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Praxeologies-in-action during struggle with problem-solving

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Engaging in productive struggle during problem-solving activities can be viewed as critical to learning mathematics. In this paper, we explore how mathematical praxeologies can be utilised to describe students' struggle with mathematical problem-solving. The results indicate that the students' productive struggle with an optimisation problem involving functions, graphs, and derivatives, contains long sequences where the students switch rapidly within and between mathematical praxeologies. During the work session as a whole, the students mix coherent arguments with tentative and, on occasion, less coherent arguments focusing on technology.

Keywords: Productive struggle, praxeologies, problem solving, derivative.

Introduction

Learning mathematics through problem-solving involves making use of mathematical knowledge and skills; strategies and techniques; control, monitoring and self-regulation; and beliefs that influence how the problem is approached (Schoenfeld, 1985). The strategies and techniques chosen by the learner can either be directly applied or be adapted to solve a specific problem or part of a problem or may be inadequate for solving the problem. In a worst-case scenario, the learner may keep pursuing various techniques without showing progress towards a solution (Schoenfeld, 1987). When learners are confronted with mathematical problems that they cannot immediately solve, they can choose to struggle with important mathematics, that is, they are engaged and grapple with key mathematical ideas that are comprehensible but not yet well formed. The aim of this struggle is to make sense of features and challenges that are embedded in the problem (Hiebert & Grouws, 2007). The struggle is an integral part of doing mathematics and is a process in which the connections among mathematical facts, ideas, and procedures are restructured (Hiebert & Grouws, 2007).

Several studies in mathematics education have put focus on mathematics instruction and strategies that teachers can implement to encourage productive struggle, as an “effort to make sense of mathematics, to figure something out that is not immediately apparent” (Hiebert and Grouws 2007, p. 287). To encourage productive struggle, teachers should be careful in selecting tasks to be used in their classrooms and employ pedagogical practices that support students' mathematical reasoning (Amidon et al., 2020). Although promoting productive struggle through challenging tasks is gaining interest, there is little research on students' productive struggle during mathematical problem-solving activities. By analysing students' productive struggle, valuable insights can be gained into how they utilise and enhance their understanding of mathematical concepts. Such insights can serve as a foundation for teachers to effectively support and encourage productive struggle in their classrooms. In this paper, we contribute to the needed research by studying students' reasoning during a problem-solving activity about an optimisation problem in Calculus, involving functions, graphs and derivatives.

The process of reasoning takes place when two or more persons engage in communication about an argument or a proof. Success depends on their common action, as the behaviour that learners

undertake with a specific intention to achieve desired goals or ends. The mutual engagement of participants in a coordinated effort to solve the problem together, that is, collaborative problem-solving (Barkley et al., 2014), supports students in creating positive peer relationships and controlling the problem-solving process.

In this paper, we report on a qualitative study in which we investigate how students organise and apply knowledge that they have previously been taught. In our analysis, the forms of reasoning, their function, and their use are interpreted through the anthropological theory of the didactic (ATD) and particularly praxeologies (Chevallard, 2020). The constructs from ATD will be defined in the next section. The purpose of the study is to explore how the constructs in ATD can be used to analyse students' struggle with mathematical problem-solving outside the classroom. We address the following research question:

How can mathematical praxeologies be utilised to describe students' struggle with mathematical problem-solving?

Related literature and theoretical perspective

The research in the field of Calculus education discusses, for instance, students' difficulties with Calculus, students' learning experiences and the meaning-making of concepts and structures in Calculus across educational levels (Thompson & Harel, 2021), students' discourse about the derivative (Park, 2013), and the derivative through the problem of the tangent to a generic function within textbooks and in teaching (Panero, 2015). A result from those studies shows that the common description of the derivative as a tangent line given by students is linked to their use of the word "derivative" for both the derivative of the function and the derivative at a point (Park, 2013). Another result shows that the discourse about the equation of the tangent "is centred on the definition of the tangent as the best linear approximation [without] allusion to pointwise and global aspects of the function" (Panero, 2015, p. 515). Such works highlight that in Calculus, calculation practices focus on algebraic computational rules, which are rarely explained (Kondratieva & Winsløw, 2018).

