx + \alpha \int_{\Gamma} |uw| \, dS(x) \\ &\leq \mu \|\nabla u\|_{0,2,\Omega} \|\nabla w\|_{0,2,\Omega} + \alpha \|u\|_{0,2,\Gamma} \|w\|_{0,2,\Gamma} \\ &\leq (\mu + \alpha M) \|u\|_{1,2,\Omega} \|w\|_{1,2,\Omega}, \end{aligned} \quad (3.8)$$

where  $M$  does not depend on  $u$ . Therefore,  $a(k, \cdot, \cdot)$  is continuous on  $H^1(\Omega) \times H^1(\Omega)$ .

Then we show the coercivity. Let  $w \in H^1(\Omega)$ , we have

$$\begin{aligned} a(k, w, w) &= \int_{\Omega} A(k) |\nabla w|^2 \, dx + \alpha \int_{\Gamma} |w|^2 \, dS(x) \geq \nu \|\nabla w\|_{0,2,\Omega}^2 + \alpha \|w\|_{0,2,\Gamma}^2 \\ &\geq \inf(\nu, \alpha) N_0(w)^2, \end{aligned} \quad (3.9)$$

hence the coercivity of  $a(k, \cdot, \cdot)$  by theorem 4.

Finally, it is clear that the linear form  $[w \mapsto \int_{\Omega} fw \, dx]$  is continuous. Problem (3.6) then admits a unique solution  $u \in H^1(\Omega)$ , by the Lax-Milgram theorem.  $\square$

### 3.2 A priori estimates for $u$

Previously, we have shown that for  $k \in W_0^{1,p}(\Omega)$ , (3.6) admits a unique solution  $u = u(k)$ . In this subsection, we give some estimates for this function  $u$ .

**Proposition 7.** *The solution  $u$  of (3.6) satisfies the inequality:*

$$\|u\|_{1,2,\Omega} \leq \frac{C \|f\|_{0,2,\Omega}}{\inf(\alpha, \nu)}. \quad (3.10)$$

Where  $C > 0$  only depends on  $\Omega$ .

*Proof.* Taking  $w = u$  as a test in (3.6) leads to

$$\int_{\Omega} A(k)|\nabla u|^2 dx + \alpha \int_{\Gamma} |u|^2 dS(x) = \int_{\Omega} fu dx. \quad (3.11)$$

We get by Cauchy-Schwarz inequality

$$\nu \|\nabla u\|_{0,2,\Omega}^2 + \alpha \|u\|_{0,2,\Gamma}^2 \leq \|f\|_{0,2,\Omega} \|u\|_{0,2,\Omega}. \quad (3.12)$$

Then by Theorem 4

$$\nu \|\nabla u\|_{0,2,\Omega}^2 + \alpha \|u\|_{0,2,\Gamma}^2 \geq \inf(\nu, \alpha) N_0(u)^2 \geq \frac{\inf(\nu, \alpha)}{C} \|u\|_{1,2,\Omega}^2. \quad (3.13)$$

Then (3.12) becomes

$$\frac{\inf(\nu, \alpha)}{C} \|u\|_{1,2,\Omega}^2 \leq \|f\|_{0,2,\Omega} \times \|u\|_{0,2,\Omega} \leq \|f\|_{0,2,\Omega} \|u\|_{1,2,\Omega}, \quad (3.14)$$

which yields (3.10).  $\square$

**Proposition 8.** Let  $k \in W_0^{1,p}(\Omega)$ , and  $u = u(k)$  the solution of (3.6). Then  $u$  and  $k$  satisfy the inequality

$$\|\sqrt{A(k)}\nabla u(k)\|_{0,2,\Omega} \leq \frac{C\|f\|_{0,2,\Omega}}{\inf(\alpha, \nu)}, \quad (3.15)$$

where  $C > 0$  only depends on  $\Omega$ .

*Proof.* Equality (3.11) can be rewritten with norms

$$\|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \alpha \|u\|_{0,2,\Gamma}^2 = \int_{\Omega} fu dx. \quad (3.16)$$

Let us bound from below the left hand side of (3.16):

$$\begin{aligned} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \alpha \|u\|_{0,2,\Gamma}^2 &= \frac{1}{2} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \frac{1}{2} \int_{\Omega} A(k)|\nabla u|^2 dx + \alpha \|u\|_{0,2,\Gamma}^2 \\ &\geq \frac{1}{2} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \frac{\nu}{2} \|\nabla u\|_{0,2,\Omega}^2 + \alpha \|u\|_{0,2,\Gamma}^2 \\ &\geq \frac{1}{2} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \frac{\inf(\alpha, \nu)}{2} N_0(u)^2 \\ &\geq \frac{1}{2} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 + \frac{\inf(\alpha, \nu)}{2C} \|u\|_{1,2,\Omega}^2. \end{aligned} \quad (3.17)$$

Then, let us apply Young inequality with weights to the right hand side of (3.16):

$$\int_{\Omega} fu dx \leq \|f\|_{0,2,\Omega} \|u\|_{1,2,\Omega} \leq \frac{\inf(\alpha, \nu)}{2C} \|u\|_{1,2,\Omega}^2 + \frac{C}{2\inf(\alpha, \nu)} \|f\|_{0,2,\Omega}^2. \quad (3.18)$$

Thus, (3.16) can be rewritten

$$\frac{1}{2} \|\sqrt{A(k)}u\|_{0,2,\Omega}^2 \leq \frac{C}{2\inf(\alpha, \nu)} \|f\|_{0,2,\Omega}^2, \quad (3.19)$$

which yields (3.15).  $\square$

These estimates will be useful in the following, to show a strong convergence in  $H^1$  for equation (3.1).

### 3.3 A first convergence result: the energy method

This subsection aims at proving the following result.

**Theorem 9** ( $H^1$  convergence). *Let  $p \in ]1, +\infty[$ , and  $(k_n)_{n \in \mathbb{N}}$  be a sequence of functions in  $W_0^{1,p}(\Omega)$ . Let us suppose that  $(k_n)$  weakly converges to  $k$  in  $W_0^{1,p}(\Omega)$ . Let  $u = u(k)$ , and  $u_n = u(k_n)$ ,  $n \in \mathbb{N}$ .*

*Then a subsequence still denoted  $(u_n)_{n \in \mathbb{N}}$  strongly converges to  $u$  in  $H^1(\Omega)$ .*

This theorem is a consequence of another, which involves  $A$  and the previous estimates.

