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# Implicit Function Theorem: Students' autonomy in learning

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*The Implicit Function Theorem is one of the rich theorems covered in multivariable calculus course with many applications and very limited lecturing time. In this paper we study the students' perspective as part of the final step of didactic transposition from taught knowledge to learnt knowledge. As a general characteristic of education at the university level, the transposition to learnt knowledge includes significant students' autonomy in learning, so with this pilot study we have studied the differences in students' learnt knowledge and different perceptions of the value of the theorem. The interviews with three students show that concrete examples are meaningful and occupy their focus more than the proof of the theorem, while considering examples as applications provides motivation for theoretical knowledge, with limited study time leading to well-known effects of compartmentalization.*

*Keywords: Multivariable calculus, implicit function theorem, didactic transposition, taught and available knowledge, students' learning autonomy.*

## Introduction

The Implicit Function Theorem (IFT), loosely speaking, addresses a problem of expressing an implicit relation between variables as a functional relation. Apart from this original *functional aspect*, there are two other aspects of the theorem: addressing the problem of solving the system of nonlinear equations, but also the problem of describing the geometric structure of the specified sets. In particular, IFT is applied in the classical differential geometry when it is needed to establish the existence of a local parametrization of curves and surfaces defined by implicit equation(s). Regarding this aspect, Zaldivar (2013), even refers to IFT as a *gem of geometry*.

The theorem, its proof and rich applications are part of the curriculum in (more theoretically oriented) mathematics programs, and typically taught in courses of multivariable calculus (if at all). On the other hand, the mathematics education literature on this topic is very sparse. A wide geographical survey presented by Martínez-Planell and Trigueros (2021) gave a broad overview of the different research directions in multivariable calculus, which might be linked to IFT, but such research does not mention IFT directly. In our previous research (Bašić & Milin Šipuš, 2022), we inspected teachers' perspective in teaching this particular piece of mathematical knowledge, especially whether there are possible tacit expectations of teachers from students related to flexible use of non-routine algebraic manipulations and geometric interpretation of equations. Here teachers argued that "IFT is an important theorem, but that there are more important topics in the course (e.g., continuity and differentiability)", therefore in the assessment IFT is "only required for a higher mark". In this situation the students motivated to achieve this knowledge are required to engage in significant autonomous learning. Such learning might be considered as a type of research-like activity (Gravesen et al., 2017), such as to construct, identify or investigate special cases, to read and reconstruct a proof or part of it, to reformulate a theorem, to formulate new proof by "mimicking" a known one (e.g., for

an analogous case), or to employ a non-formal “heuristic” representation of a mathematical object to investigate it. Our aim is to study students’ knowledge of IFT as one part of a more overarching study of understanding university students’ perspective in autonomous learning.

## **Theoretical framework and research question**

The Anthropological Theory of the Didactic (ATD) provides a relevant framework for our research (Chevallard, 1991). ATD postulates that the production and dissemination of knowledge is institutionally dependent. In the process of didactic transposition, knowledge is transformed from academic (scholar) knowledge, through knowledge to be taught and (actually) taught knowledge, to learnt/available knowledge (the knowledge actually learnt by the students). To study the organization of knowledge, ATD recognizes the structure of a praxeology consisting of the practical block (type of tasks and the techniques to solve them) and the theoretical block (theory and technology supporting the technique). Researchers develop a reference epistemological model (REM) consisting of praxeologies, as a tool to study the process of didactic transposition and to describe, organize and analyse knowledge.

Dissemination of knowledge is essentially realized through two different paradigms, the paradigm of visiting monuments and the paradigm of questioning the world. The first is the prevailing and rather unchanging paradigm in many university mathematics study programmes, which, in addition to the dense and often compartmentalized organization of knowledge, and the fast pace of the teacher's presentation, leaves the student with a great deal of responsibility for acquiring the required content, often through autonomous learning, i.e., self-study (González-Martín et al., 2014).

