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Lovász-Schrijver PSD-operator and the Stable Set Polytope of Claw-Free Graphs*

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Abstract

The subject of this work is the study of LS_+ -perfect graphs defined as those graphs G for which the stable set polytope $STAB(G)$ is achieved in one iteration of Lovász-Schrijver PSD-operator LS_+ , applied to its edge relaxation $ESTAB(G)$. In particular, we look for a polyhedral relaxation of $STAB(G)$ that coincides with $LS_+(ESTAB(G))$ and $STAB(G)$ if and only if G is LS_+ -perfect. An according conjecture has been recently formulated (LS_+ -Perfect Graph Conjecture); here we verify it for the well-studied class of claw-free graphs.

1 Introduction

The notion of a perfect graph was introduced by Berge in the early 1960s [1]. A graph is called *perfect* if each of its induced subgraphs has chromatic number equal to the size of a maximum cardinality clique in the subgraph.

Perfect graphs turned out to be an interesting and important class with a rich structure and a nice algorithmic behavior [24] and inspired numerous very interesting contributions to the literature for the past fifty years.

An early polyhedral characterization of perfect graphs was given by Chvátal in [10]: G is perfect if and only if $STAB(G)$ coincides with its corresponding clique relaxation $QSTAB(G)$. However, solving the Maximum Weight Stable Set Problem (MWSSP) for a perfect graph G by maximizing a linear objective

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function over $\text{QSTAB}(G)$ does not work directly, but only via a detour involving a geometric representation of graphs [28] and the resulting semi-definite relaxation $\text{TH}(G)$, the *theta body* of G . This is one of the main results in the seminal paper of Grötschel, Lovász and Schrijver [23]. Later on, the same authors proved a related result which connects a purely graph theoretic notion to polyhedrality of a typically nonlinear convex relaxation and to the integrality and equality of two fundamental polytopes:

Theorem 1. (Grötschel, Lovász and Schrijver [23, 24]) *For every graph G , the following are equivalent:*

1. G is perfect;
2. $\text{STAB}(G) = \text{QSTAB}(G)$;
3. $\text{TH}(G) = \text{STAB}(G)$;
4. $\text{TH}(G) = \text{QSTAB}(G)$;
5. $\text{TH}(G)$ is polyhedral.

Some years later, Lovász and Schrijver [29] introduced a semidefinite relaxation of $\text{STAB}(G)$ which is stronger than $\text{TH}(G)$ and it is obtained after applying once the LS_+ lift-and-project operator (N_+ in [29]) to the edge relation of $\text{STAB}(G)$, see Section 2.3 for details.

We denote this relaxation by $\text{LS}_+(G)$. Following the same line of reasoning used for perfect graphs, they proved that the MWSSP can be solved in polynomial-time for the class of graphs G for which $\text{LS}_+(G) = \text{STAB}(G)$. Those graphs have been called LS_+ -perfect graphs in [5] (originally N_+ -perfect in [3]). The fact that solving the MWSSP on LS_+ -perfect graphs can be done in polynomial-time encourages the search of a complete description of their stable set polytope in terms of linear inequalities.

In this context, one of our goals is to obtain a characterization of LS_+ -perfect graphs similar to the one given in Theorem 1 for perfect graphs. More precisely, we look for an appropriate polyhedral relaxation of $\text{STAB}(G)$ playing the role of $\text{QSTAB}(G)$ in Theorem 1, when we replace $\text{TH}(G)$ by $\text{LS}_+(G)$.

Also, in [29] it is proved that every valid inequality for $\text{STAB}(G)$ having as support a near-bipartite graph is valid for $\text{LS}_+(G)$ (a graph is *near-bipartite* if the non-neighbors of every node induce a bipartite graph [37]).