**Theorem 10** ( $L^2$  convergence). *With the same notations and assumptions as those of Theorem 9, we have:*

$$\sqrt{A(k_n)} \nabla u_n \xrightarrow[n \rightarrow +\infty]{L^2} \sqrt{A(k)} \nabla u. \quad (3.20)$$

*Proof of Theorem 10.* As  $k_n \xrightarrow[n \rightarrow +\infty]{W_0^{1,p}} k$ , the compact embedding  $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$  yields  $k_n \xrightarrow[n \rightarrow +\infty]{L^1} k$ . By the Lebesgue inverse theorem, there is a subsequence (still named  $(k_n)$ ) such that  $k_n \xrightarrow[n \rightarrow +\infty]{\text{ae.}} k$ . Since  $A$  is continuous, we deduce  $A(k_n) \xrightarrow[n \rightarrow +\infty]{\text{ae.}} A(k)$ .

We recall that  $u_n$  and  $u$  satisfy

$$\forall \varphi \in H^1(\Omega), \forall n \in \mathbb{N}, \quad \int_{\Omega} A(k_n) \nabla u_n \cdot \nabla \varphi \, dx + \alpha \int_{\Gamma} u_n \varphi \, dS(x) = \int_{\Omega} f \varphi \, dx. \quad (3.21)$$

$$\forall \varphi \in H^1(\Omega), \quad \int_{\Omega} A(k) \nabla u \cdot \nabla \varphi \, dx + \alpha \int_{\Gamma} u \varphi \, dS(x) = \int_{\Omega} f \varphi \, dx. \quad (3.22)$$

Inequality (3.10) yields that  $(u_n)$  is bounded in  $H^1(\Omega)$ , which ensures the existence of a subsequence still denoted  $(u_n)_{n \in \mathbb{N}}$  such that

$$u_n \xrightarrow[n \rightarrow +\infty]{H^1} w \quad \text{and} \quad u_n \xrightarrow[n \rightarrow +\infty]{L^2} w, \quad \text{hence} \quad \nabla u_n \xrightarrow[n \rightarrow +\infty]{L^2} \nabla w, \quad (3.23)$$

where  $w \in H^1(\Omega)$ . This yields in particular

$$\forall \varphi \in H^1(\Omega), \quad \int_{\Gamma} u_n \varphi \, dS(x) \xrightarrow[n \rightarrow +\infty]{} \int_{\Gamma} w \varphi \, dS(x). \quad (3.24)$$

Then, in order to show the convergence of the first term in (3.21), an integration lemma is needed.

**Lemma 11** (Convergence of integrals). *Let  $(f_n)_{n \in \mathbb{N}}$  and  $(g_n)_{n \in \mathbb{N}}$  two sequences of  $L^2(\Omega)$  functions, and  $f, g \in L^2(\Omega)$  satisfying  $f_n \xrightarrow[n \rightarrow +\infty]{L^2} f$  and  $g_n \xrightarrow[n \rightarrow +\infty]{L^2} g$ . Then we have*

$$\int_{\Omega} f_n g_n \, dx \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} f g \, dx. \quad (3.25)$$

This result will be proved after finishing the proof of Theorem 10.

The assumptions of Lemma 11 have to be checked.

- As the sequence  $(A(k_n))_{n \in \mathbb{N}}$  converges almost everywhere to  $A(k)$ , the domination  $A(k_n)|\nabla\varphi| \leq \mu|\nabla\varphi|$  and the Lebesgue theorem yield  $A(k_n)\nabla\varphi \xrightarrow[n \rightarrow +\infty]{L^2} A(k)\nabla\varphi$ .
- By (3.23):  $\nabla u_n \xrightarrow[n \rightarrow +\infty]{L^2} \nabla w$ .

Therefore, Lemma 11 yields

$$\forall \varphi \in H^1(\Omega), \int_{\Omega} A(k_n)\nabla u_n \cdot \nabla\varphi \, dx \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} A(k)\nabla w \cdot \nabla\varphi \, dx. \quad (3.26)$$

Both convergences (3.24) and (3.26) ensure that the limit  $w$  satisfies (3.22). And since the solution of (3.22) is unique (by Theorem 6), we deduce:  $w = u = u(k)$ .

Hence,  $\nabla u_n \xrightarrow[n \rightarrow +\infty]{L^2} \nabla u$ , and (3.15) ensures that there exists a subsequence of  $(\sqrt{A(k_n)}\nabla u_n)_{n \in \mathbb{N}}$  that weakly converges in  $L^2$ . Therefore, Lemma 11 yields

$$\sqrt{A(k_n)}\nabla u_n \xrightarrow[n \rightarrow +\infty]{L^2} \sqrt{A(k)}\nabla u. \quad (3.27)$$

In order to obtain the strong convergence of this sequence, the convergence of the norms remains to be proved. Let us write (3.21) with  $\varphi = u_n$  and (3.22) with  $\varphi = u$ .

$$\forall n \in \mathbb{N}, \int_{\Omega} A(k_n)|\nabla u_n|^2 \, dx + \alpha \int_{\Gamma} |u_n|^2 \, dS(x) = \int_{\Omega} f u_n \, dx. \quad (3.28)$$

$$\int_{\Omega} A(k)|\nabla u|^2 \, dx + \alpha \int_{\Gamma} |u|^2 \, dS(x) = \int_{\Omega} f u \, dx. \quad (3.29)$$

The strong convergence of  $(u_n)$  in  $L^2$  directly yields

$$\int_{\Omega} f u_n \, dx \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} f u \, dx. \quad (3.30)$$

And since  $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$  is compact:  $(u_n)|_{\Gamma} \xrightarrow[n \rightarrow +\infty]{L^2} u|_{\Gamma}$ . In particular

$$\int_{\Gamma} |u_n|^2 \, dS(x) \xrightarrow[n \rightarrow +\infty]{} \int_{\Gamma} |u|^2 \, dS(x). \quad (3.31)$$

Convergences (3.30) and (3.31) yield:

$$\int_{\Omega} A(k_n)|\nabla u_n|^2 \, dx \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} A(k)|\nabla u|^2 \, dx. \quad (3.32)$$

The result of Theorem 10 is then obtained.  $\square$

Finally, Lemma 11 needs to be proved.

*Proof of Lemma 11.* Let us decompose the difference between the two integrals:

$$\int_{\Omega} (f_n g_n - f g) dx = \int_{\Omega} f (g_n - g) dx + \int_{\Omega} (f_n - f) g_n dx. \quad (3.33)$$

On one hand, the weak convergence of  $(g_n)$  yields:  $\int_{\Omega} f (g_n - g) dx \xrightarrow{n \rightarrow +\infty} 0$ .

