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# Knowledge mobilised in a teaching experiment about conjecturing and proving by elementary preservice teachers

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*This paper presents preliminary results of the implementation of a teaching experiment designed to promote reasoning and proving processes with elementary preservice teachers. The teaching experiment deals with the inscribed angle theorem. It has been designed following the principles of task design for conjecturing and proving of Lin et al. (2012), and it tries to generate the intellectual need of deductive reasonings. Using the Mathematics Teachers' Specialised Knowledge (MTSK) model as an analytical tool, we have analysed the productions of the participants to detect and characterise the mathematical knowledge deployed by them during the task. Then, we use these results to reflect about the potential and limitations of the teaching experiment. Here, we present the results of the analysis of a pair of preservice teachers, whose behaviour was more aligned with the objective of the teaching experiment.*

*Keywords: Preservice teachers, teaching experiment, practices in mathematics, reasoning and proving, MTSK.*

## Introduction and research objectives

There is an agreement amongst the researchers on Mathematics Education about the importance and relevance of the processes of reasoning and proving (Stylianides et al., 2016), as fundamental processes in mathematics (NCTM, 2000). For example, the new Spanish curriculum sets reasoning and proving as one of the axes for the specific mathematical competences at all education levels. As a result, mathematics teachers, as a key figure in the teaching and learning of mathematics, must programme, manage, and exploit learning opportunities linked with reasoning and proving.

To make this possible, it is a necessary (but not sufficient) condition to have enough mathematical knowledge about these processes (Carrillo-Yañez et al., 2018). The learning of reasoning and proving processes is difficult for students, and also for preservice teachers (Stylianides & Stylianides, 2009). Hence, there is a need to design and implement effective tasks to revise and develop the mathematical knowledge of preservice teachers about these processes (Stylianides et al., 2016). In this report, we present preliminary results of the implementation of a teaching experiment about the inscribed angle theorem with elementary preservice teachers (hereafter, EPT), focusing on conjecturing and proving. The research objectives are:

- Detect and characterize the mathematical knowledge about reasoning and proving processes deployed by EPT during the teaching experiment and its stages.
- Reflect about the relationships between this mathematical knowledge deployed and the teaching experiment designed.

## **Theoretical framework**

### **Knowledge of mathematics teachers with a focus on reasoning and proving processes**

Building on the seminal work of Shulman (1986), in the last decades different models that try to characterise types and components of mathematics teacher knowledge have been developed. One example is the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008), which recognises a special kind of mathematical knowledge that mathematics teachers need in a teaching and learning environment.

Another example is the Mathematics Teacher's Specialised Knowledge (MTSK) model, developed by Carrillo-Yañez et al. (2018). This model focuses its attention on the knowledge that makes the teacher a specialist in the teaching of mathematics, leaving out of its focus some relevant aspects for professional tasks (such as the management of participation), but which are not closely related to the act of specifically teaching mathematics (Scheiner et al., 2019). The MTSK model, as well as MKT model, distinguishes two domains of knowledge: mathematical knowledge (MK) and pedagogical content knowledge (PCK), but this model acknowledges the central role of teacher beliefs in the deployment and use of his/her knowledge. Within the domain of MK, the MTSK model defines three subdomains in terms of mathematics itself, and it includes reasoning and proving processes in one of the subdomains, so we use MTSK as an analytical tool. Due to the objective of our work, we have focused on two subdomains of MK. One of them, Knowledge of Topics (KoT), was used to analyse the knowledge of mathematical objects that EPT used in the activities. The other subdomain, Knowledge of Practices in Mathematics (KPM), was used to analyse meta-mathematical knowledge of reasoning and proving processes. The definitions of both subdomains are detailed below:

KoT comprises a thoroughgoing knowledge of mathematical contents. This subdomain brings together four categories of knowledge: definitions, properties, and foundations; procedures; registers of representation; and phenomenology and applications (Carrillo-Yañez et al., 2018).