The focus of our study is the last step of the didactic transposition regarding the mathematical knowledge of IFT and our wider intention is to initiate an analysis of students’ practices when they are left “to their own devices”. In this pilot study we explore the method of interview as a mean to probe student’s knowledge and gather ideas about possible further research actions. For this purpose, the research question we wish to address is:

What are the main characteristics of students’ available knowledge in relation to IFT and what aspects of the didactic transposition from taught knowledge to available knowledge in the context of students’ autonomous learning would students share in an interview?

## **Context of the study and methods**

In this study we consider the students’ apprehension of the Implicit Function Theorem as being taught at the end of the second-year course *Differential Calculus of Multivariable Functions* in the mathematics study programme of the largest mathematical department in Croatia. The course has three hours of lectures and two hours of exercise classes weekly, and roughly covers the following topics: open sets in  $\mathbb{R}^n$ , compactness and connectedness, continuous functions, differentiability, Taylor’s theorem, Lagrange’s theorem, Inverse and Implicit function theorem, and basics on curves and surfaces in  $\mathbb{R}^n$ . The topic of IFT and geometry is covered (only) in the final week of the semester. Based on the interviews with teachers and our previous analysis of the course materials (Bašić & Milin Šipuš, 2022), we can describe that the curriculum of the course is extensive, but the teachers are aware of this and try to compensate in various ways: the lecture notes are regularly revised and

supplemented with more examples and the final part of the course (including IFT) is only required in assessment for the highest marks. In particular, the teachers are aware of the high complexity and the richness of the material that is covered in the final week. Their decision is to provide all the details of the proof of IFT and to try to build some intuition by several examples of various flavours.

The first example is the example of the circle in the plane given by the equation  $x^2 + y^2 = 1$ , in which one can notice that (any part of) the upper semicircle can be described as a graph of a function of  $y$  on  $x$ , while such functional dependency is not possible in the neighbourhood of the points  $(1,0)$  and  $(-1,0)$ . Other geometrical examples include a similar consideration for the sphere and one more example considering the intersection of the two spheres in  $\mathbb{R}^3$ .

The functional aspect of the theorem can be illustrated by examples in which one uses the existence part of the theorem to obtain a smooth function locally, and the uniqueness part of the theorem to “glue together” local functions and obtain a global function. The algebraic aspect is represented by the following example with the non-linear system of three equations with six variables:

Consider if the system

$$\begin{aligned}3x + 2y + z^2 + u + v^2 &= 0, \\4x + 3y + z + u^2 + v + w + 2 &= 0, \\x + z + w + u^2 + 2 &= 0.\end{aligned}$$

is solvable in  $u, v, w$  in a neighbourhood of a point  $x = y = z = u = v = 0, w = -2$ .

From the epistemological perspective, and in terms of ATD, we may describe our REM for IFT as a collection of three praxeologies with the common theoretical block, characterized by the types of tasks representing the described three aspects (geometrical, functional and algebraic). This structure is further refined by the dialectic between the local and the global aspect in the considered tasks.

To better understand the student perspective, we have interviewed three students that are aiming for the highest marks in this course. The students volunteered and have been interviewed separately by the first author. There was no connection between the interviews and the course as the researchers are not teachers to these students (at any course). Each interview lasted between 15 and 20 minutes and was audio recorded with the consent of the student. At the beginning of the interview, students were given the copy of the theorem statement from the course lecture notes.

The interview was semi-structured based on the following questions:

1. What would you say is the meaning of the IFT/ What would you say, what does the IFT state? What is the most important result in your opinion that comes from it?
2. What do you remember about IFT from the lectures? Which examples do you remember?
3. Could you apply the theorem to  $2x^2 + y^2 = 6$  ?
4. Could you apply the theorem to  $x^2 + y^2 = 3z$  and  $4x + 4y + 3z = 1$ ?
5. Do you remember the procedure of implicit differentiation from some previous course(s) and do you connect it with IFT?
6. Do you have any other comments?

