Coupled system involving eddy coefficients and a right hand side in L 1 : Review chapter

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Contents

1 Introduction

This chapter is focused on the following system:

$$\begin{cases} -\mathrm{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -\mathrm{div}(B(k)\nabla k) = A(k)|\nabla u|^2 & \text{in } \Omega, \\ -A(k)\dfrac{\partial u}{\partial n} = \alpha u & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Where Ω is an open bounded set in $\mathbb{R}^N$ ($N \geq 2$), and $\Gamma = \partial\Omega$ its boundary. The viscosity functions are bounded and satisfy:

$$\exists\ \nu, \mu > 0,\ \forall x \geq 0, \quad \nu \leq A(x), B(x) \leq \mu. \tag{1.2}$$

We also assume that $f \in L^2(\Omega)$ and $\alpha > 0$.

The aim of this chapter is proving the existence of a couple (u, k) of distributional solutions of (1.1). This is not the only kind of solutions commonly searched. Indeed, some articles deal with renormalized solutions, which are defined and studied in [3]. This notion is interesting for the study of a system having one source term in $L^1(\Omega)$, as shown in [6].

The source term of the equation in k in (1.1) is in $L^1(\Omega)$. Without any better hypothesis,it remains possible to find distributional solutions of (1.1).

This figures in the works of L. Boccardo and T. Gallouët in [1], on which is based this chapter. The associated result, proved in section 4, is a consequence of the washer Lemma, stated and proved in section 2. This Lemma brings a bound for a solution in $W_0^{1,q}(\Omega)$, where $q < \frac{N}{N-1}$. Consequently, some particular function sets are needed.

Then, in section 4, we search weak solutions in $V = \bigcap_{1<q<N'} W_0^{1,q}(\Omega)$, where N' refers to $\frac{N}{N-1}$; with test functions in $W = \bigcup_{q>N} W_0^{1,q}(\Omega)$. This method was also used by T. Gallouët and R. Herbin in [4].

Along this chapter, the viscosities A and B are bounded (assumption (1.2)). This does not really correspond to the physical situation, but is a right way to establish some interesting mathematical results, as a priori estimates, such as in section 3 and 4. The assumption of bounded viscosities appears in [4], but also in the works of S. Clain and R. Touzani (see [2]), in which an existence result is proved in the two-dimensional case. In the present work, the existence results are proved when $N = 3$.

Of course it is possible to deal with unbounded viscosities, such as in [6], where they are only bounded from below. Otherwise, if they are not, other assumptions are required, concerning the decreasing. In addition, other function spaces need to be developped, like in [5].

In order to find the existence of a solution of (1.1), this work will be divided into two steps, that correspond to sections 3 and 4. Section 3 is devoted to the equation in u, which amounts to a linear problem. But when the problem is obviously non-linear, like the equation in k considered in section 4, another method is required. This consists in regularizing the system, then solving it by a linearization and a fixed-point theorem. This method is already used in [2].

Given $m \in \mathbb{N}$ and $p \in [1, +\infty]$, the norm of $W^{m,p}(\Omega)$ will be simply written: $\|\cdot\|_{m,p,\Omega}$. In particular, $\|\cdot\|_{0,p,\Omega}$ is the $L^p(\Omega)$ norm.

2 Preliminary results

2.1 The washer lemma

This section aims at proving the following result (already proved by Boccardo-Gallouët in [1]).

Lemma 1 (Washer lemma)**.** *Let $q \in]1, N'[$ where $N' = \dfrac{N}{N-1}$, and $u \in W_0^{1,q}(\Omega)$. Let us write:*

$$B_n = \{x \in \Omega \ / \ n \leq |u(x)| < n+1\}. \tag{2.1}$$

We assume that there is a constant $C > 0$ such that:

$$\forall n \in \mathbb{N}, \quad \int_{B_n} |\nabla u(x)|^2 \, dx \leq C. \tag{2.2}$$

Then we have the inequality:

$$\|u\|_{1,q,\Omega} \leq A. \tag{2.3}$$

Where A depends on Ω, q and C that is introduced by (2.2).

Remark 1. The most important in this result is the fact that the two constants C and A are independent of u. Actually, the result that will be used is: let $(u_j)_{j\in J}$ a family of functions in $W_0^{1,q}(\Omega)$ with $q \in]1, N'[$ satisfying:

$$\forall j \in J, \ \forall n \in \mathbb{N}, \quad \int_{B_n^j} |\nabla u_j(x)|^2 \, dx \leq C, \tag{2.4}$$

where B_n^j refers to: $B_n^j = \{x \in \Omega \ / \ n \leq |u_j(x)| < n+1\}$.

Then there exists a constant $A > 0$ only depending on Ω, q and C such that:

$$\forall j \in J, \quad \|u_j\|_{1,q,\Omega} \leq A. \tag{2.5}$$

The proof of lemma 1 is based on a decomposition given by the following lemma.

Lemma 2. *Let u satisfying the assumptions of lemma 1. Let q^* be the critical exponent in Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$, and any index $n_0 \in \mathbb{N}$. There exist two constants λ_1 and λ_2 depending on n_0, q and Ω satisfying:*

$$\|\nabla u\|_{0,q,\Omega}^q \leq C^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{0,q^*,\Omega}^{\frac{q^*(2-q)}{2}} \right). \tag{2.6}$$

Moreover, we have: $\lim\limits_{n_0 \to +\infty} \lambda_2(n_0) = 0$.

Proof of lemma 2. At first, we decompose $\int_\Omega |\nabla u(x)|^q \, dx$ as a sum:

$$\int_\Omega |\nabla u(x)|^q \, dx = \sum_{n=0}^{+\infty} \int_{B_n} |\nabla u(x)|^q \, dx. \tag{2.7}$$

Let us apply Hölder inequality, recalling that Ω is bounded, thus for $n \in \mathbb{N}$, B_n is bounded.

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \int_{B_n} |\nabla u(x)|^q \, dx &\leq \left(\int_{B_n} |\nabla u(x)|^2 \, dx \right)^{q/2} \left(\int_{B_n} dx \right)^{1-\frac{q}{2}} \\ &= \mathrm{Meas}(B_n)^{\frac{2-q}{2}} \left(\int_{B_n} |\nabla u(x)|^2 \, dx \right)^{q/2} \\ &\leq \mathrm{Meas}(B_n)^{\frac{2-q}{2}} C^{q/2}. \end{aligned} \tag{2.8}$$

Where "Meas" refers to the Lebesgue measure of any borelian subset of $\mathbb{R}^N$. The last inequality comes from (2.2).

We set $n_0 \in \mathbb{N}$, and we focus on the case $n \leq n_0$, then $n > n_0$.

On one hand, let $n \leq n_0$, we have

$$\int_{B_n} |\nabla u(x)|^q \, dx \leq \text{Meas}(B_n)^{\frac{2-q}{2}} C^{q/2} \leq \text{Meas}(\Omega)^{\frac{2-q}{2}} C^{q/2}, \tag{2.9}$$

then by addition

$$\sum_{n=0}^{n_0} \int_{B_n} |\nabla u(x)|^q \, dx \leq (n_0+1)\text{Meas}(\Omega)^{\frac{2-q}{2}} C^{q/2}. \tag{2.10}$$

On the other hand, let $n > n_0$, we have

$$\int_{B_n} |u(x)|^{q^*} \, dx \geq \int_{B_n} n^{q^*} \, dx = n^{q^*}\text{Meas}(B_n), \text{ then } \text{Meas}(B_n) \leq \frac{1}{n^{q^*}} \int_{B_n} |u(x)|^{q^*} \, dx. \tag{2.11}$$

Inequalities (2.8) and (2.11) yield

$$\forall n > n_0, \quad \int_{B_n} |\nabla u(x)|^q \, dx \leq C^{q/2} \left(\frac{1}{n^{q^*}}\right)^{\frac{2-q}{2}} \left(\int_{B_n} |u(x)|^{q^*} \, dx\right)^{\frac{2-q}{2}}, \tag{2.12}$$

which yields by addition

$$\sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u(x)|^q \, dx \leq C^{q/2} \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left(\int_{B_n} |u(x)|^{q^*} \, dx\right)^{\frac{2-q}{2}}. \tag{2.13}$$

We then apply the discrete Hölder inequality on the right hand side of (2.13) to obtain

$$\sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left(\int_{B_n} |u(x)|^{q^*} \, dx\right)^{\frac{2-q}{2}} \leq \left[\sum_{n=n_0+1}^{+\infty} \int_{B_n} |u(x)|^{q^*} \, dx\right]^{\frac{2-q}{2}} \left[\sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}}\right]^{q/2}. \tag{2.14}$$

We need to prove that the series $\sum_{n \geq n_0+1} n^{\frac{N(q-2)}{N-q}}$ converges, which is equivalent to $\dfrac{N(2-q)}{N-q} > 1$.

By simple calculations, this inequality is equivalent to $q < N'$.