KPM encompasses the knowledge about “any mathematical activity carried out systematically, which represents a pillar of mathematical creation and which conforms to a logical basis from which rules can be extracted” (Carrillo-Yañez et al., 2018, p. 244). Delgado-Rebolledo et al. (2022) propose categories of knowledge based on different mathematical practices: proof and proving, defining, problem solving, and the role of mathematical language.

We understand proof in the sense of Stylianides (2007), as a connected sequence of assertions that uses a set of accepted statements, employs valid and known modes of argumentation, and is communicated with appropriate modes of expression. Reasoning and proving processes can involve a multiplicity of actions, that can explain its difficulty for many students, including EPT (Rodrigues et al., 2021). Arzarello (2008) distinguishes two different activities within these processes: first, developing an enquiry to raise and establish conjectures; and second, developing a chain of logical consequences to justify and validate the conjecture if it is true. Within the category of proof and proving practices, MTSK model recognises the importance of both activities, because it includes the knowledge on how to develop proofs (detection of regularities and patterns, construction of conjectures, role of examples and counterexamples), the knowledge on methods and types of proof and its validity, and the knowledge on functions of proof (Delgado-Rebolledo et al., 2022).

Students and preservice teachers tend to have different schemes of justification. We use the concept of personal proof scheme (hereafter, PS, Harel & Sowder, 2007), which consists of what constitutes ascertaining and persuading for a person about the validity of a mathematical statement. This is related to the knowledge of what constitutes a valid proof in mathematics. Harel and Sowder (2007) distinguish between three kind of proof schemes: external conviction PS (in which ascertaining and persuading come from reasons other than reasoning), empirical PS (which could be perceptual PS if are based in perception, or inductive PS in which conjectures are validated by the observation and check of one or more specific examples), and analytical PS (which are based on deductive reasoning). Research about proof and proving processes in prospective teachers found common difficulties like the persistent presence of inductive empirical PS (Harel & Sowder, 2007; Stylianides & Stylianides, 2009) and the knowledge about the role of examples and counterexamples in proving processes (Rodrigues et al., 2021; Stylianides et al., 2016). These difficulties could compromise the learning opportunities of reasoning and proving processes that EPT could design and implement.

### **Task design for conjecturing and proving**

We have designed and carried out a teaching experiment with the purpose of revise and develop the mathematical knowledge of a group of EPT about reasoning and proving processes. One of its objectives was the generation of the intellectual need of deductive reasonings in mathematics, promoting the evolution of empirical PS to analytical PS. This teaching experiment is focused on angle relationships in the circle because this topic is usually unknown or not remembered by EPT and this fact promotes genuine processes of conjecturing and proving. A dynamic geometry software (DGS), GeoGebra, is used because it facilitates the drawing of exact geometrical constructions and the exploration of those constructions looking for conjectures and relationships. Nevertheless, Lin et al. (2012) stress the need of a thoughtful design of the tasks, because the use of DGS could reinforce students' misconception that empirical arguments constitute proofs; and sometimes DGS does not help the construction of deductive arguments (Rodríguez & Gutiérrez, 2006). We have followed the principles of task design for conjecturing and proving of Lin et al. (2012) as a guide for our design.

Lin et al. (2012) set forth principles for the conjecturing phase, for the transition from conjecturing to proving, and for the proving phase. In the conjecturing phase, opportunities to engage in observation and construction, and opportunities to reflect on the conjecturing process and the resultant conjectures, must be provided. In the transition from conjecturing to proving, the task should generate a need to engage in mathematical proof. This need could be intellectual, via the emergence of cognitive conflicts or the need of an explanation about why a relation happens, and/or social, which means that norms must be established to accept or reject conjectures based on the logical structure of the mathematical system. In the proving phase, opportunities to use different representations to express arguments, to develop own proofs, and to evaluate the proofs of classmates must be promoted.