In the next section we present either full transcripts or rephrase parts of the conversation for easier reading. In the analysis we considered students' full answers and interpret them in relation to our described REM. Using the structure of the REM we position students' knowledge in terms of the three praxeologies by observing the types of tasks that the students solve, the techniques used and the parts of the theoretical block activated in this process. The answers to the final questions have been discussed between the authors and summarized with the indications for potential future research.

## Results

The first student discusses the statement of the theorem in terms of "big" and "small" function. They say that we seek a "small function" defined by the given "big function" and this justifies the name "implicitly defined function theorem". At first the student emphasizes that the theorem states that there is a unique "small" function and that it is of the same class ( $C^n$ ) as the "big function". The student is a bit uncertain after being asked to comment on the existence of the "small function", and changes their mind: "Maybe, it is about the existence actually and then we prove uniqueness easily".

Concerning the teaching approach, the student first recalls the example with the circle, where "the dependence of  $y$  on  $x$  is considered". It was talked about the critical points and that one would "fall off the circle" at these points [if we moved along the  $x$ -axis]. The student recalls there were other examples: "For the sphere there was a similar discussion, also with two critical points, and it was seen that we can change which variables depend on other".

For the first example of specialization (ellipsoid), the student first defines the function  $F(x, y) = 2x^2 + y^2 - 6$  and comments that "we will have functional dependency of  $y$  on  $x$ , except in the points for which  $y = 0$ ". In the second example, the student starts again by defining a function  $F(x, y, z) = (x^2 + y^2 - 3z, 4x + 4y + 3z - 1)$  and comments: "now, depending which variable we want to express, e.g. to express  $z$ , we will look at partial derivative  $\frac{\partial F}{\partial z}$  [sic]. It is different than  $[0, 0]$ , so we get a regular matrix." When asked about the dimension of the matrix they say it should be  $n \times m$  and in this case we would have  $n = 2, m = 1$ . Then the student becomes confused with the dimensions of the matrix as it is not a square matrix: "It should be a  $1 \times 1$  matrix. I've made some mistake. Aha, maybe I can plug in  $z$  from one equation to the other?"

As the student becomes more confused, the interview continues with the question on the geometric interpretation. They draw a cylinder and mention that the second equation represents a plane. The student then discusses that the intersection of the surfaces is the curve and IFT would give us the parametrization of that curve.

Interviewer: How many variables would be expressed with how many other variables?

Student: There would be  $m$  variables expressed by  $n$  variables.

Interviewer: So, in this case one variable would be expressed by two?

Student: Hmm, yes, but that is not possible because we cannot solve one equation with three variables.

[They do not see that we can use one of the coordinate variables]

In the end of the interview the student expresses that the course material was difficult at some points, but very interesting. The professors try hard to explain, write a lot on the board and show many examples.

For the second student the theorem expresses that “big  $F$ ” determines a functional dependency which is of class  $C^1$ , so that “the solution is not only scattered - without the theorem, we could construct the solution, but the theorem guarantees that locally the function is continuous, it is nice”. For this student, it is not difficult to deduce IFT from the Inverse function theorem. The examples provide a new way of thinking. Before, there were no tools to prove local existence. They remember an example about the existence of a smooth function that is defined on the whole  $\mathbb{R}$ . The theorem guarantees smoothness locally, but by uniqueness one can see that there is a function defined on the whole  $\mathbb{R}$  which agrees with the locally defined continuous functions on parts of the domain.

In the first example, they recognize an equation of an ellipsis, draw the picture and discuss “we can locally find the dependency of  $y$  on  $x$ , except in these two points” (and points to the points with  $y = 0$  in the picture they draw). When asked why we would be interested in this dependency, they are not sure and mention they have seen some complicated applications about inverses of these implicit functions, but they are not sure about the reasons...