Then we set $\lambda_2(n_0, q) = \left[\sum_{n=n_0+1}^{+\infty} n^{\frac{N(q-2)}{N-q}}\right]^{q/2}$. And since the series converges, we directly have: $\lambda_2(n_0, q) \underset{n_0 \to +\infty}{\longrightarrow} 0$. With this notation, (2.14) becomes

$$\begin{aligned} \sum_{n=n_0+1}^{+\infty} n^{\frac{(q-2)q^*}{2}} \left(\int_{B_n} |u(x)|^{q^*} \, dx\right)^{\frac{2-q}{2}} &\leq \lambda_2(n_0, q) \left[\sum_{n=n_0+1}^{+\infty} \int_{B_n} |u(x)|^{q^*} \, dx\right]^{\frac{2-q}{2}} \\ &\leq \lambda_2(n_0, q) \left(\int_{\Omega} |u(x)|^{q^*} \, dx\right)^{\frac{2-q}{2}}. \end{aligned} \tag{2.15}$$

Hence, (2.15) combined with (2.13) yield

$$\sum_{n=n_0+1}^{+\infty} \int_{B_n} |\nabla u(x)|^q \, dx \leq C^{q/2} \lambda_2(n_0, q) \left(\int_\Omega |u(x)|^{q^*} \, dx \right)^{\frac{2-q}{2}} = C^{q/2} \lambda_2(n_0, q) \|u\|_{0,q^*,\Omega}^{\frac{q^*(2-q)}{2}}. \tag{2.16}$$

We set $\lambda_1(n_0, q) = (n_0 + 1)\text{Meas}(\Omega)^{\frac{2-q}{2}}$, and we add (2.10) to (2.16) to obtain, by (2.7)

$$\int_\Omega |\nabla u(x)|^q \, dx \leq C^{q/2} \left(\lambda_1(n_0, q) + \lambda_2(n_0, q) \|u\|_{0,q^*,\Omega}^{\frac{q^*(2-q)}{2}} \right). \tag{2.17}$$

That corresponds to (2.6), which concludes the proof. □

Then we use this result to prove the washer lemma.

Proof of lemma 1. We start with the case $N = 2$, which is easier. In that case, we have $\frac{q^*(2-q)}{2} = \frac{Nq(2-q)}{2(N-q)} = q$. Then (2.6) becomes

$$\|\nabla u\|_{0,q,\Omega}^q \leq C^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{0,q^*,\Omega}^q \right). \tag{2.18}$$

As $q < 2$, we can apply Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ on the right hand side of (2.18), and then the Poincaré-Friedrichs inequality

$$\|u\|_{0,q^*,\Omega}^q \leq K \|\nabla u\|_{0,q,\Omega}^q \leq K C^{q/2} \left(\lambda_1 + \lambda_2 \|u\|_{0,q^*,\Omega}^q \right), \tag{2.19}$$

where $K > 0$ depends on Ω and q. We recall that if $b < 1$, the following equivalence stands

$$x \leq a + bx \Leftrightarrow x \leq \frac{a}{1-b}. \tag{2.20}$$

Since $\lambda_2(n_0, q) \underset{n_0 \to +\infty}{\longrightarrow} 0$, it is possible to choose $n_0 \in \mathbb{N}$ large enough to have $KC^{q/2}\lambda_2 < 1$. Then by (2.20), we obtain

$$\|u\|_{0,q^*,\Omega}^q \leq \frac{KC^{q/2}\lambda_1}{1 - KC^{q/2}\lambda_2}. \tag{2.21}$$

Thus, from (2.18):

$$\|\nabla u\|_{0,q,\Omega}^q \leq C^{q/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right). \tag{2.22}$$

And the Poincaré-Friedrichs inequality yields

$$\|u\|_{1,q,\Omega}^q \leq LC^{q/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right), \tag{2.23}$$

where L depends on q and Ω. This yields the result with

$$A = L^{1/q} C^{1/2} \left(\lambda_1 + \frac{KC^{q/2}\lambda_1\lambda_2}{1 - KC^{q/2}\lambda_2} \right)^{1/q}. \tag{2.24}$$

Let us focus now on the case $N > 2$. Since $q < N' < N$, the Sobolev embedding used before yields

$$\|u\|^q_{0,q^*,\Omega} \leq K\|\nabla u\|^q_{0,q,\Omega} \leq KC^{q/2}\left(\lambda_1 + \lambda_2\|u\|^{\frac{q^*(2-q)}{2}}_{0,q^*,\Omega}\right). \tag{2.25}$$

Inequality (2.25) is of the form $x \leq a + bx^\gamma$, with

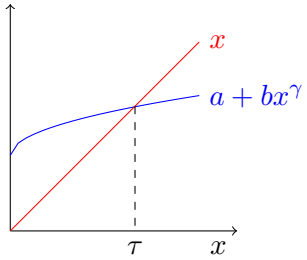
$$\gamma = \frac{q^*(2-q)}{2q} = \frac{N(2-q)}{2(N-q)}. \tag{2.26}$$

We have already shown that if $q \in]1, N'[$, then:

$$1 < \frac{N(2-q)}{N-q} < N' \text{ hence } \frac{1}{2} < \gamma < \frac{N'}{2} < 1. \tag{2.27}$$

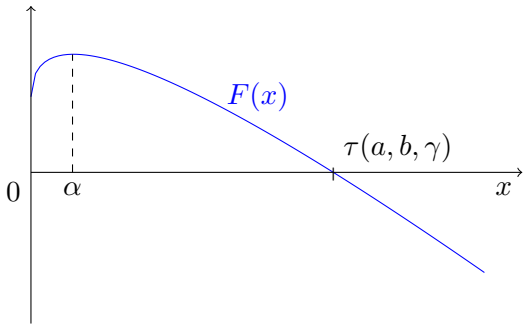
Thus, the inequation we need to solve is of the form: $x \leq a + bx^\gamma$, with $\gamma \in]0,1[$ and $a, b > 0$.

The graph on the side shows the existence of $\tau \in]0, +\infty[$ depending on a, b and γ such that $x \leq a + bx^\gamma \Leftrightarrow x \leq \tau$.



Let us show it rigorously by studying the variations of $F : x > 0 \mapsto a + bx^\gamma - x$. This function is $\mathcal{C}^\infty$ on $]0; +\infty[$ and $\left|\begin{array}{l} F'(x) = b\gamma x^{\gamma-1} - 1 \\ F''(x) = b\gamma(\gamma-1)x^{\gamma-2} \end{array}\right.$, therefore $F''(x) < 0$. We obtain the following variations

x	0		α		$+\infty$
$F''(x)$		$-$		$-$	
$F'(x)$	$+\infty$	$\searrow$	0	$\searrow$	-1
$F(x)$	a	$\nearrow$	$F(\alpha)$	$\searrow$	$-\infty$



Inequality (2.25) finally ensures the existence of $\tau = \tau(\lambda_1, \lambda_2, C) > 0$ such that

$$\|u\|^q_{0,q^*,\Omega} \leq \tau. \tag{2.28}$$

Thus, (2.6) and Poincaré-Friedrichs inequality yield

$$\|\nabla u\|^q_{0,q,\Omega} \leq C^{q/2}\left(\lambda_1 + \lambda_2\tau\right) \quad \text{hence} \quad \|u\|^q_{1,q,\Omega} \leq LC^{q/2}\left(\lambda_1 + \lambda_2\tau\right), \tag{2.29}$$

where L only depends on Ω and q. With $A = L^{1/q} C^{1/2} (\lambda_1 + \lambda_2 \tau)^{1/q}$, lemma 1 is proved in case $N > 2$. □

The first hypothesis of lemma 1 is $q \in]1, N'[$, where $N' = \frac{N}{N-1}$. It has been used many times, and we are able to show that is impossible to obtain similar results with higher values of q.

Indeed, considering the case $N > 2$, we recall that α is the only point where F' vanishes, which means

$$b\gamma\alpha^{\gamma-1} = 1 \quad \text{thus} \quad \alpha = (b\gamma)^{\frac{1}{1-\gamma}}. \tag{2.30}$$

Replacing b by the constants of lemma 2 yields $b(q) = M(q)\lambda_2(q)$, where $M(q)$ is bounded if $q \in]1, N'[$. Consequently, $b(q) \underset{q \to N'}{\longrightarrow} +\infty$ and then

$$\gamma(q) = \frac{N(2-q)}{2(N-q)} \underset{q \to N'}{\longrightarrow} 2 \quad \text{hence} \quad \frac{1}{1-\gamma} \underset{q \to N'}{\longrightarrow} \frac{1}{2}. \tag{2.31}$$

Therefore, (2.30) yields $\alpha \underset{q \to N'}{\longrightarrow} +\infty$. And since $\tau \geq \alpha$, we obtain $\tau \underset{q \to N'}{\longrightarrow} +\infty$, thus $A \underset{q \to N'}{\longrightarrow} +\infty$.

There is a similar result in the case $N = 2$. The proof of lemma 1 requires to choose an index n_0 great enough to have $b < 1$. But as $\lambda_2(n_0, q) \underset{q \to N'}{\longrightarrow} +\infty$, an increase of q implies an increase of n_0.

Thus, as $\frac{2-q}{2} = 1 - \frac{q}{2} \underset{q \to N'}{\longrightarrow} 1 - \frac{N'}{2} = 0$, it follows

$$\lambda_1(n_0, q) = (n_0 + 1)\text{Meas}(\Omega)^{\frac{2-q}{2}} \underset{\substack{q \to N' \\ n_0 \to +\infty}}{\longrightarrow} +\infty \quad \text{hence} \quad A \underset{q \to N'}{\longrightarrow} +\infty. \tag{2.32}$$

We have shown that q cannot be larger than the assumptions of lemma 1.

The result shown in this first section will be useful in the following. Generally, solving a PDE system is divided into two steps:

- simplifying the problem (for example by moving to a finite-dimension space, or by using a truncation function),
- coming back to the initial problem with the help of compacity results.