On the other hand,  $(g_n)_{n \in \mathbb{N}}$  weakly converges, then is bounded in  $L^2(\Omega)$ . We deduce

$$\left| \int_{\Omega} (f_n - f) g_n dx \right| \leq \|f_n - f\|_{0,2,\Omega} \|g_n\|_{0,2,\Omega} \leq M \|f_n - f\|_{0,2,\Omega} \xrightarrow{n \rightarrow +\infty} 0. \quad (3.34)$$

□

Theorem 9 can be proved now.

*Proof of Theorem 9.* We have already shown that  $u_n \xrightarrow{n \rightarrow +\infty} u$ , we have to prove that  $(\nabla u_n)$  strongly converges. Let  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \sqrt{\nu} \|\nabla u_n - \nabla u\|_{0,2,\Omega} &\leq \left\| \sqrt{A(k_n)} \nabla u_n - \sqrt{A(k_n)} \nabla u \right\|_{0,2,\Omega} \\ &\leq \left\| \sqrt{A(k_n)} \nabla u_n - \sqrt{A(k)} \nabla u \right\|_{0,2,\Omega} + \left\| \left( \sqrt{A(k_n)} - \sqrt{A(k)} \right) \nabla u \right\|_{0,2,\Omega} \xrightarrow{n \rightarrow +\infty} 0, \end{aligned} \quad (3.35)$$

by Theorem 10 and Lebesgue Theorem. □

## 4 Analysis of the second equation

From now until the end,  $N = 3$ . In this section, we consider the system:

$$\begin{cases} -\operatorname{div}(B(k) \nabla k) = A(k) |\nabla u(k)|^2 & \text{in } \Omega, \\ k = 0 & \text{on } \Gamma. \end{cases} \quad (4.1)$$

In order to lighten the writing, we define:

$$D(k) = A(k) |\nabla u(k)|^2. \quad (4.2)$$

### 4.1 Variational formulation and description of the method

Let us define two function spaces:

$$V = \bigcap_{1 < r < N'} W_0^{1,r}(\Omega) \quad \text{and} \quad W = \bigcup_{r > N} W_0^{1,r}(\Omega).$$

Before stating the variational formulation, we need a result.

**Proposition 12.** *Let  $k \in V$  and  $p \in W$ . Then  $B(k) \nabla k \cdot \nabla p \in L^1(\Omega)$  and  $D(k)p \in L^1(\Omega)$ .*

*Proof.* Let  $k \in V$  and  $p \in W$ . By applying the Hölder inequality with  $r > N$  such that  $p \in W_0^{1,r}(\Omega)$ , we obtain

$$\int_{\Omega} B(k)|\nabla k \cdot \nabla p| dx \leq \mu \int_{\Omega} |\nabla k \cdot \nabla p| dx \leq \mu \|\nabla k\|_{0,r',\Omega} \times \|\nabla p\|_{0,r,\Omega}. \quad (4.3)$$

Since  $r > N$ ,  $r' < N'$ , thus  $k \in W_0^{1,r'}(\Omega)$ . This yields the first result.

The second result comes from (3.15), that yields:  $\|D(k)\|_{0,1,\Omega} \leq \left(\frac{C\|f\|_{0,2,\Omega}}{\inf(\alpha,\nu)}\right)^2$ , then  $D(k) \in L^1(\Omega)$  and  $p \in L^\infty(\Omega)$ , by Sobolev embedding. Hence  $D(k)p \in L^1(\Omega)$ .  $\square$

It is important to notice that if  $N = 3$ ,  $V \hookrightarrow L^2(\Omega)$ . Indeed, we have the equivalence

$$W^{1,q}(\Omega) \hookrightarrow L^2(\Omega) \Leftrightarrow q^* > 2 \Leftrightarrow \frac{1}{q} < \frac{1}{2} + \frac{1}{N} \Leftrightarrow q > \frac{2N}{N+2}.$$

Then the condition that needs to be satisfied is:  $N' > \frac{2N}{N+2}$ , which is equivalent to  $N < 4$ .

We are now able to state the variational formulation of (4.1) ;  $k$  is called a weak solution of (4.1) if

$$\begin{cases} k \in V \\ \forall p \in W, \int_{\Omega} B(k)\nabla k \cdot \nabla p dx = \int_{\Omega} D(k)p dx. \end{cases} \quad (4.4)$$

As a priori  $D(k) \in L^1(\Omega)$ , it is not possible to solve directly (4.4). Indeed, the method that will be used consists in bringing (4.4) back to a Lax-Milgram problem, which requires being in a Hilbert space. Yet,  $W^{1,p}(\Omega)$  with  $p \neq 2$  is not a Hilbert space. And  $D(k)$  is not in  $L^2(\Omega)$ .

More precisely, the method is divided into three steps:

- Regularizing the equation, by changing the source term.
- Linearizing it to solve the regularized problem with a fixed-point theorem.
- Taking the limit when the regularizing parameter goes to zero.

## 4.2 Two first steps: solving the regularized problem

In order to set the regularized version of (4.4), several tools must be defined.

**Definition 1.** If  $w \in L^r(\Omega)$  with  $r \in [1, +\infty]$ ,  $\tilde{w} \in L^r(\mathbb{R}^N)$  refers to the extension by zero

$$\tilde{w} : x \mapsto \begin{cases} w(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

**Definition 2** (Mollifier). Let  $\rho : \mathbb{R}^N \rightarrow \mathbb{R}$  a function such that  $\text{Supp}(\rho) = \bar{B}(0, 1)$ ,  $\rho \geq 0$  and  $\int_{\mathbb{R}^N} \rho(y)dy = 1$ . We define the family  $(\rho_\varepsilon)_{\varepsilon>0}$  by

$$\forall x \in \mathbb{R}^N, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right).$$



Let  $\varepsilon > 0$ , and  $D_\varepsilon(k)$  given by:  $D_\varepsilon(k) = A(k) \left| \widetilde{\nabla u(k)} * \rho_\varepsilon \right|^2$ , and the following regularized problem:

$$\begin{cases} k \in H_0^1(\Omega) \\ \forall p \in H_0^1(\Omega), \quad \int_{\Omega} B(k) \nabla k \cdot \nabla p \, dx = \int_{\Omega} D_\varepsilon(k) p \, dx. \end{cases} \quad (4.5)$$

Linearizing is only useful if the assumptions of the Lax-Milgram theorem are satisfied. This is what will be proved now, beginning by a continuity result on the right hand side of (4.5).