In the second example, they say we could consider one variable and express functional dependency of the other two variables. In the application of the theorem, we would have  $n = 1$  and  $m = 2$ . Now, we would need to calculate partial derivatives and then see in which points this makes sense, i.e., to check that the determinant is non-zero.

About the geometric interpretation, the student argues: “I’m not sure, I would use linear algebra and approximate by Taylor polynomials. By considering the ranks, we could see when the system has a solution. The equations represent a plane and a paraboloid, and their intersection. So, the theorem implies that we could locally express the curve in the intersection by using one variable as parameter.”

To the question about implicit differentiation, the answer is: “I’m not sure that I have seen something like this before, maybe, but I did not consider that seriously.”

Finally, the student comments that it was discussed in the lectures that the proofs of IFT and Inverse Function Theorem are equivalent, but to them IFT is a more natural and more useful statement.

The third student discusses the theorem:

The theorem states that there is a solution to a system of algebraic equations. In the neighbourhood of the solution. It can show that there is a solution on a larger domain. The theorem shows there is a unique connection between two (or more) variables.

They mention that in the lectures and lecture notes there are many examples. The student does not remember the proof, and comments that this says something about its presentation:

When I like a proof, I can recall it. After the examples, I have understood the meaning of the theorem. I remember a task with a system of equations with five variables, in which one should discuss dependency of three variables on the other two. I also remember a task about the existence of a function on a global domain. Without the theorem we can prove the existence of a function, but by the theorem we see that this function is locally of class  $C^p$ . That is the strength of the theorem.

After being asked explicitly about the circle, the student recalls that it was discussed with the emphasis on the critical points (1,0) and (-1,0). For example, to the right of point (1,0) the function would not be defined.

The final part of the interview concerns implicit differentiation:

Interviewer: Why do we want to express one variable as a function of the other?  
Student: Hmm, I don't know.  
Interviewer: Do you remember the procedure of implicit differentiation?  
Student: Yes, we did this in the first year course.  
Interviewer: Is that connected to the IFT?  
Student: I am not sure what is the question or what I should say.  
Interviewer: What is the connection between the theorem and the procedure of implicit derivation?  
Student: There is a formula that we obtain after making the derivation.  
Interviewer: Why are we allowed to derive in this way?  
Student: I am not sure.

The interview continues with the question about the specialization of the theorem in the case in  $\mathbb{R}^3$ . The student first defined  $F(x, y, z) = (x^2 + y^2 - 3z, 4x + 4y + 3z - 1)$  and comments: "then we calculate partial derivatives with respect to  $y$  and  $z$ . We get a  $2 \times 2$  matrix and then the regularity of this matrix gives us the criterion".

When asked about the geometric interpretation, this student express that they have difficulties with curves and surfaces. Then they interpret the equations correctly: "The second equation represents a plane, the first equation a paraboloid ... and we have some curve in the intersection. The theorem would state that the curve depends on the variable  $x$ ."

The overall impression about the teaching of this topic is elaborated in more details:

This topic is at the very end of the course, so I did not have enough time to learn all the details in just one week. In addition, the topic is very interesting and it motivates me a lot. I did not see the point of the previous topics. I would like to have more exercise classes, and my colleagues agree. We had double more time in exercise classes in the first year. I lacked motivation in the earlier parts of the course, so I did not learn on my own. Now I see, I miss that. The motivation comes from applications and I am sorry that this came only at the very end.

## Discussion

For all three students it is evident that the topic is new to them, but that they are confident about the use of mathematical language to understand the theorem. They all talk fluently about the various terms that appear in the theorem, such as functions, equations, points, and different conditions such as continuity or belonging to class  $C^p$ . At the same time, the main understanding of the theorem comes from the examples, and each of the three students emphasizes a different aspect of the theorem.