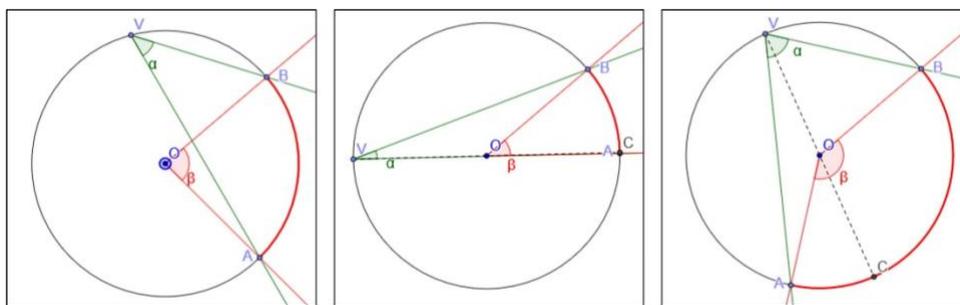
### **The teaching experiment. Data analysis**

This research is framed within the methodology of design research (Cobb et al., 2015), in which there is a combination of a theoretical and pragmatic orientation. There exists an interdependence between the research about teaching-learning processes and the design of a proposal trying to improve and understand these processes, with iterative cycles of design, implementation, and analysis. We have

carried out a *teaching experiment*, understood as a sequence of teaching episodes in which the participants can be a teacher-researcher, one or more students and other observer-researchers.

It was carried out with a group of EPT of the Degree in Primary Education (University of Valladolid, Spain), in a semester subject about geometry and its teaching and learning (2<sup>nd</sup> year). Due to geometry is a natural area for the development of reasoning and proving skills (NCTM, 2000), we specifically place different opportunities to revise and develop the knowledge about these processes during the subject. This teaching experiment is the first opportunity to do it: it was situated in the first part of the subject, covered a session of 90 minutes, and it is centred on the inscribed angle theorem. Previously, the participants had known GeoGebra and worked with it and with the definitions of different types of angles in a circle, but not with its angle relationships.

The teaching experiment was based on Lin et al. (2012) principles. Task statements (translated into English) can be seen at <https://www.geogebra.org/m/d9twyuhp>. It was divided in three stages, the first two dealt with the conjecture phase, and the third with the transition from conjecturing to proving and the proving phase. In the first stage, each EPT worked individually. It consisted in drawing inscribed angles on circles and its corresponding central angles using drawing tools and paper, and establishing a conjecture about its angular relationship. After that, each EPT should reflect on his/her conjecture, answering three questions about his/her certainty of the conjecture, and if he/she thought that the conjecture is sufficiently proved or not, and why. These questions are similar to the conceptual awareness pillars of Stylianides and Stylianides (2009), and they gave us information about the EPT's proof schemes. In the second stage, the task was the same (including the questions) but now it was developed in pairs and using a GeoGebra applet in which EPT had the construction (Figure 1, left), to drag and inquire about the relationship with the aid of the graphics view toolbar.



**Figure 1. Images of GeoGebra applets of the teaching experiment**

The third stage, in pairs and with other applet, started with two questions seeking to generate an intellectual need to engage in mathematical proof, asking for an explanation about why they think that the conjecture is true and what relationships between the two angles can promote it. Later, some questions are posed for guiding the construction of a proof of the conjecture and for mobilising the knowledge of topics needed for it. First, it was considered the easier case in which one of the angle's sides contains the centre point of the circle (Figure 1, centre), with these questions: How is the triangle VOB according to their sides, and why? What is the measure of each angle of VOB? What is the relationship between these angles of VOB and the corresponding central angle? Then, they have to justify the other cases using the previous one (Figure 1, right). Lastly, each group should again answer the questions about the certainty of the conjecture, and if it is sufficiently proved or not, and why.