**Theorem 13.** *Let  $\varepsilon > 0$  and  $D_\varepsilon(k)$  defined as in (4.5). Then  $D_\varepsilon(k) \in L^2(\Omega)$  and there exists a constant  $C > 0$  only depending of the data of the problem such that*

$$\forall k \in H_0^1(\Omega), \quad \|D_\varepsilon(k)\|_{0,2,\Omega} \leq \frac{C}{\varepsilon^2}. \quad (4.6)$$

The proof of Theorem 13 is based on a general result.

**Lemma 14.** *Let  $v \in H^m(\Omega)$ ,  $\ell \in \mathbb{N}$  and  $(\rho_\varepsilon)_{\varepsilon>0}$  a mollifier defined as previously. For  $\varepsilon > 0$  sufficiently close to 0, we have*

$$\|v * \rho_\varepsilon\|_{m+\ell,2} \leq \frac{C(m,\ell)}{\varepsilon^\ell} \|v\|_{m,2}. \quad (4.7)$$

*Proof of Lemma 14.* Let us recall the definition of the Sobolev norm  $\|v * \rho_\varepsilon\|_{m+\ell,2}^2 = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq \ell}} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2$ . Let  $\alpha, \beta \in \mathbb{N}^d$  such that  $|\alpha| \leq m$  and  $|\beta| \leq \ell$ .

Plancherel theorem yields

$$\|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2 = \frac{1}{(2\pi)^N} \|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{0,2}^2, \quad (4.8)$$

where  $\mathcal{F}$  refers to the Fourier transform operator.

Since  $v \in H^m(\Omega)$ , the following derivation formula stands  $\partial^{\alpha+\beta}(v * \rho_\varepsilon) = (\partial^\alpha v) * (\partial^\beta \rho_\varepsilon)$ . And by Fourier transform, we get

$$\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon)) = \mathcal{F}(\partial^\alpha v) \mathcal{F}(\partial^\beta \rho_\varepsilon). \quad (4.9)$$

Then with norms:

$$\|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{0,2}^2 = \int_{\mathbb{R}^N} |\mathcal{F}(\partial^\alpha v)(\xi)|^2 \cdot |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 d\xi. \quad (4.10)$$

The definition of  $\rho_\varepsilon$  with respect to  $\rho$  yields, by a change of variables

$$\mathcal{F}(\rho_\varepsilon)(\xi) = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} e^{-ix\xi} \rho\left(\frac{x}{\varepsilon}\right) dx = \int_{\mathbb{R}^N} e^{-i\varepsilon y \xi} \rho(y) dy = \mathcal{F}(\rho)(\varepsilon\xi). \quad (4.11)$$

Then we obtain

$$\begin{aligned} |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 &= |\xi^\beta \mathcal{F}(\rho_\varepsilon)(\xi)|^2 = |\xi^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 \\ &= \frac{1}{\varepsilon^{2|\beta|}} |(\varepsilon\xi)^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 = \frac{1}{\varepsilon^{2|\beta|}} |\mathcal{F}(\partial^\beta \rho)(\varepsilon\xi)|^2. \end{aligned} \quad (4.12)$$

Yet,  $\partial^\beta \rho \in \mathcal{D}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ , which yields  $\forall \xi \in \mathbb{R}^N$ ,  $|\mathcal{F}(\partial^\beta \rho)(\xi)| \leq \|\partial^\beta \rho\|_{0,1}$ . This reported in the inequality (4.10) ensure the existence of a constant  $c_0 > 0$  independent of  $v$  such that

$$\|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\mathcal{F}(\partial^\alpha v)\|_{0,2}^2. \quad (4.13)$$

Plancherel Theorem then yields

$$\|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\partial^\alpha v\|_{0,2}^2. \quad (4.14)$$

By addition for  $|\alpha| \leq m$ :

$$\sum_{|\alpha| \leq m} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|v\|_{m,2}^2. \quad (4.15)$$

Taking  $\varepsilon \in ]0, 1[$  yields  $\frac{1}{\varepsilon^{2|\beta|}} \leq \frac{1}{\varepsilon^{2\ell}}$ . Thus, adding for  $|\beta| \leq \ell$  brings the inequality

$$\|v * \rho_\varepsilon\|_{m+\ell,2}^2 \leq \frac{c(\ell)}{\varepsilon^{2\ell}} \|v\|_{m,2}^2, \quad (4.16)$$

hence (4.7).  $\square$

We can now apply this result to prove Theorem 13.

*Proof of Theorem 13.* The definition of  $D_\varepsilon$  yields

$$\|D_\varepsilon(k)\|_{0,2,\Omega} = \left( \int_\Omega |\sqrt{A(k)} \widetilde{\nabla u(k)} * \rho_\varepsilon|^4 dx \right)^{1/2} = \left\| \sqrt{A(k)} \widetilde{\nabla u(k)} * \rho_\varepsilon \right\|_{0,4,\Omega}^2. \quad (4.17)$$

And since  $N = 3$ , the Sobolev embedding  $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega)$  yields

$$\left\| \sqrt{A(k)} \widetilde{\nabla u(k)} * \rho_\varepsilon \right\|_{0,4,\Omega} \leq c_1 \left\| \sqrt{A(k)} \widetilde{\nabla u(k)} * \rho_\varepsilon \right\|_{1,2,\Omega} \leq \frac{c_2}{\varepsilon} \left\| \sqrt{A(k)} \nabla u(k) \right\|_{0,2,\Omega}. \quad (4.18)$$

Finally, (3.15) yields the result.  $\square$

In order to solve the regularized problem (4.5), we first linearize it, as said before.

$$\begin{cases} q = q(k) \in H_0^1(\Omega) \\ \forall p \in H_0^1(\Omega), \int_\Omega B(k) \nabla q \cdot \nabla p dx = \int_\Omega D_\varepsilon(k) p dx. \end{cases} \quad (4.19)$$

We have already shown that  $[p \mapsto \int_\Omega D_\varepsilon(k) p dx]$  is a continuous linear form on  $H_0^1(\Omega)$ , and then it is clear that  $[(q, p) \mapsto \int_\Omega B(k) \nabla q \cdot \nabla p dx]$  is a bilinear form on  $H_0^1(\Omega)$ .