In a problem-solving process within an educational institution, there appears several patterns of action connected to the process of construction or reconstruction of a mathematical praxeology. This process starts with a specific task that can be interpreted as belonging to a type of task (T). The type of task (T) groups together the tasks that can be accomplished by the same technique (τ), explained by a technology (θ) which itself is justified by a theory (Θ) (Chevallard, 2020). These four components constitute the four elements of a praxeology. Type of task and technique constitute the praxis block of the praxeology, while technology and theory constitute the logos block. For the same type of task, there may be several techniques, technologies, and theories that constitute different praxeologies. Institutional rules or constraints have an important role on how those praxeologies are shaped. In this paper we particularly analyse how techniques and technologies are utilised by students with respect to different subtasks connected to the task that those students are engaged in. As mentioned above, in our study the type of task is an optimisation problem in Calculus. The techniques consist of the actions that the students apply when they attempt to solve the problem, while the technologies consist of students' explanations, as justifications of their use of techniques. When the students' explanations also function to justify the technologies, we interpret these justifications as elements of theory. For

example, when a student calculates the derivative of e^{3x} , the student implements a technique. If this student explains how the formula for the derivative for e^{ax} works, this explanation is interpreted as technology. In our study, where students work together with mathematical problem-solving, we put focus on distinguishing between their technical work and technological discussions where the students explain their use of techniques.

Method

The study takes place within an educational institution of university mathematics. The participating students were enrolled in an introductory Calculus course, with lectures and problem-solving sessions, offered in a teacher education for upper secondary school programme.

In this paper, we describe a part of the analysis of a problem-solving session with two undergraduate students (one male, one female) who collaborate to solve an optimisation problem involving functions, graphs, and derivatives.

The students were instructed by the teacher to work in pairs with selected problems, without assistance from the teacher. The students had not previously attempted to solve these problems. The researcher (the first author of this paper) invited the students to participate in the research study. Two pairs volunteered and one pair was selected for further analysis. The session lasted 1 hour and 45 minutes.

The task that the students worked on (Figure 1, translated from Swedish to English) was selected by the researcher among the problems recommended to the students by the teacher.

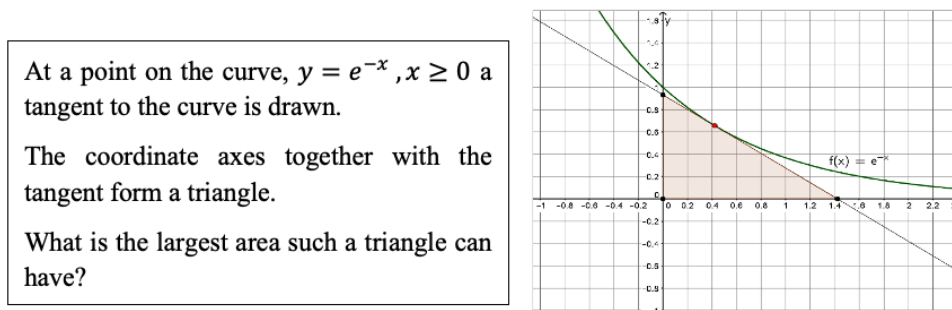


Figure 1: The task, as presented to the students

The participating students' prior knowledge included the concepts of tangent and derivative, as well as how to solve optimization problems by using derivatives, from upper secondary school and their regular course work. The task that the students worked on was provided by the teacher at the time of the data collection. Therefore, the students had no opportunity to prepare any solution to the task in advance. The students worked on the problem by drawing and writing on a touchscreen laptop while reasoning about their work. In addition, one of the students explored problem-related constructions in GeoGebra on her computer. These constructions were also discussed between the students regarding both technical and mathematical aspects. We have chosen not to put focus on these aspects in our analysis since these discussions did not contribute significantly towards solving the problem.