2.2 A general result on norms

Prop./Def. 3 (Trace norm on H^1)**.** *Let us define the application:* $N_0(u) = \left(\|\nabla u\|^2_{0,2,\Omega} + \|u\|^2_{0,2,\Gamma} \right)^{1/2}$, *"$u_{|\Gamma}$" referring to the trace of u on Γ.*

Then N_0 is a norm on $H^1(\Omega)$.

Proof. N_0 is non negative, homogeneous and satisfies triangular inequality (direct consequence of the fact that the norms L^2 also do). Let us consider $u \in H^1(\Omega)$ such that $N_0(u) = 0$. It comes $\|\nabla u\|_{0,2,\Omega} = 0$ and $\|u\|_{0,2,\Gamma} = 0$.

First, this yields that $\nabla u = 0$ almost everywhere on Ω. Thus, we write: $u \overset{ae.}{=} \lambda$, where λ is a real constant. And if u is constant almost everywhere, its trace has the same value almost everywhere on Γ. Yet, $\mathrm{tr}(u) = 0$, therefore $u \overset{ae.}{=} 0$.

Consequently, N_0 is a norm on $H^1(\Omega)$. □

Theorem 4 (Two equivalent norms)**.** *Let us recall the definition of the classical Sobolev norm on $H^1(\Omega)$:*

$$\|u\|_{1,2,\Omega} = \left(\|u\|^2_{0,2,\Omega} + \|\nabla u\|^2_{0,2,\Omega}\right)^{1/2}. \tag{2.33}$$

Then this norm and N_0 previously defined are equivalent.

Proof. At first, the trace operator $\mathrm{tr} : H^1(\Omega) \to H^{1/2}(\Gamma)$ is surjective (see [7]), and by Sobolev embedding, $H^{1/2}(\Gamma) \hookrightarrow L^q(\Gamma)$, with $\frac{1}{q} = \frac{1}{2} - \frac{1}{2N}$, thus $q = \frac{2N}{N-1} \geq 2$.

Then, since Ω is bounded, we have: $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$. Hence

$$\exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|^2_{0,2,\Gamma} \leq C\|u\|^2_{1,2,\Omega}. \tag{2.34}$$

We deduce

$$\forall u \in H^1(\Omega), \quad N_0(u)^2 \leq (1 + C)\|u\|^2_{1,2,\Omega}. \tag{2.35}$$

Therefore N_0 is dominated by $\|\cdot\|_{1,2,\Omega}$.

Conversely, we intend to show

$$\exists C > 0, \forall u \in H^1(\Omega), \quad \|u\|^2_{1,2,\Omega} \leq CN_0(u)^2. \tag{2.36}$$

By contradiction, let us suppose the negation of (2.36), which means

$$\forall C > 0, \exists u \in H^1(\Omega), \quad \|u\|^2_{1,2,\Omega} > CN_0(u)^2. \tag{2.37}$$

Then in particular

$$\forall n \in \mathbb{N}, \exists u_n \in H^1(\Omega), \quad \|u_n\|^2_{1,2,\Omega} > nN_0(u_n)^2. \tag{2.38}$$

Inequality (2.38) yields that $\forall n \in \mathbb{N}^*, \quad \|u_n\|_{0,2,\Omega} > 0$. Then, both sides of (2.38) van be divided by $\|u_n\|_{0,2,\Omega}$. Which means, keeping the same notation

$$\forall n \in \mathbb{N}, \exists u_n \in S, \quad \|u_n\|^2_{1,2,\Omega} > nN_0(u_n)^2, \tag{2.39}$$

where S refers to the "L^2-sphere" defined by: $S = \{v \in H^1(\Omega) \ / \ \|v\|_{0,2,\Omega} = 1\}$. Then (2.39) can be rewritten as

$$\forall n \geq 2, \quad \frac{1}{n} + \frac{1}{n}\|\nabla u_n\|^2_{0,2,\Omega} > \|u_n\|_{0,2,\Gamma} + \|\nabla u_n\|^2_{0,2,\Omega}. \tag{2.40}$$

Since $\|u_n\|^2_{0,2,\Gamma} \geq 0$, (2.40) yields in particular

$$\left(1 - \frac{1}{n}\right)\|\nabla u_n\|^2_{0,2,\Omega} \leq \frac{1}{n}, \quad \text{then } \|\nabla u_n\|^2_{0,2,\Omega} \leq \frac{1}{n-1}. \tag{2.41}$$

Therefore, the sequence $(\nabla u_n)_{n\in\mathbb{N}^*}$ is bounded in $L^2(\Omega)$; and as $(u_n)_{n\in\mathbb{N}^*}$ is a sequence of points of S, $(u_n)_{n\in\mathbb{N}}$ is also bounded in $L^2(\Omega)$. We deduce (u_n) is bounded in $H^1(\Omega)$. Thus, Banach-Alaoglu theorem ensures the existence of a subsequence still denoted $(u_n)_{n\in\mathbb{N}^*}$ satisfying $u_n \underset{n\to+\infty}{\overset{H^1}{\rightharpoonup}} u$.

But (2.40) yields

$$\forall n \geq 2, \quad \frac{1}{n} > \|u_n\|^2_{0,2,\Gamma} + \left(1 - \frac{1}{n}\right) \|\nabla u_n\|^2_{0,2,\Omega} \geq \frac{1}{2} N_0(u_n)^2, \tag{2.42}$$

hence $N_0(u_n) \underset{n\to+\infty}{\longrightarrow} 0$, then $u = 0$.

Last, by Rellich theorem: $u_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} 0$, which implies: $\|u_n\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0$. But we have assumed that $\forall n \in \mathbb{N}, \quad \|u_n\|_{0,2,\Omega} = 1$, which is a contradiction. □

3 Analysis of the first equation

In this subsection, given $p \in]1, +\infty[$ and $k \in W_0^{1,p}(\Omega)$, we consider the system:

$$\begin{cases} -\mathrm{div}(A(k)\nabla u) = f & \text{in } \Omega, \\ -A(k)\dfrac{\partial u}{\partial n} = \alpha u & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Where $f \in L^2(\Omega)$, and $\alpha > 0$. We recall that $A : \mathbb{R} \to \mathbb{R}_+$ is bounded from above and below:

$$\forall k \in \mathbb{R}, \quad 0 < \nu \leq A(k) \leq \mu < +\infty. \tag{3.2}$$

3.1 Variational formulation

Let us write the variational form of this equation. To do so, we need the Green formula.

Theorem 5 (Green formula)**.** *Let $f \in H^1(\Omega)$ and $\mathbf{g} \in H^1(\Omega)^N$. Assume that $\mathbf{n}$ is the normal vector of Ω, then the following formula stands*

$$\int_\Omega \mathrm{div}(\mathbf{g}) f \, dx = -\int_\Omega \mathbf{g} \cdot \nabla f \, dx + \int_\Gamma (\mathbf{g} \cdot n) f \, dS(x). \tag{3.3}$$

Let us consider $w \in H^1(\Omega)$ and multiply the first equation of (3.1) by w, then integrate.

$$-\int_\Omega \mathrm{div}(A(k)\nabla u) w \, dx = \int_\Omega f w \, dx. \tag{3.4}$$

The Green formula (3.3) yields

$$\begin{aligned} -\int_\Omega \mathrm{div}(A(k)\nabla u) w \, dx &= \int_\Omega A(k)\nabla u \cdot \nabla w \, dx - \int_\Gamma A(k)\frac{\partial u}{\partial n} w \, dS(x) \\ &= \int_\Omega A(k)\nabla u \cdot \nabla w \, dx + \alpha \int_\Gamma uw \, dS(x). \end{aligned} \tag{3.5}$$

That leads to the variational formulation of (3.1):

$$\begin{cases} u \in H^1(\Omega) \\ \forall w \in H^1(\Omega), \quad \displaystyle\int_\Omega A(k)\nabla u \cdot \nabla w \, dx + \alpha \int_\Gamma uw \, dS(x) = \int_\Omega fw \, dx. \end{cases} \tag{3.6}$$

All the terms of (3.6) are well defined, since $u, w \in H^1(\Omega)$, hence $\nabla u, \nabla w \in L^2(\Omega)$. And A is bounded. Moreover, by the trace theorem (see [7]), $u, w \in H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$.

The main result of this section is the following.

Theorem 6. *There exists a unique function $u \in H^1(\Omega)$ satisfying (3.6).*

Proof. This proof uses the Lax-Milgram theorem on the Hilbert space $H^1(\Omega)$. Let $a(k, \cdot, \cdot)$ be the bilinear form given by

$$\forall u, w \in H^1(\Omega), \quad a(k, u, w) = \int_\Omega A(k)\nabla u \cdot \nabla w \, dx + \alpha \int_\Gamma uw \, dS(x). \tag{3.7}$$

Let $u, w \in H^1(\Omega)$

$$\begin{aligned} |a(k, u, w)| &\leq \int_\Omega A(k)|\nabla u \cdot \nabla w| \, dx + \alpha \int_\Gamma |uw| \, dS(x) \\ &\leq \mu \|\nabla u\|_{0,2,\Omega} \|\nabla w\|_{0,2,\Omega} + \alpha \|u\|_{0,2,\Gamma} \|w\|_{0,2,\Gamma} \\ &\leq (\mu + \alpha M) \|u\|_{1,2,\Omega} \|w\|_{1,2,\Omega}, \end{aligned} \tag{3.8}$$

where M does not depend on u. Therefore, $a(k, \cdot, \cdot)$ is continuous on $H^1(\Omega) \times H^1(\Omega)$.