- Let  $q, p \in H_0^1(\Omega)$ :

$$\left| \int_\Omega B(k) \nabla q \cdot \nabla p dx \right| \leq \mu \|\nabla q\|_{0,2,\Omega} \|\nabla p\|_{0,2,\Omega} \leq \mu \|q\|_{1,2,\Omega} \|p\|_{1,2,\Omega}. \quad (4.20)$$

Then continuity is proved.

- Let  $p \in H_0^1(\Omega)$ :

$$\begin{aligned} \int_{\Omega} B(k)|\nabla p|^2 dx &\geq \nu \|\nabla p\|_{0,2,\Omega}^2 \geq \frac{\nu}{2} \left( \|\nabla p\|_{0,2,\Omega}^2 + \frac{1}{c_{\Omega}^2} \|p\|_{0,2,\Omega}^2 \right) \\ &\geq \frac{\nu}{2} \inf \left( 1, \frac{1}{c_{\Omega}^2} \right) \|p\|_{1,2,\Omega}^2. \end{aligned} \quad (4.21)$$

Where  $c_{\Omega}$  refers to the Poincaré-Friedrichs constant for  $\Omega$ . Then coercivity is proved.

Until the end,  $c_{\Omega}$  keeps the same meaning than in (4.21).

Consequently, problem (4.19) is a Lax-Milgram problem, hence by the same-named theorem, (4.19) has a unique solution  $q = q(k) \in H_0^1(\Omega)$ . Our goal is applying the Leray-Schauder fixed-point theorem on the operator  $[k \mapsto q(k)]$ .

Taking  $p = q(k)$  in (4.19) yields

$$\int_{\Omega} B(k)|\nabla q(k)|^2 dx = \int_{\Omega} D_{\varepsilon}(k)q(k) dx, \text{ hence } \nu \|\nabla q(k)\|_{0,2,\Omega} \leq c_{\Omega} \|D_{\varepsilon}(k)\|_{0,2,\Omega}. \quad (4.22)$$

Then Theorem 13 yields

$$\|\nabla q(k)\|_{0,2,\Omega} \leq \frac{c_{\Omega} \|D_{\varepsilon}(k)\|_{0,2,\Omega}}{\nu} \leq \frac{c_{\Omega} C}{\nu \varepsilon^2} =: M. \quad (4.23)$$

The constant  $M > 0$  does not depend on  $k$ . Since  $\Omega$  is bounded, the norms  $\|\nabla \cdot\|_{0,2,\Omega}$  and  $\|\cdot\|_{1,2,\Omega}$  are equivalent.

The end of this subpart aims at showing the existence of a solution to (4.5). This needs the Leray-Schauder fixed-point theorem (see [9]).

**Theorem 15** (Leray-Schauder). *Let  $X$  a Banach space, and  $B$  a convex closed bounded nonempty subset of  $X$ . Let  $f : B \rightarrow B$  a continuous function with  $f(B)$  compact.*

*Then  $f$  admits at least one fixed point in  $B$ .*

In our problem, we set  $X = H_0^1(\Omega)$ ,  $B = \bar{B}(0, M)$  and the function

$$Q : \begin{array}{l} B \rightarrow B \\ k \mapsto q(k) \end{array}.$$

It is clear that  $B$  is a convex closed bounded nonempty subset of  $H_0^1(\Omega)$ , and  $Q(B) \subset B$ . Remains to be proved that  $Q$  is continuous and  $Q(B)$  is compact.

**Let us show that  $Q(B)$  is compact.** Let  $(k_n)_{n \in \mathbb{N}}$  a sequence of elements of  $B$ . Since  $(k_n)$  is bounded in  $H^1(\Omega)$ , a subsequence still named  $(k_n)$  can be extracted, which satisfies:  $k_n \xrightarrow[n \rightarrow +\infty]{H^1} k$ . By theorem 10 with  $p = 2$ , this weak convergence is sufficient to obtain (3.20). The Young convolution inequality consequently yields

$$\begin{aligned} & \|\sqrt{A(k_n)}\nabla u(k_n) * \rho_\varepsilon - \sqrt{A(k)}\nabla u(k) * \rho_\varepsilon\|_{0,2} \\ & \leq \|\sqrt{A(k_n)}\nabla u(k_n) - \sqrt{A(k)}\nabla u(k)\|_{0,2} \times \underbrace{\|\rho_\varepsilon\|_{0,1}}_{=1} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned} \quad (4.24)$$

This strong convergence in  $L^2$  and the Lebesgue inverse theorem ensure the existence of a subsequence of  $(k_n)_{n \in \mathbb{N}}$  still named the same way satisfying

$$\sqrt{A(k_n)}\nabla u(k_n) * \rho_\varepsilon \xrightarrow[n \rightarrow +\infty]{\text{ae.}} \sqrt{A(k)}\nabla u(k) * \rho_\varepsilon, \text{ hence } D_\varepsilon(k_n) \xrightarrow[n \rightarrow +\infty]{\text{ae.}} D_\varepsilon(k). \quad (4.25)$$

By theorem 9,  $(u(k_n))_{n \in \mathbb{N}}$  strongly converges in  $H^1(\Omega)$ , then  $(\nabla u(k_n))_{n \in \mathbb{N}}$  strongly converges in  $L^2(\Omega)^N$ . Thus, the Lebesgue inverse theorem yields the existence of  $h \in L^2(\Omega)$  such that

$$\forall n \in \mathbb{N}, \quad |\nabla u(k_n)| \leq h \text{ ae. } \Omega, \quad (4.26)$$

where  $(k_n)_{n \in \mathbb{N}}$  refers to another subsequence. Therefore, we obtain

$$|\widetilde{\nabla u(k_n)} * \rho_\varepsilon(x)| \leq \tilde{h} * \rho_\varepsilon(x). \quad (4.27)$$

And since  $h$  does not depend on  $n$  and  $h * \rho_\varepsilon$  is continuous with compact support,  $D_\varepsilon(k_n)$  is bounded by a constant that does not depend on  $n$ , then (4.25) and Lebesgue theorem yield

$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} D_\varepsilon(k_n)p \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} D_\varepsilon(k)p \, dx. \quad (4.28)$$

Reporting (4.28) in (4.19) yields:

$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} B(k_n)\nabla q(k_n) \cdot \nabla p \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} B(k)\nabla q(k) \cdot \nabla p \, dx. \quad (4.29)$$