This result enables the authors in [3] to establish the following:

Conjecture 2 (LS_+ -Perfect Graph Conjecture). *For every graph G , if $\text{LS}_+(G) = \text{STAB}(G)$ then every facet defining inequality of $\text{STAB}(G)$ has near-bipartite support.*

According to the findings in [37], if G is a near-bipartite graph then every facet defining inequality of $\text{STAB}(G)$ has a complete join of antiwebs as a support graph, see Section 2 for details. Hence, Conjecture 2 can be restated as:

Conjecture 3. *If G is an LS_+ -perfect graph, the support graph of any facet defining inequality of $\text{STAB}(G)$ is a complete join of antiwebs.*

Later, in [42] Wagler defined for a given graph G the relaxation $\text{ASTAB}^*(G)$ described by inequalities whose support are complete joins of antiweb subgraphs in G . In the same context, graphs for which $\text{STAB}(G)$ coincides with $\text{ASTAB}^*(G)$ are called *joined a -perfect* [11, 40]. Hence Conjecture 2 establishes that $\text{ASTAB}^*(G)$ corresponds to the relaxation we are looking for to play the role of $\text{QSTAB}(G)$ in Theorem 1, when we replace $\text{TH}(G)$ by $\text{LS}_+(G)$.

Some progress has been made towards proving its validity since it was proposed in [3]. Conjecture 2 has been already verified for near-perfect graphs [3], for *fs-perfect graphs* [4, 5], for *webs* [16], for *line graphs* [17], and for several graph classes defined by clique cutsets [43, 44], see Section 2.3 for details and more precise definitions.

The aim of this contribution is to verify Conjecture 3 for a well studied graph class containing all webs, all line graphs and the complements of near-bipartite graphs: the class of claw-free graphs. Claw-free graphs attracted much attention due to their seemingly asymmetric behavior w.r.t. the stable set problem. On the one hand, the first combinatorial algorithms to solve the problem in polynomial time for claw-free graphs [30, 35] date back to 1980. Therefore, the polynomial equivalence of optimization and separation due to [24] implies that it is possible to optimize over the stable set polytope of a claw-free graph in polynomial time. On the other hand, the problem of characterizing the stable set polytope of claw-free graphs in terms of an explicit description by means of a facet-defining system, originally posed in [24], was open for several decades. This motivated the study of claw-free graphs and its different subclasses, that finally answered this long-standing problem only recently (see Section 2.4 for details).

To prove the validity of the conjecture on claw-free graphs we use two different approaches. On one hand, we take advantage of the known description of the stable set polytope for quasi-line graphs. On the other hand, we characterize the structure of the connected LS_+ -perfect claw-free not quasi-line graphs without clique cutsets.

The paper is organized as follows: In Section 2.1 we present notation and basic results that will be used throughout the paper. In Section 2.2 we present the different relaxations we use of the well-known stable set polytope, and also families of graphs which are defined in terms of the facet defining inequalities in their stable set polytope. In Section 2.3, we present the State-of-the-Art on LS_+ -perfect graphs (including families of LS_+ -imperfect graphs needed for the subsequent proofs) and the results concerning the facet-description of their stable set polytopes from the literature. We devote a subsection to claw-free graphs (Section 2.4) and their relevant subclasses, presenting known valid inequalities describing the stable set polytope of certain claw-free graphs. In Section 3, we verify Conjecture 3 for the class of claw-free graphs. Also, in Section 4 we describe the family of minimally LS_+ -imperfect graphs that are claw-free. Finally, in Section 5 we present the conclusions and future lines of research on this topic, establishing connections with the previous results.

Parts of the here presented results appeared without proofs in [6, 7].

2 Definitions and preliminary results

2.1 Graph theory

Throughout this work, $G = (V, E)$ stands for a simple graph with node set V and edge set E . When we need to emphasize the relationship between the node set and the edge set with the graph, we may write $V(G)$ and $E(G)$. The *complementary graph* of G is denoted by \overline{G} .

Given $U \subset V$, $G[U]$ is the subgraph of G induced by the nodes in U . i.e. having node set U and edge set $\{uv : uv \in E, \{u, v\} \subset U\}$. Given a graph G' , if there is a subset of nodes of G inducing G' we say that G' is a node induced subgraph of G and write $G' \subset G$. Also, we denote by $G - U$ the subgraph of G induced by the nodes in $V \setminus U$. For simplicity, we write $G - u$ instead of $G - \{u\}$.