In the a-priori analysis of the task presented to the students, we could identify multiple subtasks, which we will briefly present. One of these subtasks involves identifying the equation of the tangent

line to the given function at a specific tangent point, which in this case can be denoted (a, e^{-a}) . A wide variety of techniques can be used by the students when working on this type of task. For instance, the slope-intercept formula for a line is $y = kx + m$, where k is the slope of the line and m is the y -intercept (standard Swedish notation); the point-slope formula for a line is $y - y_1 = k(x - x_1)$ where (x_1, y_1) is a point on the line and k is the slope of the line; calculating the first derivative $y' = f'(x) = -e^{-x}$ and calculating $f'(a) = -e^{-a}$ to find the value for k . When these calculations are connected to the formula for the area of a right triangle (base times height divided by 2) the area A can be expressed in terms of a and algebraically simplified to $A = \frac{1}{2}e^{-a}(1 + a)^2$. The remaining part of the solution can – from a mathematical point of view – be considered as straightforward, by computing the derivative A' by applying the product rule for derivatives, algebraically simplifying the result to $A' = \frac{1}{2}e^{-a}(1 + a)(1 - a)$, identifying the relevant zero $a = 1$ of the derivative and analyzing the outcome to confirm the maximum value $A = 2e^{-1}$.

The data for this paper consists of a video-recorded session, and students' written answers. The researcher (first author) observed the session. A two-phase qualitative content analysis was applied to investigate the collected data (Bryman, 2020). During the first phase, data was analysed to identify the subtasks that the students chose to address. During the second phase, we characterised students' actions in terms of technical, technological, and theoretical moments, for example technical work and technological discussions. During our analysis we observed a connection between students' struggles with different subtasks and their shifts between praxeologies. For this reason, we particularly put focus on identifying these shifts. We identified these shifts based on students' change of subtasks or their use of new techniques. Figure 2 maps the connections within and between different elements of the mathematical praxeology which are illustrated by solid lines. The dashed line delimits the praxis from logos and at the same time indicates the chronological order between different actions. Rectangular shapes are used to indicate headings referring to parts of the praxeology in longer episodes, while oval shapes are used to mark the beginning or end of an action. For this paper, the relevant transcriptions have been translated from Swedish to English. Square brackets [...] are used to clarify what the students are referring to when it is not clear what they are talking about, as well as to mark that some part of the citation is excluded.

Results and analysis

The discernible aim of the initial part of the students' work with the task (Figure 1) is to determine the area of the right triangle. This initial part consists of students addressing several subtasks. Student 1 initiates the work session by denoting the tangent point as $A(a, f(a))$ [verbatim], representing the equation for the tangent line as $g(x) = kx + m$, and expressing the slope as $k = f'(a)$. All this is part of a technological moment, a sequence where $k = f'(a)$ appears as a technological conviction justified by the students as "The first derivative is an equation for the slope of a tangent line to a curve at an indicated point". Student 1 tries to express the tangent in the point $A(a, f(a))$ as $f(a) = f'(a) \cdot x + m$ that is corrected to $f(a) = f'(a) \cdot a + m$. This work sequence can be seen as an initiation of a technique τ for the subtask of finding m . The students are aware of the importance of solving this subtask for determining the area of the triangle. At this stage, they switch to looking at the screen of Student 2's computer, which shows the graphic window in

GeoGebra. The students use GeoGebra to construct a sliding tangent line which allows them to dynamically measure the area of the triangle. They agree that they have solved the problem in GeoGebra. In the next section, we will focus on students' struggle towards finding m .

Students' struggle

In this section we will present a sequence from the students' struggle. We put focus on a part where the students struggle with expressing the point of intersection of the tangent line with the y-axis. We first present an overview of this struggle (Figure 2) that will be elaborated below.