Then we show the coercivity. Let $w \in H^1(\Omega)$, we have

$$\begin{aligned} a(k, w, w) &= \int_\Omega A(k)|\nabla w|^2 \, dx + \alpha \int_\Gamma |w|^2 \, dS(x) \geq \nu \|\nabla w\|^2_{0,2,\Omega} + \alpha \|w\|^2_{0,2,\Gamma} \\ &\geq \inf(\nu, \alpha) N_0(w)^2, \end{aligned} \tag{3.9}$$

hence the coercivity of $a(k, \cdot, \cdot)$ by theorem 4.

Finally, it is clear that the linear form $[w \mapsto \int_\Omega fw \, dx]$ is continuous. Problem (3.6) then admits a unique solution $u \in H^1(\Omega)$, by the Lax-Milgram theorem. □

3.2 A priori estimates for u

Previously, we have shown that for $k \in W_0^{1,p}(\Omega)$, (3.6) admits a unique solution $u = u(k)$. In this subsection, we give some estimates for this function u.

Proposition 7. *The solution u of (3.6) satisfies the inequality:*

$$\|u\|_{1,2,\Omega} \leq \frac{C\|f\|_{0,2,\Omega}}{\inf(\alpha, \nu)}. \tag{3.10}$$

Where $C > 0$ only depends on Ω.

Proof. Taking $w = u$ as a test in (3.6) leads to

$$\int_\Omega A(k)|\nabla u|^2\,dx + \alpha\int_\Gamma |u|^2\,dS(x) = \int_\Omega fu\,dx. \tag{3.11}$$

We get by Cauchy-Schwarz inequality

$$\nu\|\nabla u\|^2_{0,2,\Omega} + \alpha\|u\|^2_{0,2,\Gamma} \le \|f\|_{0,2,\Omega}\|u\|_{0,2,\Omega}. \tag{3.12}$$

Then by Theorem 4

$$\nu\|\nabla u\|^2_{0,2,\Omega} + \alpha\|u\|^2_{0,2,\Gamma} \ge \inf(\nu,\alpha)N_0(u)^2 \ge \frac{\inf(\nu,\alpha)}{C}\|u\|^2_{1,2,\Omega}. \tag{3.13}$$

Then (3.12) becomes

$$\frac{\inf(\nu,\alpha)}{C}\|u\|^2_{1,2,\Omega} \le \|f\|_{0,2,\Omega}\times\|u\|_{0,2,\Omega} \le \|f\|_{0,2,\Omega}\|u\|_{1,2,\Omega}, \tag{3.14}$$

which yields (3.10). □

Proposition 8. *Let $k \in W_0^{1,p}(\Omega)$, and $u = u(k)$ the solution of (3.6). Then u and k satisfy the inequality*

$$\|\sqrt{A(k)}\nabla u(k)\|_{0,2,\Omega} \le \frac{C\|f\|_{0,2,\Omega}}{\inf(\alpha,\nu)}, \tag{3.15}$$

where $C > 0$ only depends on Ω.

Proof. Equality (3.11) can be rewritten with norms

$$\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \alpha\|u\|^2_{0,2,\Gamma} = \int_\Omega fu\,dx. \tag{3.16}$$

Let us bound from below the left hand side of (3.16):

$$\begin{aligned}
\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \alpha\|u\|^2_{0,2,\Gamma} &= \frac{1}{2}\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \frac{1}{2}\int_\Omega A(k)|\nabla u|^2\,dx + \alpha\|u\|^2_{0,2,\Gamma}\\
&\ge \frac{1}{2}\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \frac{\nu}{2}\|\nabla u\|^2_{0,2,\Omega} + \alpha\|u\|^2_{0,2,\Gamma}\\
&\ge \frac{1}{2}\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \frac{\inf(\alpha,\nu)}{2}N_0(u)^2\\
&\ge \frac{1}{2}\|\sqrt{A(k)}u\|^2_{0,2,\Omega} + \frac{\inf(\alpha,\nu)}{2C}\|u\|^2_{1,2,\Omega}.
\end{aligned} \tag{3.17}$$

Then, let us apply Young inequality with weights to the right hand side of (3.16):

$$\int_\Omega fu\,dx \le \|f\|_{0,2,\Omega}\|u\|_{1,2,\Omega} \le \frac{\inf(\alpha,\nu)}{2C}\|u\|^2_{1,2,\Omega} + \frac{C}{2\inf(\alpha,\nu)}\|f\|^2_{0,2,\Omega}. \tag{3.18}$$

Thus, (3.16) can be rewritten

$$\frac{1}{2}\|\sqrt{A(k)}u\|^2_{0,2,\Omega} \le \frac{C}{2\inf(\alpha,\nu)}\|f\|^2_{0,2,\Omega}, \tag{3.19}$$

which yields (3.15). □

These estimates will be useful in the following, to show a strong convergence in H^1 for equation (3.1).

3.3 A first convergence result: the energy method

This subsection aims at proving the following result.

Theorem 9 (H^1 convergence)**.** *Let $p \in]1, +\infty[$, and $(k_n)_{n\in\mathbb{N}}$ be a sequence of functions in $W_0^{1,p}(\Omega)$. Let us suppose that (k_n) weakly converges to k in $W_0^{1,p}(\Omega)$. Let $u = u(k)$, and $u_n = u(k_n), n \in \mathbb{N}$.*

Then a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ strongly converges to u in $H^1(\Omega)$.

This theorem is a consequence of another, which involves A and the previous estimates.

Theorem 10 (L^2 convergence)**.** *With the same notations and assumptions as those of Theorem 9, we have:*

$$\sqrt{A(k_n)}\nabla u_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} \sqrt{A(k)}\nabla u. \tag{3.20}$$

Proof of Theorem 10. As $k_n \underset{n\to+\infty}{\overset{W_0^{1,p}}{\longrightarrow}} k$, the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ yields $k_n \underset{n\to+\infty}{\overset{L^1}{\longrightarrow}} k$. By the Lebesgue inverse theorem, there is a subsequence (still named (k_n)) such that $k_n \underset{n\to+\infty}{\overset{\text{ae.}}{\longrightarrow}} k$. Since A is continuous, we deduce $A(k_n) \underset{n\to+\infty}{\overset{\text{ae.}}{\longrightarrow}} A(k)$.

We recall that u_n and u satisfy

$$\forall \varphi \in H^1(\Omega), \forall n \in \mathbb{N}, \quad \int_\Omega A(k_n)\nabla u_n \cdot \nabla\varphi \, dx + \alpha \int_\Gamma u_n \varphi \, dS(x) = \int_\Omega f\varphi \, dx. \tag{3.21}$$

$$\forall \varphi \in H^1(\Omega), \quad \int_\Omega A(k)\nabla u \cdot \nabla\varphi \, dx + \alpha \int_\Gamma u \varphi \, dS(x) = \int_\Omega f\varphi \, dx. \tag{3.22}$$

Inequality (3.10) yields that (u_n) is bounded in $H^1(\Omega)$, which ensures the existence of a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n \underset{n\to+\infty}{\overset{H^1}{\rightharpoonup}} w \quad \text{and} \quad u_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} w, \quad \text{hence} \quad \nabla u_n \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} \nabla w, \tag{3.23}$$

where $w \in H^1(\Omega)$. This yields in particular

$$\forall \varphi \in H^1(\Omega), \quad \int_\Gamma u_n \varphi \, dS(x) \underset{n\to+\infty}{\longrightarrow} \int_\Gamma w\varphi \, dS(x). \tag{3.24}$$

Then, in order to show the convergence of the first term in (3.21), an integration lemma is needed.

Lemma 11 (Convergence of integrals)**.** *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ two sequences of $L^2(\Omega)$ functions, and $f, g \in L^2(\Omega)$ satisfying $f_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} f$ and $g_n \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} g$. Then we have*

$$\int_\Omega f_n g_n \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega fg \, dx. \tag{3.25}$$

This result will be proved after finishing the proof of Theorem 10.

The assumptions of Lemma 11 have to be checked.

- As the sequence $(A(k_n))_{n\in\mathbb{N}}$ converges almost everywhere to $A(k)$, the domination $A(k_n)|\nabla\varphi| \leq \mu|\nabla\varphi|$ and the Lebesgue theorem yield $A(k_n)\nabla\varphi \xrightarrow[n\to+\infty]{L^2} A(k)\nabla\varphi$.

- By (3.23): $\nabla u_n \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} \nabla w$.

Therefore, Lemma 11 yields

$$\forall \varphi \in H^1(\Omega), \int_\Omega A(k_n)\nabla u_n \cdot \nabla\varphi \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega A(k)\nabla w \cdot \nabla\varphi \, dx. \tag{3.26}$$

Both convergences (3.24) and (3.26) ensure that the limit w satisfies (3.22). And since the solution of (3.22) is unique (by Theorem 6), we deduce: $w = u = u(k)$.