Given $v \in V$, $N_G(v)$ is the *neighbourhood* of v and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v . When the graph is clear from the context, we simply write $N(v)$ or $N[v]$.

A *stable set* in G is a subset of mutually nonadjacent nodes in G and a *clique* is a subset of pairwise adjacent nodes in G . The cardinality of a stable set of maximum cardinality in G is denoted by $\alpha(G)$.

A *clique cutset* of a connected graph G is a clique whose removal disconnects G .

For $n \in \mathbb{N}$, $[n]$ will denote the additive group defined on the set $\{1, \dots, n\}$, with integer addition modulo n . Given $a, b \in [n]$, let $b - a$ be the minimum non-negative integer t such that $a + t = b \pmod{n}$. We denote by $[a, b]_n$ the *circular interval* defined by the set $\{a + s : 0 \leq s \leq b - a\}$. Similarly, $(a, b]_n$, $[a, b)_n$, and $(a, b)_n$ correspond to $[a, b]_n \setminus \{a\}$, $[a, b]_n \setminus \{b\}$, and $[a, b]_n \setminus \{a, b\}$, respectively.

For any positive integer number n , C_n denotes the cycle with set of nodes $[n]$ and such that for all $i \in [n]$, i is adjacent to $i + 1$. Similarly, P_n denotes the path obtained by deleting the edge $n1$ in C_n .

A *hole* in the graph G is an induced cycle in the graph. Similarly, an *antihole* in G is an induced cycle in \overline{G} . The hole (antihole) is *odd* if it has an odd number of nodes.

Chudnosvsky et al. proved the well-known Strong Perfect Graph Theorem [8] (previously stated as a conjecture by Berge [1]) showing that perfect graphs are exactly those graphs having neither odd holes nor odd antiholes.

A wheel is a graph formed by a cycle and one extra node outside the cycle, connected to every node in it.

Given integer numbers k and n such that $n \geq 2(k + 1)$, W_n^k denotes the *web* graph with set of nodes $[n]$ and edges $i(i + t)$ for all $i \in [n]$ and $t \in \{1, \dots, k\}$. Observe that, for all $i \in [n]$, $[i, i + k]_n$ is a maximal clique of W_n^k . We say that a set of nodes $U \subset [n]$ is a set of $r \geq 1$ consecutive nodes in W_n^k if there exists $i \in [n]$ such that $U = [i, i + r)_n$. The *antiweb* A_n^{k+1} is the complementary graph of a web W_n^k . It is easy to see that $C_n = W_n^1 = A_n^k$ and $\overline{C}_n = W_n^{k-1} = A_n^2$ with $k = \lfloor \frac{n}{2} \rfloor$. Also, complete graphs of n nodes are the antiwebs A_n^1 .

A graph is *near-bipartite* [37] if the non-neighbors of every node induce a bipartite graph. Observe that antiwebs are near-bipartite graphs.

The line graph of a graph G , denoted by $L(G)$, is obtained by turning adjacent edges of G into adjacent nodes of the line graph. Then we say that a graph H is a *line graph* if there is a graph G such that $H = L(G)$.

A graph is *claw-free* if it does not contain a claw as a node induced subgraph, where a *claw* is a four node graph with one node connected to a stable set of size three.

Let us now recall some operations on graphs that will be used in this paper.

Given two node disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ the *complete join* of them is the graph $G_1 \vee G_2$ having node set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. For a given graph $G = (V, E)$ and $U_1, U_2 \subset V$ such that $U_1 \cap U_2 = \emptyset$, we say that U_1 is *completely joined to* U_2 (in G) if $G[U_1 \cup U_2] = G[U_1] \vee G[U_2]$.

The graph G' is obtained from G after an *odd subdivision of the edge* e of G if it is obtained from G by replacing e by a path of odd length ([45]).