Data collected are the productions of EPT in each stage (on paper and in a GeoGebra classroom environment), and the field notes of the teacher-researcher and one observer researcher. These data correspond to the implementation in the first cycle. The content analysis method is used (Cohen et al., 2018), for the rigorous examination, replication, inference, and verification of the contents of written data. The productions of EPT have been analysed by each author, identifying elements of mathematical knowledge deployed and used by EPT in each stage of the teaching experiment (using MTSK subdomains and categories, and the proof schemes categories). Then, these results were shared between the authors to reach an agreement. Finally, the comparison of the productions of the participants, and the elements of knowledge detected, allow us to state some reflections about relationships between the mathematical knowledge deployed and the teaching experiment designed.

### **Findings: the case of Cynthia and Juan**

Our expectations are that the EPT can conjecture and formulate the angle relationship and reflect, throughout the teaching experiment, on what is needed to justify a mathematical conjecture. We expected that some groups of EPT showed inductive empirical PS but other groups could express the need of a deductive reasoning with the aid of the teaching experiment. We presented in this paper one example of the latter: the results of the analysis of the group of Cynthia and Juan (pseudonyms), whose behaviour and productions are more aligned to the purpose of the teaching experiment and have greater richness than others based exclusively on inductive reasonings.

During the first stage, each EPT correctly drew different examples of the configuration and measured the angles with the protractor (KoT deployed: definitions, procedures). Both established, based on an inductive proof scheme of several cases, the conjecture that the amplitude of the central angle is always bigger than the inscribed one (KPM deployed: how to develop proofs – detection of regularities and patterns, construction and formulation of conjectures–, types of proof), and they are sure about its certainty. Furthermore, each EPT stated the conjecture about the angular relationship, but not in the same way:

Juan:           The inscribed angle is half of the central one.  
Cynthia:       The central angle is approximately twice the amplitude of the inscribed one (manually measured).

Both EPT expressed doubts about the generalization of this relation, due to apparent counterexamples that they had found (KPM deployed: how to develop proofs – role of examples and counterexamples). Cynthia thought that it is only true for inscribed and central angles whose angle bisectors are the same, and Juan thought that it is not true for small angles. Both said that they need a more accurate measurement tool to improve their certainty about the conjecture. Cynthia added that she thinks “it would be necessary to demonstrate this logically”, showing that Cynthia can have characteristics of an analytical PS (KPM deployed: what constitutes a valid proof).

In the second stage, when they could use GeoGebra, the need for accuracy of the measurement was overcome and both EPT were able to generalize the conjecture through the checking of different and doubtful cases (KPM deployed: how to develop proofs – detection of regularities and patterns). For example, Cynthia said that: “Thanks to GeoGebra, I can generalize the conjecture from the case in which angles share the angle bisector to any inscribed angle and its corresponding central one”. In

this group, GeoGebra has been shown as a useful tool for improving certainty of students because they can check the conjecture in those cases in which they had doubts (Rodríguez & Gutiérrez, 2006). Also in this stage, they affirmed that they are sure about the validity of the conjecture because it is “empirical enough proved”, but a logical or mathematical reason is needed. This is consistent with the way mathematicians develop new results and it confirms that this group has an analytical PS (KPM deployed: how to develop proofs, what constitutes a valid proof).

In the third stage, when they were asked to justify why they think the conjecture is true, they developed an abductive reasoning conjecturing that there is a relationship between the amplitudes of the angles and the distance from the points B and C to the angle vertex and to the centre of the circumference (see Figure 1, centre and left) (KoT deployed: definitions, properties; KPM deployed: formulation of conjectures, need of a general justification). However, this pair of EPT did not concrete the relationship and they continued with the subsequent questions of the third stage.