Hence, $\nabla u_n \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} \nabla u$, and (3.15) ensures that there exists a subsequence of $\left(\sqrt{A(k_n)}\nabla u_n\right)_{n\in\mathbb{N}}$ that weakly converges in L^2. Therefore, Lemma 11 yields

$$\sqrt{A(k_n)}\nabla u_n \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} \sqrt{A(k)}\nabla u. \tag{3.27}$$

In order to obtain the strong convergence of this sequence, the convergence of the norms remains to be proved. Let us write (3.21) with $\varphi = u_n$ and (3.22) with $\varphi = u$.

$$\forall n \in \mathbb{N}, \quad \int_\Omega A(k_n)|\nabla u_n|^2 \, dx + \alpha \int_\Gamma |u_n|^2 \, dS(x) = \int_\Omega f u_n \, dx. \tag{3.28}$$

$$\int_\Omega A(k)|\nabla u|^2 \, dx + \alpha \int_\Gamma |u|^2 \, dS(x) = \int_\Omega f u \, dx. \tag{3.29}$$

The strong convergence of (u_n) in L^2 directly yields

$$\int_\Omega f u_n \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega f u \, dx. \tag{3.30}$$

And since $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact: $(u_n)_{|\Gamma} \xrightarrow[n\to+\infty]{L^2} u_{|\Gamma}$. In particular

$$\int_\Gamma |u_n|^2 \, dS(x) \underset{n\to+\infty}{\longrightarrow} \int_\Gamma |u|^2 \, dS(x). \tag{3.31}$$

Convergences (3.30) and (3.31) yield:

$$\int_\Omega A(k_n)|\nabla u_n|^2 \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega A(k)|\nabla u|^2 \, dx. \tag{3.32}$$

The result of Theorem 10 is then obtained. □

Finally, Lemma 11 needs to be proved.

Proof of Lemma 11. Let us decompose the difference between the two integrals:

$$\int_\Omega (f_n g_n - fg)\,dx = \int_\Omega f(g_n - g)\,dx + \int_\Omega (f_n - f)g_n\,dx. \tag{3.33}$$

On one hand, the weak convergence of (g_n) yields: $\int_\Omega f(g_n - g)\,dx \underset{n\to+\infty}{\longrightarrow} 0$.
On the other hand, $(g_n)_{n\in\mathbb{N}}$ weakly converges, then is bounded in $L^2(\Omega)$. We deduce

$$\left|\int_\Omega (f_n - f)g_n\,dx\right| \leq \|f_n - f\|_{0,2,\Omega}\|g_n\|_{0,2,\Omega} \leq M\|f_n - f\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.34}$$

□

Theorem 9 can be proved now.

Proof of Theorem 9. We have already shown that $u_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} u$, we have to prove that (∇u_n) strongly converges. Let $n \in \mathbb{N}$, we have

$$\begin{aligned}\sqrt{\nu}\|\nabla u_n - \nabla u\|_{0,2,\Omega} &\leq \left\|\sqrt{A(k_n)}\nabla u_n - \sqrt{A(k_n)}\nabla u\right\|_{0,2,\Omega} \\ &\leq \left\|\sqrt{A(k_n)}\nabla u_n - \sqrt{A(k)}\nabla u\right\|_{0,2,\Omega} + \left\|\left(\sqrt{A(k_n)} - \sqrt{A(k)}\right)\nabla u\right\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned} \tag{3.35}$$

by Theorem 10 and Lebesgue Theorem. □

4 Analysis of the second equation

From now until the end, $N = 3$. In this section, we consider the system:

$$\begin{cases} -\mathrm{div}(B(k)\nabla k) = A(k)|\nabla u(k)|^2 & \text{in } \Omega, \\ k = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

In order to lighten the writing, we define:

$$D(k) = A(k)|\nabla u(k)|^2. \tag{4.2}$$

4.1 Variational formulation and description of the method

Let us define two function spaces:

$$V = \bigcap_{1<r<N'} W_0^{1,r}(\Omega) \quad \text{and} \quad W = \bigcup_{r>N} W_0^{1,r}(\Omega).$$

Before stating the variational formulation, we need a result.

Proposition 12. *Let $k \in V$ and $p \in W$. Then $B(k)\nabla k \cdot \nabla p \in L^1(\Omega)$ and $D(k)p \in L^1(\Omega)$.*

Proof. Let $k \in V$ and $p \in W$. By applying the Hölder inequality with $r > N$ such that $p \in W_0^{1,r}(\Omega)$, we obtain

$$\int_\Omega B(k)|\nabla k \cdot \nabla p|\, dx \leq \mu \int_\Omega |\nabla k \cdot \nabla p|\, dx \leq \mu \|\nabla k\|_{0,r',\Omega} \times \|\nabla p\|_{0,r,\Omega}. \tag{4.3}$$

Since $r > N$, $r' < N'$, thus $k \in W_0^{1,r'}(\Omega)$. This yields the first result.

The second result comes from (3.15), that yields: $\|D(k)\|_{0,1,\Omega} \leq \left(\frac{C\|f\|_{0,2,\Omega}}{\inf(\alpha,\nu)}\right)^2$, then $D(k) \in L^1(\Omega)$ and $p \in L^\infty(\Omega)$, by Sobolev embedding. Hence $D(k)p \in L^1(\Omega)$. □

It is important to notice that if $N = 3$, $V \hookrightarrow L^2(\Omega)$. Indeed, we have the equivalence

$$W^{1,q}(\Omega) \hookrightarrow L^2(\Omega) \Leftrightarrow q^* > 2 \Leftrightarrow \frac{1}{q} < \frac{1}{2} + \frac{1}{N} \Leftrightarrow q > \frac{2N}{N+2}.$$

Then the condition that needs to be satisfied is: $N' > \frac{2N}{N+2}$, which is equivalent to $N < 4$.

We are now able to state the variational formulation of (4.1) ; k is called a weak solution of (4.1) if

$$\begin{cases} k \in V \\ \forall p \in W, \quad \displaystyle\int_\Omega B(k)\nabla k \cdot \nabla p\, dx = \int_\Omega D(k)p\, dx. \end{cases} \tag{4.4}$$

As a priori $D(k) \in L^1(\Omega)$, it is not possible to solve directly (4.4). Indeed, the method that will be used consists in bringing (4.4) back to a Lax-Milgram problem, which requires being in a Hilbert space. Yet, $W^{1,p}(\Omega)$ with $p \neq 2$ is not a Hilbert space. And $D(k)$ is not in $L^2(\Omega)$.

More precisely, the method is divided into three steps:

- Regularizing the equation, by changing the source term.
- Linearizing it to solve the regularized problem with a fixed-point theorem.
- Taking the limit when the regularizing parameter goes to zero.

4.2 Two first steps: solving the regularized problem

In order to set the regularized version of (4.4), several tools must be defined.

Definition 1. If $w \in L^r(\Omega)$ with $r \in [1, +\infty]$, $\tilde{w} \in L^r(\mathbb{R}^N)$ refers to the extension by zero

$$\tilde{w} : x \mapsto \begin{cases} w(x) \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

Definition 2 (Mollifier)**.** Let $\rho : \mathbb{R}^N \to \mathbb{R}$ a function such that $\text{Supp}(\rho) = \bar{B}(0,1)$, $\rho \geq 0$ and $\int_{\mathbb{R}^N} \rho(y)dy = 1$. We define the family $(\rho_\varepsilon)_{\varepsilon>0}$ by

$$\forall x \in \mathbb{R}^N, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right).$$

Let $\varepsilon > 0$, and $D_\varepsilon(k)$ given by: $D_\varepsilon(k) = A(k)\left|\widetilde{\nabla u(k)} * \rho_\varepsilon\right|^2$, and the following regularized problem:

$$\begin{cases} k \in H_0^1(\Omega) \\ \forall p \in H_0^1(\Omega), \quad \displaystyle\int_\Omega B(k)\nabla k \cdot \nabla p \, dx = \int_\Omega D_\varepsilon(k) p \, dx. \end{cases} \tag{4.5}$$

Linearizing is only useful if the assumptions of the Lax-Milgram theorem are satisfied. This is what will be proved now, beginning by a continuity result on the right hand side of (4.5).

Theorem 13. *Let $\varepsilon > 0$ and $D_\varepsilon(k)$ defined as in (4.5). Then $D_\varepsilon(k) \in L^2(\Omega)$ and there exists a constant $C > 0$ only depending of the data of the problem such that*

$$\forall k \in H_0^1(\Omega), \quad \|D_\varepsilon(k)\|_{0,2,\Omega} \leq \frac{C}{\varepsilon^2}. \tag{4.6}$$

The proof of Theorem 13 is based on a general result.

Lemma 14. *Let $v \in H^m(\Omega)$, $\ell \in \mathbb{N}$ and $(\rho_\varepsilon)_{\varepsilon>0}$ a mollifier defined as previously. For $\varepsilon > 0$ sufficiently close to 0, we have*

$$\|v * \rho_\varepsilon\|_{m+\ell,2} \leq \frac{C(m,\ell)}{\varepsilon^\ell}\|v\|_{m,2}. \tag{4.7}$$

Proof of Lemma 14. Let us recall the definition of the Sobolev norm $\|v * \rho_\varepsilon\|^2_{m+\ell,2} = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq \ell}} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|^2_{0,2}$. Let $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| \leq m$ and $|\beta| \leq \ell$. Plancherel theorem yields

$$\|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|^2_{0,2} = \frac{1}{(2\pi)^N}\|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|^2_{0,2}, \tag{4.8}$$

where $\mathcal{F}$ refers to the Fourier transform operator.