The *stretching of a node* is defined in [27]. Let v be a node of G with neighborhood $N(v)$ and let A_1 and A_2 be nonempty subsets of $N(v)$ such that $A_1 \cup A_2 = N(v)$, and $A_1 \cap A_2 = \emptyset$. A stretching of a node v is obtained as follows: remove v , introduce three nodes instead, called v_1, v_2 and u , and add an edge between v_i and every node in $\{u\} \cup A_i$ for $i \in \{1, 2\}$. Also, we say that H is a *stretching of* G if it is obtained from G after a finite number of node-stretching operations. It is not hard to see that an odd-subdivision of an edge is a sequence of node stretching operations.

2.2 On the Stable Set Polytope

Given $G = (V, E)$, the *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of G .

The *support graph* of a valid inequality of $\text{STAB}(G)$ is the subgraph of G induced by the nodes with nonzero coefficient in the inequality. A *full-support* inequality has G as its support graph. A graph G is said to be *facet-defining* if $\text{STAB}(G)$ has a full-support facet-defining inequality.

The following is an immediate consequence of Theorem 4.1 in [10].

Corollary 4. *If G is a facet-defining graph then G does not contain a clique cutset.*

If $G' \subset G$, the inequality

$$\sum_{i \in V(G')} x_i \leq \alpha(G')$$

is known as the *rank constraint* associated with G' and it is clearly valid for $\text{STAB}(G)$.

A more general type of valid inequalities for $\text{STAB}(G)$ is obtained from subgraphs of G which are complete join of graphs, called *joined constraints*: if

$G' \subset G$ and G' is the complete join of G_j with $j = 1, \dots, k$, the constraint

$$\sum_{j=1}^k \frac{1}{\alpha(G_j)} \sum_{i \in V(G_j)} x_i \leq 1,$$

is also valid for $\text{STAB}(G)$.

Note that the inequalities are scaled to have right hand side 1 and they include rank constraints when $k = 1$. From complete joins of antiwebs, the *joined antiweb constraints* are obtained.

Two canonical relaxations of $\text{STAB}(G)$ are defined in terms of rank constraints: the *edge relaxation*

$$\text{ESTAB}(G) = \{x \in [0, 1]^V : x_i + x_j \leq 1, ij \in E\},$$

and the *clique relaxation*

$$\text{QSTAB}(G) = \{x \in [0, 1]^V : \sum_{i \in Q} x_i \leq 1, Q \text{ maximal clique of } G\}.$$

We have $\text{STAB}(G) \subset \text{QSTAB}(G) \subset \text{ESTAB}(G)$ for any graph. In addition, $\text{STAB}(G)$ equals $\text{ESTAB}(G)$ for bipartite graphs only, and $\text{QSTAB}(G)$ for perfect graphs only [10].

A graph G is called *near-perfect graphs* (defined in [36]) if $\text{STAB}(G)$ is obtained after adding to $\text{QSTAB}(G)$ the rank inequality associated with the whole graph. Moreover, G is *rank-perfect* ([40]) if and only if $\text{STAB}(G)$ is described by rank constraints only. By definition, rank-perfect graphs include all perfect and near-perfect graphs.

As subclasses of rank-perfect graphs, the class of *a-perfect graphs* is introduced in [41] as graphs G where $\text{STAB}(G)$ is given by nonnegativity constraints and rank constraints associated with antiwebs. In particular, antiwebs are *a-perfect* by [37, 41].

We denote by $\text{ASTAB}^*(G)$ the linear relaxation of $\text{STAB}(G)$ obtained by nonnegativity constraints and all joined antiweb constraints. By construction, we see that

$$\text{STAB}(G) \subset \text{ASTAB}^*(G) \subset \text{QSTAB}(G) \subset \text{ESTAB}(G).$$

Recall that graphs G with $\text{STAB}(G) = \text{ASTAB}^*(G)$ are called *joined a-perfect*. Results from Shepherd [37] imply that all near-bipartite graphs are *joined a-perfect*, and results from Wagler [43, 44] show that several graph classes defined by clique cutsets are further subclasses of *joined a-perfect graphs*.