The first student mostly discusses the functional aspect, with the emphasis on the dialectic of existence and uniqueness. Of all the examples, the ones that are singled out are geometric. The geometric interpretation (that the theorem provides a parametrization of the intersection curve) in the selected tasks has been immediate for this student, although they did not know how to draw one of the surfaces. In addition, the application of the theorem in the special case was not straightforward as

it was confusing how many variables depends on how many other variables. Even the meaningless result of obtaining a  $1 \times 2$  matrix instead of a  $2 \times 2$  matrix (to check its regularity) did not help to become aware of the mistake. We find that this episode is insightful as it shows that there are many details to consider how to improve self-study, even when the students are aware of the conceptual issues. The student expresses that the teacher is good at explaining and provides many examples, yet that is not enough for all the aspects to be successfully absorbed in a short time if the topic is difficult.

The second student emphasizes the dialectic between local and global definition of the function. As the main result, they mention that the theorem guarantees nice properties (e.g., continuity) of the implicit function. The student emphasizes the beauty of the result and that it brings a new way of thinking. Moreover, they critically discuss the relation of implicit function theorem and inverse function theorem, showing inclination for applications and aesthetics of mathematics. This student easily specializes the theorem to the geometric context in low dimensions and interprets its result as the *local* parametrization of the curve by a coordinate variable. We find very interesting that this student did not make any connection between the theorem and the procedure of implicit differentiation learned in the first year.

The third student mostly considers the theorem through the discourse about the solutions of the algebraic system of equations. This interview also confirms two points from the previous – one is the emphasis of the local-global perspective, and another is the lack of the link between the learned technique of implicit differentiation and the theorem (IFT). The student very clearly discusses their autonomous learning experience; it is led mostly by motivation stemming from applications of mathematics which is hidden in the course until the very end. It is also pointed out that the difficult topics require more time than it is left for autonomous learning, which is a consequence of planning the curriculum according to the customary deductive sequence of presenting the course results.

## **Conclusion**

In this paper we have focused on the university students' perspective related to learning a specific mathematical theorem with the aim to describe students' available knowledge as the final result of the process of didactic transposition. The discussed theorem has several specific characteristics: the practical block is related to applications that are rich, connected to and achieved by concrete procedures in special cases, and important in the fields of multivariable calculus and differential geometry, while the proof of the theorem (the theoretical part) is difficult and lengthy. Our previous studies described the teachers' perspective of the didactic transposition in the considered institution, from which we observed that the content is huge, time in lecturing is limited and the expectations of teachers regarding this piece of knowledge are restricted only to students aiming for higher marks. In addition, for the majority of the students, the situation remains full of tacit expectations as they progress to the following courses without solid understanding of the theorem and its consequences.

Highly motivated students are thus in the situation to engage in significant amount of self-study, which in general might consist of activities that carry research potential. Such activities are performed in relation to many other mathematical works and could be considered part of the students' training to become professional mathematicians. However, from the similar activities in the interviews with these students it is visible that the outcomes of their self-study are varied, which we might consider



as an indication that the didactic transposition is undirected and uncontrolled. The specialization is not always straightforward and even the awareness of one's flawed reasoning does not necessarily help in autonomous building of conceptual understanding. To some students, the study of individual examples is meaningful and occupy their focus more than the proof of the theorem, while considering examples as applications provides motivation for the theoretical knowledge. On the other hand, the limited time for study leads to compartmentalization of knowledge, seen in the lack of connections that the students make between certain procedures and theoretical results that justify them.

From the considered cases we have observed that the focus of their independent study differs and hence the available knowledge is personal. Each student emphasized a different aspect of the theorem as the most important and, even though they have shown significant mathematical abilities and understanding of the theorem, it can be observed that the students rely on different representations. This indicates that the last step of the didactic transposition is indeed highly dependent on the institution and the students, and their results can be hardly assumed. In addition, students' ability to explicate their ways of reasoning and willingness to report supports the idea to use the format of an interview in more elaborate research (including additional data collection) on students' ways of autonomous learning. We believe such efforts would be valuable as the results might inform researchers in mathematics education and university teachers about the support that is needed and they might provide.

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