With the aid of the guidance questions, Cynthia and Juan justified the easier case (Figure 1, centre):

Both: [VOB is] Isosceles and obtuse triangle. Because two sides are the radii. All the angles of the triangle add up to  $180^\circ$ . The obtuse angle VOB + the corresponding central angle add up to  $180^\circ$ . Which leads us to justify that: The corresponding central angle is equal to the sum of the two acute angles of the triangle VOB. Therefore, angle alpha is half the angle beta.

The guidance questions seem to allow this pair of EPT to deploy the knowledge needed (KoT deployed: definitions, properties) in the construction of a deductive proof for this case, and to correctly construct them (KPM deployed: methods and types of proof). After that, they wrote how to develop a proof when the centre point of the circle is inside the inscribed angle (Figure 1, right), applying the previous case in both sides of the ray VC (KPM deployed: methods and types of proof), but they did not write it in detail. They did not properly develop a proof for the remaining case.

When they were asked about their certainty after this stage, they affirmed that it has been improved because they know they have logical arguments about its validity, and not only empirical ones. This shows that these students possibly differentiate between an inductive reasoning, and its limitations, and a deductive one. Nevertheless, when they were asked if the conjecture is now sufficiently proved, they compared the empirical reasoning and the deductive one, apparently giving the same importance to both reasonings.

## **Reflections about the potential and limitations of the teaching experiment**

This teaching experiment allowed us to explore the way in which the type of reasoning used by EPT when solving a geometric task evolved. Although a great part of the EPT that solved the task specifically used inductive arguments, Cynthia and Juan’s group noticed, from the outset, that this type of procedure was not enough to prove their conjectures. At this point we want to highlight the importance of questions such as “Do you think that the conjecture is sufficiently proved?”. These questions, inspired by Stylianides and Stylianides (2009), enable the EPT to reflect about the reasoning and proving processes of each stage, deploying and verbalizing elements of their KPM.

Although the formal proof was not achieved by the EPT autonomously, various elements were put into play by proposing tools that improved the conjecture process, as was the case of the designs in

GeoGebra. In the first stage, Cynthia and Juan were able to think of possible counterexamples, critical aspect in conjecturing and proving processes (Stylianides et al., 2016), and then, with GeoGebra, they could verify their (non)existence exploring borderline cases (e.g., very small angles). The first two stages of the teaching experiment were useful to mobilise the MK needed to detect and formulate the conjecture of the angle relationship, first part of reasoning and proving processes (Arzarello, 2008).

At the beginning of the third stage, the EPT developed reasonings and established relationships between elements of the figure that were incorrect and unexpected, like the relationship between the angles and the length of the segments that Cynthia and Juan discussed. The current version of the teaching experiment does not exploit the potential of these abductive reasonings, and in future versions it would be interesting to add means (GeoGebra applets, group discussions) so that EPT could explore these detected relationships. Then, the guiding questions enabled most groups of EPT to mobilise the required Knowledge of Topics to construct a proof in the easier case. Cynthia and Juan, who show evidence of an analytical PS, could use this KoT to construct this proof applying elements of KPM. However, other groups were not able to do that, showing the persistence of inductive proof schemes also among EPT (Stylianides & Stylianides, 2009) and as a widespread difficulty, not exclusive of the Spanish context of formation of EPT. This teaching experiment by itself is not enough, in most cases, to generate the intellectual need of a deductive proof. There is a need of a careful design of sets of formative tasks in order to promote opportunities to move EPTs towards analytical proof schemes and to strengthen advances in their KPM. The teaching experiment shows a possible structure for a task, following Lin et al. (2012) principles, but there are some challenges to move from one task to a set: the order and the connection between tasks, the role of the previous knowledge of EPT of different mathematical topics (KoT) or the connection of the tasks with school practices and didactical knowledge. More research about these aspects is needed.

Finally, this work meets the need expressed by Carrillo-Yañez et al. (2018) with respect to testing the analytical functionality of the characterizations on the KPM, finding that the proposal made by Delgado-Rebolledo et al. (2022) has been useful for our purposes, at least for the practice of proving.

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