2.3 On the semidefinite relaxation $\text{LS}_+(G)$ of $\text{STAB}(G)$

In order to introduce the LS_+ -operator we denote by e_0, e_1, \dots, e_n the vectors of the canonical basis of \mathbb{R}^{n+1} (where the first coordinate is indexed zero) and

S_+^n the convex cone of symmetric and positive semi-definite $(n \times n)$ -matrices with real entries. Let $K \subset [0, 1]^n$ be a convex set and

$$\text{cone}(K) = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : x = x_0 y; \ y \in K \right\}.$$

Then, the convex set $M_+(K)$ is defined as:

$$M_+(K) = \left\{ Y \in S_+^{n+1} : \begin{array}{l} Y e_0 = \text{diag}(Y), \\ Y e_i \in \text{cone}(K), \\ Y(e_0 - e_i) \in \text{cone}(K), \ i = 1, \dots, n \end{array} \right\},$$

where $\text{diag}(Y)$ denotes the vector whose i -th entry is Y_{ii} , for every $i = 0, \dots, n$. Projecting this lifting back to the space \mathbb{R}^n results in

$$\text{LS}_+(K) = \left\{ x \in [0, 1]^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y e_0, \text{ for some } Y \in M_+(K) \right\}.$$

In [29], Lovász and Schrijver proved that $\text{LS}_+(K)$ is a relaxation of the convex hull of integer solutions in K and that

$$\text{LS}_+^n(K) = \text{conv}(K \cap \{0, 1\}^n),$$

where $\text{LS}_+^0(K) = K$ and $\text{LS}_+^k(K) = \text{LS}_+(\text{LS}_+^{k-1}(K))$ for every $k \geq 1$.

In this work we focus on the behavior of a single application of the LS_+ -operator to the edge relaxation $\text{ESTAB}(G)$ of the stable set polytope of a graph. Recall that $\text{LS}_+(G) = \text{LS}_+(\text{ESTAB}(G))$.

If $\text{LS}_+(G) = \text{STAB}(G)$ then G is LS_+ -perfect, and all other graphs are LS_+ -imperfect.

According to Lemma 1.5 in [29], given a graph G , every valid inequality whose support graph is near-bipartite is also valid for $\text{LS}_+(G)$. Then, this result and the definition of $\text{ASTAB}^*(G)$ imply that

$$\text{LS}_+(G) \subseteq \text{ASTAB}^*(G) \tag{1}$$

and Conjecture 3 establishes that the equality in equation (1) characterizes LS_+ -perfect graphs or, in other words, that LS_+ -perfect graphs coincide with joined a-perfect graphs.

It is known from [5] that every subgraph of an LS_+ -perfect graph is also LS_+ -perfect. Thus, exhibiting one LS_+ -imperfect subgraph G' in a graph G certifies the LS_+ -imperfection of G . This motivates the definition of *minimally LS_+ -imperfect graphs* as the LS_+ -imperfect graphs whose proper induced subgraphs are all LS_+ -perfect. The two smallest such graphs were found by [15] and [27] and are called G_{LT} and G_{EMN} , see Figure 1.

Observe that the graph G_{LT} is obtained after replicating a node in C_5 . It is clear that odd cycles and their complements are LS_+ -perfect. Thus, node replication does not preserve LS_+ -perfection. Moreover, we easily obtain:

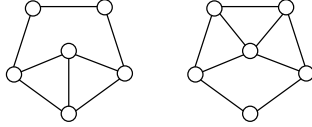


Figure 1: The graphs G_{LT} (on the left) and G_{EMN} (on the right).

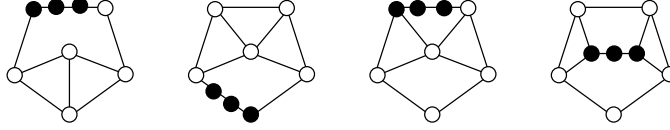


Figure 2: Some node-stretchings (v_1, w, v_2 in black) of G_{LT} and G_{EMN} .