We have proved that  $B(k_n)\nabla q(k_n) \xrightarrow[n \rightarrow +\infty]{L^2} B(k)\nabla q(k)$ . The Lebesgue inverse theorem and the continuity of  $1/B$  yield  $\frac{1}{B(k_n)} \xrightarrow[n \rightarrow +\infty]{\text{ae.}} \frac{1}{B(k)}$ , and in addition  $\frac{1}{B(k_n)}|\nabla p| \leq \frac{1}{\nu}|\nabla p|$ . We then obtain

$$\frac{1}{B(k_n)}\nabla p \xrightarrow[n \rightarrow +\infty]{L^2} \frac{1}{B(k)}\nabla p. \quad (4.30)$$

The assumptions of Lemma 11 are satisfied, so that it yields

$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} \nabla q(k_n) \cdot \nabla p \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla q(k) \cdot \nabla p \, dx, \text{ hence } \nabla q(k_n) \xrightarrow[n \rightarrow +\infty]{L^2} \nabla q(k). \quad (4.31)$$

We deduce by compact embedding  $q(k_n) \xrightarrow[n \rightarrow +\infty]{L^2} q(k)$ . We now intend to show

$$\int_{\Omega} D_\varepsilon(k_n)q(k_n) \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} D_\varepsilon(k)q(k) \, dx. \quad (4.32)$$

Another decomposition is needed:

$$\begin{aligned} \|D_\varepsilon(k_n)q(k_n) - D_\varepsilon(k)q(k)\|_{0,1,\Omega} & \leq \|(D_\varepsilon(k_n) - D_\varepsilon(k))q(k)\|_{0,1,\Omega} \\ & \quad + \|D_\varepsilon(k_n)(q(k_n) - q(k))\|_{0,1,\Omega}. \end{aligned} \quad (4.33)$$

On the one hand, the Lebesgue theorem yields  $\|(D_\varepsilon(k_n) - D_\varepsilon(k))q(k)\|_{0,1,\Omega} \xrightarrow{n \rightarrow +\infty} 0$ .

On the other hand, there exist two constants  $c_1$  and  $c_2$  due to the domination of  $(D_\varepsilon(k_n))$  and the embedding  $L^2(\Omega) \hookrightarrow L^1(\Omega)$  such that

$$\|D_\varepsilon(k_n)(q(k_n) - q(k))\|_{0,1,\Omega} \leq c_1 \|q(k_n) - q(k)\|_{0,1,\Omega} \leq c_2 \|q(k_n) - q(k)\|_{0,2,\Omega} \xrightarrow{n \rightarrow +\infty} 0. \quad (4.34)$$

Hence, (4.32) is proved. Reporting in (4.19) yields

$$\int_{\Omega} B(k_n) |\nabla q(k_n)|^2 dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} B(k) |\nabla q(k)|^2 dx. \quad (4.35)$$

We need a last decomposition:

$$\begin{aligned} \int_{\Omega} (B(k_n) |\nabla q(k_n)|^2 - B(k) |\nabla q(k)|^2) dx &= \int_{\Omega} B(k) (|\nabla q(k_n)|^2 - |\nabla q(k)|^2) dx \\ &+ \int_{\Omega} (B(k_n) - B(k)) |\nabla q(k_n)|^2 dx. \end{aligned} \quad (4.36)$$

Given that the sequence  $(B(k_n) |\nabla q(k_n)|^2)_{n \in \mathbb{N}}$  is bounded, (4.35) yields for another subsequence still named  $(k_n)_{n \in \mathbb{N}}$ :

$$\sqrt{B(k_n)} \nabla q(k_n) \xrightarrow[n \rightarrow +\infty]{L^2} \sqrt{B(k)} \nabla q(k). \quad (4.37)$$

Then, by the Lebesgue inverse theorem, a subsequence still named the same way is dominated. That means  $(|\nabla q(k_n)|^2)_{n \in \mathbb{N}}$  is dominated by a  $L^1$  function. Then the Lebesgue theorem yields

$$\int_{\Omega} (B(k_n) - B(k)) |\nabla q(k_n)|^2 dx \xrightarrow{n \rightarrow +\infty} 0. \quad (4.38)$$

Consequently, (4.36) yields

$$\int_{\Omega} B(k) (|\nabla q(k_n)|^2 - |\nabla q(k)|^2) dx \xrightarrow{n \rightarrow +\infty} 0. \quad (4.39)$$

Let  $N_B$  be defined by

$$N_B : f \in H_0^1(\Omega) \mapsto \|B(k) \nabla f\|_{0,2,\Omega}. \quad (4.40)$$

The two norms  $\|\cdot\|_{1,2,\Omega}$  and  $N_B$  are equivalent, and (4.39) ensures that  $(q(k_n))_{n \in \mathbb{N}}$  strongly converges for the norm  $N_B$ . Thus, it also strongly converges in  $H_0^1(\Omega)$ .

We obtain

$$q(k_n) \xrightarrow[n \rightarrow +\infty]{H^1} q(k). \quad (4.41)$$

Therefore, we have shown that  $Q(B)$  is compact, and by a very similar way, the continuity of  $Q$  is deduced. Consequently, Theorem 15 ensures the existence of a fixed-point of  $Q$  in  $B$ . Which means there exists a function  $k \in B$  that satisfies (4.5).

### 4.3 Last step: Taking the limit when $\varepsilon$ goes to zero

Let us define a family of intermediary solutions with respect to what has just been done. For  $\varepsilon > 0$ , the intermediary solution  $k_\varepsilon$  satisfies

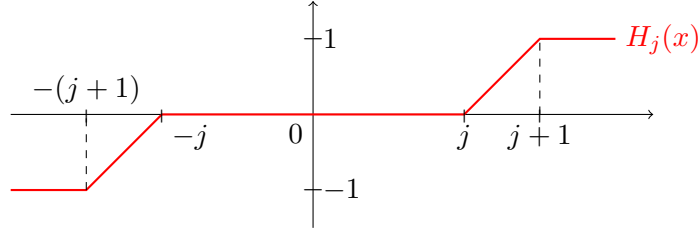
$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} B(k_\varepsilon) \nabla k_\varepsilon \cdot \nabla p \, dx = \int_{\Omega} D_\varepsilon(k_\varepsilon) p \, dx. \quad (4.42)$$

Our next goal is showing that  $(k_\varepsilon)$  is bounded uniformly with respect to  $\varepsilon$ . For this we use the washer lemma, (Lemma 1). But a general theorem from Stampacchia (see [8]) is needed before.