Since $v \in H^m(\Omega)$, the following derivation formula stands $\partial^{\alpha+\beta}(v * \rho_\varepsilon) = (\partial^\alpha v) * (\partial^\beta \rho_\varepsilon)$. And by Fourier transform, we get

$$\mathcal{F}\left(\partial^{\alpha+\beta}(v * \rho_\varepsilon)\right) = \mathcal{F}(\partial^\alpha v)\mathcal{F}(\partial^\beta \rho_\varepsilon). \tag{4.9}$$

Then with norms:

$$\|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|^2_{0,2} = \int_{\mathbb{R}^N} |\mathcal{F}(\partial^\alpha v)(\xi)|^2 \cdot |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 d\xi. \tag{4.10}$$

The definition of ρ_ε with respect to ρ yields, by a change of variables

$$\mathcal{F}(\rho_\varepsilon)(\xi) = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} e^{-ix\xi}\rho\left(\frac{x}{\varepsilon}\right) dx \underset{y=x/\varepsilon}{=} \int_{\mathbb{R}^N} e^{-i\varepsilon y\xi}\rho(y)dy = \mathcal{F}(\rho)(\varepsilon\xi). \tag{4.11}$$

Then we obtain

$$\begin{aligned} |\mathcal{F}(\partial^\beta \rho_\varepsilon)(\xi)|^2 &= |\xi^\beta \mathcal{F}(\rho_\varepsilon)(\xi)|^2 = |\xi^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 \\ &= \frac{1}{\varepsilon^{2|\beta|}}|(\varepsilon\xi)^\beta \mathcal{F}(\rho)(\varepsilon\xi)|^2 = \frac{1}{\varepsilon^{2|\beta|}}|\mathcal{F}(\partial^\beta \rho)(\varepsilon\xi)|^2. \end{aligned} \tag{4.12}$$

Yet, $\partial^\beta \rho \in \mathscr{D}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, which yields $\forall \xi \in \mathbb{R}^N, \quad |\mathcal{F}(\partial^\beta \rho)(\xi)| \leq \|\partial^\beta \rho\|_{0,1}$. This reported in the inequality (4.10) ensure the existence of a constant $c_0 > 0$ independent of v such that

$$\|\mathcal{F}(\partial^{\alpha+\beta}(v * \rho_\varepsilon))\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\mathcal{F}(\partial^\alpha v)\|_{0,2}^2. \tag{4.13}$$

Plancherel Theorem then yields

$$\|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|\partial^\alpha v\|_{0,2}^2. \tag{4.14}$$

By addition for $|\alpha| \leq m$:

$$\sum_{|\alpha| \leq m} \|\partial^{\alpha+\beta}(v * \rho_\varepsilon)\|_{0,2}^2 \leq \frac{c_0}{\varepsilon^{2|\beta|}} \|v\|_{m,2}^2. \tag{4.15}$$

Taking $\varepsilon \in]0,1[$ yields $\frac{1}{\varepsilon^{2|\beta|}} \leq \frac{1}{\varepsilon^{2\ell}}$. Thus, adding for $|\beta| \leq \ell$ brings the inequality

$$\|v * \rho_\varepsilon\|_{m+\ell,2}^2 \leq \frac{c(\ell)}{\varepsilon^{2\ell}} \|v\|_{m,2}^2, \tag{4.16}$$

hence (4.7). □

We can now apply this result to prove Theorem 13.

Proof of Theorem 13. The definition of D_ε yields

$$\|D_\varepsilon(k)\|_{0,2,\Omega} = \left(\int_\Omega |\widetilde{\sqrt{A(k)}\nabla u(k)} * \rho_\varepsilon|^4 \, dx \right)^{1/2} = \left\| \widetilde{\sqrt{A(k)}\nabla u(k)} * \rho_\varepsilon \right\|_{0,4,\Omega}^2. \tag{4.17}$$

And since $N = 3$, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega)$ yields

$$\left\| \widetilde{\sqrt{A(k)}\nabla u(k)} * \rho_\varepsilon \right\|_{0,4,\Omega} \leq c_1 \left\| \widetilde{\sqrt{A(k)}\nabla u(k)} * \rho_\varepsilon \right\|_{1,2,\Omega} \leq \frac{c_2}{\varepsilon} \left\| \sqrt{A(k)}\nabla u(k) \right\|_{0,2,\Omega}. \tag{4.18}$$

Finally, (3.15) yields the result. □

In order to solve the regularized problem (4.5), we first linearize it, as said before.

$$\begin{cases} q = q(k) \in H_0^1(\Omega) \\ \forall p \in H_0^1(\Omega), \quad \displaystyle\int_\Omega B(k)\nabla q \cdot \nabla p \, dx = \int_\Omega D_\varepsilon(k) p \, dx. \end{cases} \tag{4.19}$$

We have already shown that $[p \mapsto \int_\Omega D_\varepsilon(k) p \, dx]$ is a continuous linear form on $H_0^1(\Omega)$, and then it is clear that $[(q,p) \mapsto \int_\Omega B(k)\nabla q \cdot \nabla p \, dx]$ is a bilinear form on $H_0^1(\Omega)$.

- Let $q, p \in H_0^1(\Omega)$:

$$\left| \int_\Omega B(k)\nabla q \cdot \nabla p \, dx \right| \leq \mu \|\nabla q\|_{0,2,\Omega} \|\nabla p\|_{0,2,\Omega} \leq \mu \|q\|_{1,2,\Omega} \|p\|_{1,2,\Omega}. \tag{4.20}$$

Then continuity is proved.

- Let $p \in H_0^1(\Omega)$:

$$\begin{aligned}\int_\Omega B(k)|\nabla p|^2\,dx \geq \nu\|\nabla p\|_{0,2,\Omega}^2 &\geq \frac{\nu}{2}\left(\|\nabla p\|_{0,2,\Omega}^2 + \frac{1}{c_\Omega^2}\|p\|_{0,2,\Omega}^2\right)\\ &\geq \frac{\nu}{2}\inf\left(1, \frac{1}{c_\Omega^2}\right)\|p\|_{1,2,\Omega}^2.\end{aligned} \tag{4.21}$$

Where c_Ω refers to the Poincaré-Friedrichs constant for Ω. Then coercivity is proved.

Until the end, c_Ω keeps the same meaning than in (4.21).

Consequently, problem (4.19) is a Lax-Milgram problem, hence by the same-named theorem, (4.19) has a unique solution $q = q(k) \in H_0^1(\Omega)$. Our goal is applying the Leray-Schauder fixed-point theorem on the operator $[k \mapsto q(k)]$.

Taking $p = q(k)$ in (4.19) yields

$$\int_\Omega B(k)|\nabla q(k)|^2\,dx = \int_\Omega D_\varepsilon(k)q(k)\,dx, \text{ hence } \nu\|\nabla q(k)\|_{0,2,\Omega} \leq c_\Omega\|D_\varepsilon(k)\|_{0,2,\Omega}. \tag{4.22}$$

Then Theorem 13 yields

$$\|\nabla q(k)\|_{0,2,\Omega} \leq \frac{c_\Omega\|D_\varepsilon(k)\|_{0,2,\Omega}}{\nu} \leq \frac{c_\Omega C}{\nu\varepsilon^2} =: M. \tag{4.23}$$

The constant $M > 0$ does not depend on k. Since Ω is bounded, the norms $\|\nabla\cdot\|_{0,2,\Omega}$ and $\|\cdot\|_{1,2,\Omega}$ are equivalent.

The end of this subpart aims at showing the existence of a solution to (4.5). This needs the Leray-Schauder fixed-point theorem (see [9]).

Theorem 15 (Leray-Schauder)**.** *Let X a Banach space, and B a convex closed bounded nonempty subset of X. Let $f : B \to B$ a continuous function with $f(B)$ compact.*
Then f admits at least one fixed point in B.

In our problem, we set $X = H_0^1(\Omega)$, $B = \bar{B}(0, M)$ and the function

$$\begin{array}{rccc} Q: & B & \to & B \\ & k & \mapsto & q(k) \end{array}.$$

It is clear that B is a convex closed bounded nonempty subset of $H_0^1(\Omega)$, and $Q(B) \subset B$. Remains to be proved that Q is continuous and $Q(B)$ is compact.

Let us show that $Q(B)$ is compact. Let $(k_n)_{n\in\mathbb{N}}$ a sequence of elements of B. Since (k_n) is bounded in $H^1(\Omega)$, a subsequence still named (k_n) can be extracted, which satisfies: $k_n \underset{n\to+\infty}{\overset{H^1}{\rightharpoonup}} k$. By theorem 10 with $p = 2$, this weak convergence is sufficient to obtain (3.20). The Young convolution inequality consequently yields

$$\|\sqrt{A(k_n)}\nabla u(k_n) * \rho_\varepsilon - \sqrt{A(k)}\nabla u(k) * \rho_\varepsilon\|_{0,2} \\ \leq \|\sqrt{A(k_n)}\nabla u(k_n) - \sqrt{A(k)}\nabla u(k)\|_{0,2} \times \underbrace{\|\rho_\varepsilon\|_{0,1}}_{=1} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.24}$$

This strong convergence in L^2 and the Lebesgue inverse theorem ensure the existence of a subsequence of $(k_n)_{n\in\mathbb{N}}$ still named the same way satisfying

$$\sqrt{A(k_n)}\nabla u(k_n) * \rho_\varepsilon \underset{n\to+\infty}{\overset{\text{ae.}}{\longrightarrow}} \sqrt{A(k)}\nabla u(k) * \rho_\varepsilon, \text{ hence } D_\varepsilon(k_n) \underset{n\to+\infty}{\overset{\text{ae.}}{\longrightarrow}} D_\varepsilon(k). \tag{4.25}$$