Remark 5. *Given a graph $G = (V, E)$ having C_5 as a node induced subgraph, the graph obtained after replication of any node in C_5 is LS_+ -imperfect.*

Further LS_+ -imperfect graphs can be obtained by applying operations preserving LS_+ -imperfection. In [27] it is shown:

Theorem 6 ([27]). *The stretching of a node preserves LS_+ -imperfection.*

Hence, all stretchings of G_{LT} and G_{EMN} are LS_+ -imperfect, see Figure 2 for some examples.

2.4 On claw-free graphs

In several respects, claw-free graphs are generalizations of line graphs. An intermediate class between line graphs and claw-free graphs form *quasi-line graphs*, where the neighborhood of any node can be partitioned into two cliques (i.e., quasi-line graphs are the complements of near-bipartite graphs).

Quasi-line graphs can be divided into two subclasses: *fuzzy circular interval graphs* and *semi-line graphs*.

Let \mathcal{C} be a circle, \mathcal{I} a collection of intervals in \mathcal{C} without proper containments and common endpoints, and V a multiset of points in \mathcal{C} . A fuzzy circular interval graph $G(V, \mathcal{I})$ has node set V and two nodes are adjacent if both belong to one interval $I \in \mathcal{I}$, where edges between different endpoints of the same interval may be omitted.

A graph is semi-line if it is either a line graph or a quasi-line graph without a representation as a fuzzy circular interval graph.

For fuzzy circular interval graphs it turned out ([14]) that so-called *clique family inequalities* suffice to describe their stable set polytope.

More specifically, Stauffer (in [38, 39]) verified a conjecture stated in [32] establishing that every facet-defining clique family inequality of a fuzzy circular

interval graph G is associated with a web in G . These inequalities have the form

$$(p-r) \sum_{i \in W} x_i + (p-r-1) \sum_{i \in W_o} x_i \leq (p-r) \left\lfloor \frac{n}{p} \right\rfloor \quad (2)$$

where $G[W]$ has a web subgraph W_n^{p-1} and $r = n - \left\lfloor \frac{n}{p} \right\rfloor p \geq 1$. Moreover, $W \setminus V(W_n^{p-1})$ is the set of nodes in G having at least $2p-1$ consecutive neighbors in W_n^{p-1} . Also, a node in W_o is adjacent to exactly $2p-2$ consecutive nodes in W_n^{p-1} .

Chudnovsky and Seymour [9] extend results of Edmonds [12] and Edmonds and Pulleyblank [13] to semi-line graphs. Actually, they prove that if G is a semi-line graph then $\text{STAB}(G)$ is given by clique constraints and rank constraints associated with the line graphs of 2-connected hypomatchable induced subgraphs (a graph is called *hypomatchable* if removing any of its nodes results in a graph having a perfect matching).

Then, semi-line graphs are rank-perfect with line graphs as only facet-defining subgraphs.

However, there are claw-free graphs which are not quasi-line. Although there are many important results on the facial structure of them [18, 19, 20, 21, 22, 26], in order to prove the conjecture on claw-free not quasi-line graphs we take into account their decomposition given in [9].

A node is *simplicial* if its neighborhood is a clique. A strip (G, a, b) is a (not necessarily connected) graph with two designated simplicial nodes a and b . Given two node-disjoint strips (G_1, a_1, b_1) and (G_2, a_2, b_2) , their *composition* is the union of $G_1 \setminus \{a_1, b_1\}$ and $G_2 \setminus \{a_2, b_2\}$ together with all edges between $N_{G_1}(a_1)$ and $N_{G_2}(a_2)$, and between $N_{G_1}(b_1)$ and $N_{G_2}(b_2)$. A claw-free strip containing a 5-wheel as induced subgraph is a *5-wheel strip*.

This composition operation can be generalized to more than two strips (see [9] for further details).