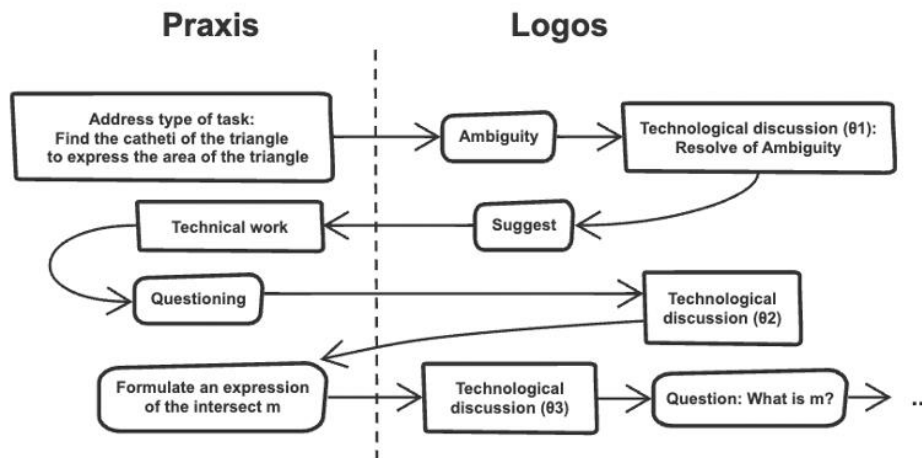


Figure 2: Overview of the process of struggle

Address type of task: The students continue to discuss how to calculate the area of the triangle. Student 2 gives the following explanation:

- Student 2: I wrote that the area is equal to y times x but then I realised that y equals $f(0)$.
 Student 1: Yes!
 Student 2: OK, but m is also equal to $f(0)$ [Pause] therefore you can take m times x divided by 2.
 Student 1: Well, I have written [...] I have thought a little differently, but it is the same.

Student 2 has, in her notes, written: Area $\frac{f(0) \cdot x}{2}$, $y = -e^{-x} \cdot x + m$, $m = f(0)$, $A = \frac{y \cdot x}{2}$. This notation is slightly different from the notation introduced by Student 1 and at this point the students seem to sense that there is an ambiguity in their reasoning which they address in the continued discussion.

Technological discussion (θ1): Student 1 initiates the following discussion.

- Student 1: Here we have a straight line $kx + m$ [points at the line] but the slope for this line should be the derivative of this function [points at the graph].
 Student 2: Yes!
 Student 1: But then I have written that $k = f'(a)$. I don't know where this point should be because it depends on where we move it. And as you said m is ... no, m is...
 Student 2: Yes, m is the same as $f(0)$.
 Student 1: m is this [points at the intersection between the y-axis and the tangent line]. The function is there [points at the intersection between the y-axis and the graph of $y = e^{-x}$], can you see?
 Student 2: Yes!
 Student 1: The function is up there when zero but m is a bit lower down... or do you mean a different m ?
 Student 2: I was thinking m for the tangent line [shows $y = -e^{-x} \cdot x + m$ and $m = f(0)$].

- Student 1: If you think... $f(0)$ is this point...but m for the line [the tangent line] is further down.
 Student 2: Yes.
 Student 1: Therefore it is not $f(0)$.

The technology involves one student justifying to the other student, using a drawn graph, that the intersect m and the intersect of the function with the y-axis are not the same. The discussion continues and Student 1 motivates the already made statement $f(a) = f'(a) \cdot a + m$. Student 1 suggests continuing by differentiating the function $f(x) = e^{-x}$.

Technical work: The students calculate $f'(x)$ and Student 1 writes $g(x): e^{-x} = -e^{-x}$ which is questioned by Student 2. Student 1 writes $g(x) = -e^{-x} \cdot x + m$. This leads to a discussion about x .

Technological discussion (θ2): This technological discussion involves the use of variables to denote mathematical concepts. The students realise that there is a conflict in how they denote the variables because they use x both as a variable of the tangent line and as a variable in the derivative of the function. Although they agree, with arguments that are not very developed, the students manage to connect (x, y) and the tangent point $(a, f(a))$. Eventually the students express the intersect m as $e^{-a}(1 + a) = m$. The students do not yet realise how to utilise this result for solving the subtask.

Technological discussion (θ3): Student 2 initiates a new discussion.

- Student 2: If you look at this [points at the expression $e^{-a} = -e^{-a} \cdot a + m$] we can use, choose any x -value and insert into the original function and then you will get a y -value, then you have an x -value and y -value at that point [points at the tangent point].
 Student 1: Hmm
 Student 2: Then you can use the same y -value to figure out what this is [points at $g(x) = -e^{-x} \cdot x + m$].
 Student 1: Hmm? Yes, but then you get m for a specific point and we want to be able to derive and find the minimum value...we need something more, we are missing something in order to move on [...]. We need to find m ! What is m ?...