By theorem 9, $(u(k_n))_{n\in\mathbb{N}}$ strongly converges in $H^1(\Omega)$, then $(\nabla u(k_n))_{n\in\mathbb{N}}$ strongly converges in $L^2(\Omega)^N$. Thus, the Lebesgue inverse theorem yields the existence of $h \in L^2(\Omega)$ such that

$$\forall n \in \mathbb{N}, \quad |\nabla u(k_n)| \leq h \quad \text{ae. } \Omega, \tag{4.26}$$

where $(k_n)_{n\in\mathbb{N}}$ refers to another subsequence. Therefore, we obtain

$$|\widetilde{\nabla u(k_n)} * \rho_\varepsilon(x)| \leq \tilde{h} * \rho_\varepsilon(x). \tag{4.27}$$

And since h does not depend on n and $h * \rho_\varepsilon$ is continuous with compact support, $D_\varepsilon(k_n)$ is bounded by a constant that does not depend on n, then (4.25) and Lebesgue theorem yield

$$\forall p \in H_0^1(\Omega), \quad \int_\Omega D_\varepsilon(k_n) p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega D_\varepsilon(k) p \, dx. \tag{4.28}$$

Reporting (4.28) in (4.19) yields:

$$\forall p \in H_0^1(\Omega), \quad \int_\Omega B(k_n)\nabla q(k_n) \cdot \nabla p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega B(k)\nabla q(k) \cdot \nabla p \, dx. \tag{4.29}$$

We have proved that $B(k_n)\nabla q(k_n) \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} B(k)q(k)$. The Lebesgue inverse theorem and the continuity of $1/B$ yield $\frac{1}{B(k_n)} \underset{n\to+\infty}{\overset{\text{ae.}}{\longrightarrow}} \frac{1}{B(k)}$, and in addition $\frac{1}{B(k_n)}|\nabla p| \leq \frac{1}{\nu}|\nabla p|$. We then obtain

$$\frac{1}{B(k_n)}\nabla p \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} \frac{1}{B(k)}\nabla p. \tag{4.30}$$

The assumptions of Lemma 11 are satisfied, so that it yields

$$\forall p \in H_0^1(\Omega), \quad \int_\Omega \nabla q(k_n) \cdot \nabla p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega \nabla q(k) \cdot \nabla p \, dx, \text{ hence } \nabla q(k_n) \underset{n\to+\infty}{\overset{L^2}{\rightharpoonup}} \nabla q(k). \tag{4.31}$$

We deduce by compact embedding $q(k_n) \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} q(k)$. We now intend to show

$$\int_\Omega D_\varepsilon(k_n) q(k_n) \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega D_\varepsilon(k) q(k) \, dx. \tag{4.32}$$

Another decomposition is needed:

$$\|D_\varepsilon(k_n)q(k_n) - D_\varepsilon(k)q(k)\|_{0,1,\Omega} \leq \|(D_\varepsilon(k_n) - D_\varepsilon(k))q(k)\|_{0,1,\Omega} \\ + \|D_\varepsilon(k_n)(q(k_n) - q(k))\|_{0,1,\Omega}. \tag{4.33}$$

On the one hand, the Lebesgue theorem yields $\|(D_\varepsilon(k_n) - D_\varepsilon(k))q(k)\|_{0,1,\Omega} \underset{n\to+\infty}{\longrightarrow} 0$.

On the other hand, there exist two constants c_1 and c_2 due to the domination of $(D_\varepsilon(k_n))$ and the embedding $L^2(\Omega) \hookrightarrow L^1(\Omega)$ such that

$$\|D_\varepsilon(k_n)(q(k_n) - q(k))\|_{0,1,\Omega} \leq c_1\|q(k_n) - q(k)\|_{0,1,\Omega} \leq c_2\|q(k_n) - q(k)\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0. \quad (4.34)$$

Hence, (4.32) is proved. Reporting in (4.19) yields

$$\int_\Omega B(k_n)|\nabla q(k_n)|^2\,dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega B(k)|\nabla q(k)|^2\,dx. \quad (4.35)$$

We need a last decomposition:

$$\begin{aligned}\int_\Omega \left(B(k_n)|\nabla q(k_n)|^2 - B(k)|\nabla q(k)|^2\right)\,dx = &\int_\Omega B(k)\left(|\nabla q(k_n)|^2 - |\nabla q(k)|^2\right)\,dx \\ &+ \int_\Omega (B(k_n) - B(k))|\nabla q(k_n)|^2\,dx.\end{aligned} \quad (4.36)$$

Given that the sequence $\left(B(k_n)|\nabla q(k_n)|^2\right)_{n\in\mathbb{N}}$ is bounded, (4.35) yields for another subsequence still named $(k_n)_{n\in\mathbb{N}}$:

$$\sqrt{B(k_n)}\nabla q(k_n) \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} \sqrt{B(k)}\nabla q(k). \quad (4.37)$$

Then, by the Lebesgue inverse theorem, a subsequence still named the same way is dominated. That means $\left(|\nabla q(k_n)|^2\right)_{n\in\mathbb{N}}$ is dominated by a L^1 function. Then the Lebesgue theorem yields

$$\int_\Omega (B(k_n) - B(k))|\nabla q(k_n)|^2\,dx \underset{n\to+\infty}{\longrightarrow} 0. \quad (4.38)$$

Consequently, (4.36) yields

$$\int_\Omega B(k)\left(|\nabla q(k_n)|^2 - |\nabla q(k)|^2\right)\,dx \underset{n\to+\infty}{\longrightarrow} 0. \quad (4.39)$$

Let N_B be defined by

$$N_B : f \in H_0^1(\Omega) \mapsto \|B(k)\nabla f\|_{0,2,\Omega}. \quad (4.40)$$

The two norms $\|\cdot\|_{1,2,\Omega}$ and N_B are equivalent, and (4.39) ensures that $(q(k_n))_{n\in\mathbb{N}}$ strongly converges for the norm N_B. Thus, it also strongly converges in $H_0^1(\Omega)$.

We obtain

$$q(k_n) \underset{n\to+\infty}{\overset{H^1}{\longrightarrow}} q(k). \quad (4.41)$$

Therefore, we have shown that $Q(B)$ is compact, and by a very similar way, the continuity of Q is deduced. Consequently, Theorem 15 ensures the existence of a fixed-point of Q in B. Which means there exists a function $k \in B$ that satisfies (4.5).

4.3 Last step: Taking the limit when ε goes to zero

Let us define a family of intermediary solutions with respect to what has just been done. For $\varepsilon > 0$, the intermediary solution k_ε satisfies

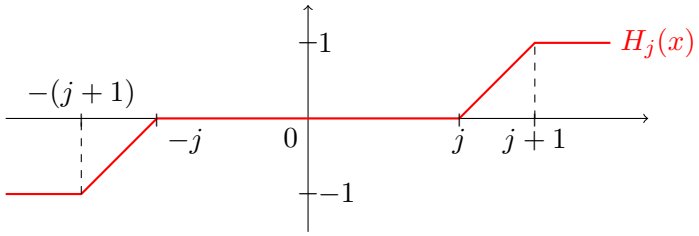
$$\forall p \in H_0^1(\Omega), \quad \int_\Omega B(k_\varepsilon)\nabla k_\varepsilon \cdot \nabla p \, dx = \int_\Omega D_\varepsilon(k_\varepsilon) p \, dx. \tag{4.42}$$

Our next goal is showing that (k_ε) is bounded uniformly with respect to ε. For this we use the washer lemma, (Lemma 1). But a general theorem from Stampacchia (see [8]) is needed before.

Theorem 16 (Regularity and truncation function)**.** *Let $H \in W^{1,\infty}(\mathbb{R})$ having a finite number of discontinuities, and such that $H(0) = 0$. Let $q \in [1, +\infty[$ and $v \in W_0^{1,q}(\Omega)$.*

Then $H(v) \in W_0^{1,q}(\Omega)$ and we have: $\nabla H(v) = H'(v)\nabla v$.

Until the end of this chapter, for $j \in \mathbb{N}$, H_j refers to the truncation function defined by



This function satisfies the assumptions of 16, so that we can take $p = H_j(k_\varepsilon)$ in (4.42) to obtain

$$\forall j \in \mathbb{N}, \quad \int_\Omega B(k_\varepsilon)|\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) \, dx = \int_\Omega D_\varepsilon(k_\varepsilon) H_j(k_\varepsilon) \, dx. \tag{4.43}$$

On one hand, by writing $B_j^\varepsilon = \{x \in \Omega \ / \ j \leq |k_\varepsilon(x)| < j+1\}$:

$$\int_\Omega B(k_\varepsilon)|\nabla k_\varepsilon|^2 H_j'(k_\varepsilon) \, dx = \int_{B_j^\varepsilon} B(k_\varepsilon)|\nabla k_\varepsilon|^2 \, dx \geq \nu \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2 \, dx. \tag{4.44}$$

One the other hand, the Young convolution inequality yields

$$\int_\Omega D_\varepsilon(k_\varepsilon) \underbrace{H_j(k_\varepsilon)}_{\leq 1} \, dx \leq \left\| \sqrt{A(k_\varepsilon)}\widetilde{\nabla u(k_\varepsilon)} * \rho_\varepsilon \right\|_{0,2}^2 \leq \left\| \sqrt{A(k_\varepsilon)}\nabla u(k_\varepsilon) \right\|_{0,2,\Omega}^2. \tag{4.45}$$

By (3.15), the right hand side is uniformly bounded with respect to ε. Both inequalities (4.44) and (4.45) reported in (4.43) yield

$$\exists C > 0, \forall \varepsilon > 0, \forall j \in \mathbb{N}, \quad \int_{B_j^\varepsilon} |\nabla k_\varepsilon|^2 \, dx \leq C. \tag{4.46}$$

This corresponds to the assumptions of lemma 1. Then for any exponent $q \in]1, N'[$, we have: $\|k_\varepsilon\|_{1,q,\Omega} \leq A$. And given that C does not depend on ε, this is the same for A (as a consequence of lemma 1).