**Theorem 16** (Regularity and truncation function). *Let  $H \in W^{1,\infty}(\mathbb{R})$  having a finite number of discontinuities, and such that  $H(0) = 0$ . Let  $q \in [1, +\infty[$  and  $v \in W_0^{1,q}(\Omega)$ .*

*Then  $H(v) \in W_0^{1,q}(\Omega)$  and we have:  $\nabla H(v) = H'(v) \nabla v$ .*

Until the end of this chapter, for  $j \in \mathbb{N}$ ,  $H_j$  refers to the truncation function defined by



This function satisfies the assumptions of 16, so that we can take  $p = H_j(k_\varepsilon)$  in (4.42) to obtain

$$\forall j \in \mathbb{N}, \quad \int_{\Omega} B(k_\varepsilon) |\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) \, dx = \int_{\Omega} D_\varepsilon(k_\varepsilon) H_j(k_\varepsilon) \, dx. \quad (4.43)$$

On one hand, by writing  $B_j^\varepsilon = \{x \in \Omega / j \leq |k_\varepsilon(x)| < j+1\}$ :

$$\int_{\Omega} B(k_\varepsilon) |\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) \, dx = \int_{B_j^\varepsilon} B(k_\varepsilon) |\nabla k_\varepsilon|^2 \, dx \geq \nu \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2 \, dx. \quad (4.44)$$

On the other hand, the Young convolution inequality yields

$$\int_{\Omega} D_\varepsilon(k_\varepsilon) \underbrace{H_j(k_\varepsilon)}_{\leq 1} \, dx \leq \left\| \sqrt{A(k_\varepsilon)} \widetilde{\nabla u(k_\varepsilon)} * \rho_\varepsilon \right\|_{0,2}^2 \leq \left\| \sqrt{A(k_\varepsilon)} \nabla u(k_\varepsilon) \right\|_{0,2,\Omega}^2. \quad (4.45)$$

By (3.15), the right hand side is uniformly bounded with respect to  $\varepsilon$ . Both inequalities (4.44) and (4.45) reported in (4.43) yield

$$\exists C > 0, \forall \varepsilon > 0, \forall j \in \mathbb{N}, \quad \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2 \, dx \leq C. \quad (4.46)$$

This corresponds to the assumptions of lemma 1. Then for any exponent  $q \in ]1, N'[,$  we have:  $\|k_\varepsilon\|_{1,q,\Omega} \leq A$ . And given that  $C$  does not depend on  $\varepsilon$ , this is the same for  $A$  (as a consequence of lemma 1).

Therefore,  $(k_\varepsilon)_{\varepsilon>0}$  is bounded in  $W^{1,q}(\Omega)$  for  $q \in ]1, N'[,$  Banach-Alaoglu theorem yields the existence of a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  such that  $\varepsilon_n \xrightarrow{n \rightarrow +\infty} 0$  and  $k_{\varepsilon_n} \xrightarrow[n \rightarrow +\infty]{W_0^{1,q}} k$ . Therefore,  $k \in V$ .

We need to show that  $k$  is a solution of (4.4), which means we need to show

$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} B(k) \nabla k \cdot \nabla p \, dx = \int_{\Omega} D(k) p \, dx. \quad (4.47)$$

The intermediary solutions  $(k_{\varepsilon_n})$  satisfy

$$\forall p \in H_0^1(\Omega), \quad \int_{\Omega} B(k_{\varepsilon_n}) \nabla k_{\varepsilon_n} \cdot \nabla p \, dx = \int_{\Omega} D_{\varepsilon_n}(k_{\varepsilon_n}) p \, dx. \quad (4.48)$$

**Convergence of the left hand side:**

We have supposed that  $k_{\varepsilon_n} \xrightarrow[n \rightarrow +\infty]{W_0^{1,q}} k$ , which means in particular

$$\forall p \in W, \quad \int_{\Omega} \nabla k_{\varepsilon_n} \cdot \nabla p \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla k \cdot \nabla p \, dx. \quad (4.49)$$

To add the term  $B(k_n)$ , it seems useful to state a variant of Lemma 11.

**Lemma 17** (Double convergence in  $V$  and  $W$ ). *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ ,  $f \in \bigcap_{1 < r < N'} L^r(\Omega)$  and  $g \in L^q(\Omega)$ , where  $q > N$ . Let us consider two sequences  $(f_n)_{n \in \mathbb{N}}$  and  $(g_n)_{n \in \mathbb{N}}$  such that*

$$\left\{ \begin{array}{l} \forall r \in ]1, N'[, \quad f_n \xrightarrow[n \rightarrow +\infty]{L^r} f \\ g_n \xrightarrow[n \rightarrow +\infty]{L^q} g \end{array} \right. \quad (4.50)$$

*Then we have the convergence*

$$\int_{\Omega} f_n g_n \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} f g \, dx. \quad (4.51)$$

*Proof.* The proof of Lemma 17 is similar to the proof of Lemma 11. We write

$$\int_{\Omega} (f_n g_n - f g) \, dx = \int_{\Omega} (f_n - f) g_n \, dx + \int_{\Omega} f (g_n - g) \, dx. \quad (4.52)$$

On one hand, the Hölder inequality yields

$$\left| \int_{\Omega} (f_n - f) g_n \, dx \right| \leq \|f_n - f\|_{0,q',\Omega} \|g_n\|_{0,q,\Omega} \leq M \|f_n - f\|_{0,q',\Omega} \xrightarrow{n \rightarrow +\infty} 0, \quad (4.53)$$

where  $q'$  refers to the conjugate of  $q$ , which is in  $]1, N'[,$

On the other hand, the weak convergence of  $(g_n)_{n \in \mathbb{N}}$  yields

$$\int_{\Omega} (g_n - g) f \, dx \xrightarrow{n \rightarrow +\infty} 0. \quad (4.54)$$

Hence the result.  $\square$

Let us now check if the assumptions of Lemma 17 are satisfied.