Thus far, student 2 has made the most of the contributions and finds a need to explain to student 1, how to use the expressions above for further work. At this point the explanations provided also function as a justification of the choice of notation needed for their continued work. The question about m marks the end of the part of the students' struggle that we have focused on and that is shown in Figure 2. However, they have not solved the problem. For example, in order to find a way to calculate m , the idea of utilising a linear transformation is suggested but with little success: "Moving the line one step up is the same as moving the line there [point to the right]".

After having spent 35 minutes on the problem, they engage in a new approach where only a line is present in the coordinate system together with b and h , as symbols for positive quantities representing base and height of a right triangle. The students initiate a new approach, where they pursue a general investigation about the equation of a straight line without involving derivatives. This move can be interpreted as a shift from a Calculus praxeology to a geometrical praxeology. After introducing new notation for the triangle's base and height, they write down the equation of the tangent line again ($g(x) = -\frac{h}{b}x + h$). This allows them to put focus on the relationship between k and m in the slope-intercept formula $y = kx + m$. The students eventually write an expression for the area $A =$

$\left(m \left(-\frac{m}{k}\right) \cdot \frac{1}{2}\right)$ and from this point the students' struggle becomes more straightforward towards the solution.

Discussions and conclusion

An important contribution of this paper is the identification of a model for analysing students' struggle in terms of mathematical praxeologies (Chevallard, 2020). A traditional classroom study with ATD is based on an object of learning connected to a praxeological reference model. Since our focus in this paper is not so much on formal mathematical technological justifications, but rather on how such justification appears in the students' discourse, we have opted for not involving any reference models as theoretical underpinnings for the praxeologies selected by the students.

The results indicate that students' struggle contains long sequences where the students switch rapidly between techniques and technology, often prematurely leaving one subtask and switching to the next. These switches are intensified, particularly between technical and technological moments, when students struggle with temporary confusion. For instance, during technological discussion (01) the students present coherent arguments about the intersection between the tangent line and the y-axis, but from different perspectives. The difference stems from confusion about the notation of the function and the tangent line, which the students manage to overcome by trying out new approaches. The students' attempts, that involve technical and technological moments, are often about strategies and techniques; control, monitoring, and self-regulation in the problem-solving processes, which, according to Schoenfeld (1985), are important elements in problem-solving. One notable exception among these attempts is a theoretical moment where the students reduce the complexity of the previous subtask by deducing the equation $g(x) = -\frac{h}{b}x + h$ without involving derivatives. From that point on, they are confident that they are on the right track and – although they keep making several technical mistakes – finally manage to solve the problem.

The results also indicate that, in the shift between subtasks the students are not fully able to continue where the previous discussion related to the same subtask ended. An example is when the students differentiate the function $f(x) = e^{-x}$ and arrive at an expression for the tangent line without realising how to utilise this result. Instead, the students keep pursuing familiar techniques, which they are able to execute albeit not necessarily being productive, without showing progress towards a solution. One explanation for this temporary confusion could be that the students do not distinguish the derivative as a function and the derivative at a point. This result is in line with Park's (2013) observation, regarding the students' description of the derivative as a tangent line.

A key concern for teachers and researchers is how to develop students' capacity to persevere in solving mathematical problems and to help students realise that they can do well in mathematics with struggle. In this paper, we show that the students' struggle can be described by analysing students reasoning in acting to solve a problem and by paying attention to how the students' switch within and between mathematical praxeologies. We also see another type of praxeology that arises when GeoGebra is used to try to solve subtasks in the task. GeoGebra is here a tool that is intended to be used to solve the task with graphical techniques that differ from the analytical techniques needed to solve the task with only pen and paper. This raises a question about the task itself, and how it can be

designed to support productive struggle (Amidon et al., 2020). The students' struggle might have been more productive if the intended technologies (in GeoGebra and by hand) had been designed to support each other. The students' struggle might also have been more productive if the students had been able to choose other techniques and technologies, for example the point-slope formula $y - f(a) = f'(a) \cdot (x - a)$.

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