Therefore, $(k_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,q}(\Omega)$ for $q \in]1, N'[$. Banach-Alaoglu theorem yields the existence of a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$ and $k_{\varepsilon_n} \overset{W_0^{1,q}}{\underset{n\to+\infty}{\rightharpoonup}} k$. Therefore, $k \in V$.

We need to show that k is a solution of (4.4), which means we need to show

$$\forall p \in H_0^1(\Omega), \quad \int_\Omega B(k)\nabla k \cdot \nabla p \, dx = \int_\Omega D(k)p \, dx. \tag{4.47}$$

The intermediary solutions (k_{ε_n}) satisfy

$$\forall p \in H_0^1(\Omega), \quad \int_\Omega B(k_{\varepsilon_n})\nabla k_{\varepsilon_n} \cdot \nabla p \, dx = \int_\Omega D_{\varepsilon_n}(k_{\varepsilon_n})p \, dx. \tag{4.48}$$

Convergence of the left hand side:

We have supposed that $k_{\varepsilon_n} \overset{W_0^{1,q}}{\underset{n\to+\infty}{\rightharpoonup}} k$, which means in particular

$$\forall p \in W, \quad \int_\Omega \nabla k_{\varepsilon_n} \cdot \nabla p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega \nabla k \cdot \nabla p \, dx. \tag{4.49}$$

To add the term $B(k_n)$, it seems useful to state a variant of Lemma 11.

Lemma 17 (Double convergence in V and W)**.** *Let Ω be an open subset of $\mathbb{R}^N$, $f \in \bigcap_{1<r<N'} L^r(\Omega)$ and $g \in L^q(\Omega)$, where $q > N$. Let us consider two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ such that*

$$\begin{cases} \forall r \in]1, N'[, & f_n \overset{L^r}{\underset{n\to+\infty}{\longrightarrow}} f \\ & g_n \overset{L^q}{\underset{n\to+\infty}{\rightharpoonup}} g \end{cases} \tag{4.50}$$

Then we have the convergence

$$\int_\Omega f_n g_n \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega fg \, dx. \tag{4.51}$$

Proof. The proof of Lemma 17 is similar to the proof of Lemma 11. We write

$$\int_\Omega (f_n g_n - fg) \, dx = \int_\Omega (f_n - f)g_n \, dx + \int_\Omega f(g_n - g) \, dx. \tag{4.52}$$

On one hand, the Hölder inequality yields

$$\left| \int_\Omega (f_n - f)g_n \, dx \right| \leq \|f_n - f\|_{0,q',\Omega} \|g_n\|_{0,q,\Omega} \leq M \|f_n - f\|_{0,q',\Omega} \underset{n\to+\infty}{\longrightarrow} 0, \tag{4.53}$$

where q' refers to the conjugate of q, which is in $]1, N'[$.

On the other hand, the weak convergence of $(g_n)_{n\in\mathbb{N}}$ yields

$$\int_\Omega (g_n - g)f \, dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.54}$$

Hence the result. □

Let us now check if the assumptions of Lemma 17 are satisfied.

The compact embedding $V \hookrightarrow L^1(\Omega)$ yields $k_{\varepsilon_n} \xrightarrow[n\to+\infty]{L^1} k$. Thus, the Lebesgue inverse theorem yields $k_{\varepsilon_n} \xrightarrow[n\to+\infty]{\text{ae.}} k$, where (ε_n) refers to a subsequence. Hence, by countinuity of B: $B(k_{\varepsilon_n}) \xrightarrow[n\to+\infty]{\text{ae.}} B(k)$. And the domination $B(k_{\varepsilon_n})|\nabla p| \leq \mu|\nabla p|$ stands. Consequently, the Lebesgue theorem yields

$$\forall r \in]1, N'[, \forall p \in W, \quad B(k_{\varepsilon_n})\nabla p \xrightarrow[n\to+\infty]{L^r} B(k)\nabla p. \tag{4.55}$$

And the weak convergence of $(k_{\varepsilon_n})_{n\in\mathbb{N}}$ is already established. Then, Lemma 17 yields the convergence of the left hand side

$$\forall p \in W, \quad \int_\Omega B(k_{\varepsilon_n})\nabla k_{\varepsilon_n} \cdot \nabla p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega B(k)\nabla k \cdot \nabla p \, dx. \tag{4.56}$$

Convergence of the right hand side:

We aim at applying the Lebesgue inverse theorem on $(D_{\varepsilon_n}(k_{\varepsilon_n}))$, so we need to prove

$$\forall p \in W, \quad D_{\varepsilon_n}(k_{\varepsilon_n})p \xrightarrow[n\to+\infty]{L^1} D(k)p. \tag{4.57}$$

Yet, it is clear that for $n \in \mathbb{N}$ and $p \in W$

$$\|D_{\varepsilon_n}(k_{\varepsilon_n})p - D(k)p\|_{0,1,\Omega} = \|D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)\|_{0,1,\Omega} \, \|p\|_{0,\infty,\Omega}. \tag{4.58}$$

Then we write a decomposition.

$$\begin{aligned} \int_\Omega |D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)| \, dx &= \int_{\mathbb{R}^N} \left| A(k_{\varepsilon_n}) |\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 - A(k) |\widetilde{\nabla u(k)}|^2 \right| dx \\ \leq \int_\Omega |A(k_{\varepsilon_n}) - A(k)| \cdot |\nabla u(k)|^2 \, dx &+ \int_{\mathbb{R}^N} A(k_{\varepsilon_n}) \left| |\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 - |\widetilde{\nabla u(k)}|^2 \right| dx. \end{aligned} \tag{4.59}$$

The same argument as in the proof of (4.55) yields

$$\int_\Omega |A(k_{\varepsilon_n}) - A(k)| \cdot |\nabla u(k)|^2 \, dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.60}$$

It remains to prove the following

$$\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n} \xrightarrow[n\to+\infty]{L^2} \widetilde{\nabla u(k)}. \tag{4.61}$$

First we write

$$\begin{aligned} \left\| \widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} \right\|_{0,2} \leq &\left\| \widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} \right\|_{0,2} \\ &+ \left\| \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} \right\|_{0,2}. \end{aligned} \tag{4.62}$$

On one hand, the Young convolution inequality yields

$$\left\| \widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} \right\|_{0,2} \leq \|\nabla u(k_{\varepsilon_n}) - \nabla u(k)\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0, \tag{4.63}$$

as a consequence of theorem 9.

On the other hand, a classical convergence result has to be recalled.

Theorem 18. *Let v a $L^p(\mathbb{R}^N)$ function with compact support ($1 \leq p < +\infty$), and (ρ_ε) a mollifier. Then*

$$\|v * \rho_\varepsilon - v\|_{0,p} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.64}$$

Theorem 18 applied to $\widetilde{\nabla u(k)}$ brings the following convergence

$$\left\| \widetilde{\nabla u(k)} * \rho_{\varepsilon_n} - \widetilde{\nabla u(k)} \right\|_{0,2} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.65}$$

This finally yields (4.61).

There is a subsequence, still named (ε_n), such that $|\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 \underset{n\to+\infty}{\longrightarrow} |\widetilde{\nabla u(k)}|^2$ and $|\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 \leq g \in L^2(\mathbb{R}^N)$ almost everywhere (g can be taken with compact support).

Therefore, Lebesgue theorem can be used for the last term of (4.59) to obtain

$$\int_{\mathbb{R}^N} A(k_{\varepsilon_n}) \left| |\widetilde{\nabla u(k_{\varepsilon_n})} * \rho_{\varepsilon_n}|^2 - |\widetilde{\nabla u(k)}|^2 \right| dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.66}$$

We finally obtain

$$\|D_{\varepsilon_n}(k_{\varepsilon_n}) - D(k)\|_{0,1,\Omega} \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.67}$$

Therefore, (4.58) yields the convergence of the right hand side

$$\forall p \in W, \quad \int_\Omega D_{\varepsilon_n}(k_{\varepsilon_n}) p \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega D(k) p \, dx. \tag{4.68}$$

We have shown that the function k given by the washer lemma satisfies (4.4). Moreover, since $k \in V$, there exists $q \in]1, N'[$ such that $q^* > 2$ (for example by taking $q = \frac{4}{3}$, $q^* = \frac{12}{5} > 2$).

As a result, the embedding $V \hookrightarrow L^2(\Omega)$ is compact. Then (k_{ε_n}) strongly converges to k in $L^2(\Omega)$. This allows us to conclude.

4.4 Conclusion: solution of the coupled problem

Let us consider the sequence (ε_n) mentioned previously, after all the extractions that were done. We have shown that (k_{ε_n}) strongly converges in $L^2(\Omega)$ to a function k.

Moreover, (k_{ε_n}) weakly converges to k in every space $W_0^{1,q}(\Omega)$ with $q \in]1, N[$. Therefore, theorem 9 yields that $(u(k_{\varepsilon_n}))$ strongly converges to $u(k)$ in $H^1(\Omega)$.

All of this ensures the existence of a couple $(u; k)$ of solutions of (1.1) at the distribution sense, with $u \in H^1(\Omega)$ and $k \in V$ (in particular $k \in L^2$).

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