The compact embedding  $V \hookrightarrow L^1(\Omega)$  yields  $k_{\varepsilon_n} \xrightarrow[n \rightarrow +\infty]{L^1} k$ . Thus, the Lebesgue inverse theorem yields  $k_{\varepsilon_n} \xrightarrow[n \rightarrow +\infty]{\text{ae.}} k$ , where  $(\varepsilon_n)$  refers to a subsequence. Hence, by continuity of  $B$ :  $B(k_{\varepsilon_n}) \xrightarrow[n \rightarrow +\infty]{\text{ae.}} B(k)$ . And the domination  $B(k_{\varepsilon_n})|\nabla p| \leq \mu|\nabla p|$  stands. Consequently, the Lebesgue theorem yields

$$\forall r \in ]1, N'[ , \forall p \in W, \quad B(k_{\varepsilon_n})\nabla p \xrightarrow[n \rightarrow +\infty]{L^r} B(k)\nabla p. \quad (4.55)$$

And the weak convergence of  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  is already established. Then, Lemma 17 yields the convergence of the left hand side

$$\forall p \in W, \quad \int_{\Omega} B(k_{\varepsilon_n})\nabla k_{\varepsilon_n} \cdot \nabla p \, dx \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} B(k)\nabla k \cdot \nabla p \, dx. \quad (4.56)$$

### Convergence of the right hand side:

We aim at applying the Lebesgue inverse theorem on  $(D_{\varepsilon_n}(k_{\varepsilon_n}))$ , so we need to prove

$$\forall p \in W, \quad D_{\varepsilon_n}(k_{\varepsilon_n})p \xrightarrow[n \rightarrow +\infty]{L^1} D(k)p. \quad (4.57)$$

Yet, it is clear that for  $n \in \mathbb{N}$  and  $p \in W$

$$\|D_{\varepsilon_n}(k_{\varepsilon_n})p - D(k)p\|_{0,1,\Omega} = \|D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)\|_{0,1,\Omega} \|p\|_{0,\infty,\Omega}. \quad (4.58)$$

Then we write a decomposition.

$$\begin{aligned} & \int_{\Omega} |D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)| \, dx = \int_{\mathbb{R}^N} \left| A(k_{\varepsilon_n})|\widetilde{\nabla u(k_{\varepsilon_n}) * \rho_{\varepsilon_n}}|^2 - A(k)|\widetilde{\nabla u(k)}|^2 \right| \, dx \\ & \leq \int_{\Omega} |A(k_{\varepsilon_n}) - A(k)| \cdot |\nabla u(k)|^2 \, dx + \int_{\mathbb{R}^N} A(k_{\varepsilon_n}) \left| |\widetilde{\nabla u(k_{\varepsilon_n}) * \rho_{\varepsilon_n}}|^2 - |\widetilde{\nabla u(k)}|^2 \right| \, dx. \end{aligned} \quad (4.59)$$

The same argument as in the proof of (4.55) yields

$$\int_{\Omega} |A(k_{\varepsilon_n}) - A(k)| \cdot |\nabla u(k)|^2 \, dx \xrightarrow[n \rightarrow +\infty]{} 0. \quad (4.60)$$

It remains to prove the following

$$\widetilde{\nabla u(k_{\varepsilon_n}) * \rho_{\varepsilon_n}} \xrightarrow[n \rightarrow +\infty]{L^2} \widetilde{\nabla u(k)}. \quad (4.61)$$

First we write

$$\begin{aligned} \left\| \widetilde{\nabla u(k_{\varepsilon_n}) * \rho_{\varepsilon_n}} - \widetilde{\nabla u(k)} \right\|_{0,2} & \leq \left\| \widetilde{\nabla u(k_{\varepsilon_n}) * \rho_{\varepsilon_n}} - \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} \right\|_{0,2} \\ & \quad + \left\| \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} \right\|_{0,2}. \end{aligned} \quad (4.62)$$



On one hand, the Young convolution inequality yields

$$\left\| \widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} \right\|_{0,2} \leq \|\nabla u(k_{\varepsilon_n}) - \nabla u(k)\|_{0,2,\Omega} \xrightarrow{n \rightarrow +\infty} 0, \quad (4.63)$$

as a consequence of theorem 9.

On the other hand, a classical convergence result has to be recalled.

**Theorem 18.** *Let  $v$  a  $L^p(\mathbb{R}^N)$  function with compact support ( $1 \leq p < +\infty$ ), and  $(\rho_\varepsilon)$  a mollifier. Then*

$$\|v * \rho_\varepsilon - v\|_{0,p} \xrightarrow{n \rightarrow +\infty} 0. \quad (4.64)$$

Theorem 18 applied to  $\widetilde{\nabla u(k)}$  brings the following convergence

$$\left\| \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} \right\|_{0,2} \xrightarrow{n \rightarrow +\infty} 0. \quad (4.65)$$

This finally yields (4.61).

There is a subsequence, still named  $(\varepsilon_n)$ , such that  $|\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 \xrightarrow{n \rightarrow +\infty} |\widetilde{\nabla u(k)}|^2$  and  $|\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 \leq g \in L^2(\mathbb{R}^N)$  almost everywhere ( $g$  can be taken with compact support).

Therefore, Lebesgue theorem can be used for the last term of (4.59) to obtain

$$\int_{\mathbb{R}^N} A(k_{\varepsilon_n}) \left| |\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 - |\widetilde{\nabla u(k)}|^2 \right| dx \xrightarrow{n \rightarrow +\infty} 0. \quad (4.66)$$

We finally obtain

$$\|D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)\|_{0,1,\Omega} \xrightarrow{n \rightarrow +\infty} 0. \quad (4.67)$$

Therefore, (4.58) yields the convergence of the right hand side

$$\forall p \in W, \quad \int_{\Omega} D_{\varepsilon_n}(k_{\varepsilon_n}) p dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} D(k) p dx. \quad (4.68)$$

We have shown that the function  $k$  given by the washer lemma satisfies (4.4). Moreover, since  $k \in V$ , there exists  $q \in ]1, N[$  such that  $q^* > 2$  (for example by taking  $q = \frac{4}{3}$ ,  $q^* = \frac{12}{5} > 2$ ).

As a result, the embedding  $V \hookrightarrow L^2(\Omega)$  is compact. Then  $(k_{\varepsilon_n})$  strongly converges to  $k$  in  $L^2(\Omega)$ . This allows us to conclude.

#### 4.4 Conclusion: solution of the coupled problem

Let us consider the sequence  $(\varepsilon_n)$  mentioned previously, after all the extractions that were done. We have shown that  $(k_{\varepsilon_n})$  strongly converges in  $L^2(\Omega)$  to a function  $k$ .

Moreover,  $(k_{\varepsilon_n})$  weakly converges to  $k$  in every space  $W_0^{1,q}(\Omega)$  with  $q \in ]1, N[$ . Therefore, theorem 9 yields that  $(u(k_{\varepsilon_n}))$  strongly converges to  $u(k)$  in  $H^1(\Omega)$ .

All of this ensures the existence of a couple  $(u; k)$  of solutions of (1.1) at the distribution sense, with  $u \in H^1(\Omega)$  and  $k \in V$  (in particular  $k \in L^2$ ).

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