According to these definitions, in [31] it is proved that every claw-free not quasi-line graph G with $\alpha(G) \geq 4$ admits a decomposition into strips, where at most one strip is quasi-line and all the remaining ones are 5-wheel strips having stability number at most 3.

There are only three “basic” types of 5-wheel strips (see Fig. 3) which can be extended by adding nodes belonging to the neighborhood of the 5-wheels (see [31] for details).

Note that a claw-free but not quasi-line graph G with $\alpha(G) \geq 4$ containing a clique cutset may have a facet-defining subgraph G' with $\alpha(G') = 3$ (inside a 5-wheel strip of type 3), see [34] for examples.

3 LS_+ -Perfect Graph Conjecture for claw free graphs

In this section we verify the validity of the LS_+ -Perfect Graph Conjecture on claw-free graphs by analyzing the behavior of the LS_+ -operator on all its rele-

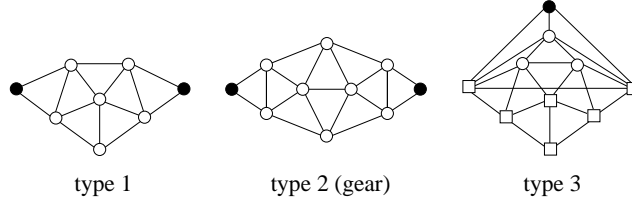


Figure 3: The three types of basic 5-wheel strips.

vant subclasses. Additionally, we characterize minimally LS_+ -imperfect graphs within these classes.

We start with those graphs having stability number two by recalling the following result:

Theorem 7 ([5]). *Let G be a graph with $\alpha(G) = 2$ such that $G - v$ is an odd antihole for some node v . G is LS_+ -perfect if and only if v is completely joined to $V(G) \setminus \{v\}$.*

From this, we can prove:

Theorem 8. *Let G be a connected imperfect graph with $\alpha(G) = 2$. If G is LS_+ -perfect, then G is a complete join of odd antiholes and (possible empty) perfect graphs.*

Proof. Since G is an imperfect graph and $\alpha(G) = 2$, G contains an odd antihole \overline{C} . If $G = \overline{C}$, we are done. Otherwise, if G is LS_+ -perfect then by Theorem 7, G is the complete join of \overline{C} and $G' = G - V(\overline{C})$.

If G' is a perfect graph then the result follows. Otherwise, G' is LS_+ -perfect and $\alpha(G') = 2$ and we apply the same argument as for G . \square

Using Corollary 5.2 in [10] on the linear description of the stable set polytope of complete joins of graphs, we conclude:

Corollary 9. *The LS_+ -Perfect Graph Conjecture is true for graphs with stability number 2.*

3.1 Quasi-line graphs

Recall that quasi-line graphs divide into the two subclasses of semi-line graphs and fuzzy circular interval graphs.

As we have already mentioned, a way to attack the conjecture is from the polyhedral point of view. Using this approach, in Lemma 7 in [17], the authors characterized LS_+ -perfect line graphs by showing:

Theorem 10 ([17]). *Let G be the line graph of a 2-connected hypomatchable graph. Then, G is LS_+ -perfect if and only if either G is a complete graph or G is an odd cycle.*

Additionally, Chudnovsky and Seymour [9] proved that the stable set polytope of a semi-line graph is given by rank constraints associated with cliques and the line graphs of 2-connected hypomatchable graphs. From these two results we directly conclude:

Corollary 11. *The LS_+ -Perfect Graph Conjecture holds for semi-line graphs.*

For fuzzy circular interval graphs, we make use of Theorem 5 in [16].

Theorem 12 ([16]). *If a web graph is LS_+ -perfect then it is either a perfect or a minimally imperfect graph.*

Using this result, we can prove:

Theorem 13. *Let G be a facet-defining fuzzy circular interval graph. Then, G is LS_+ -perfect if and only if it is an odd cycle or its complement.*

Proof. If G is an odd cycle or its complement then clearly G is LS_+ -perfect. Let us prove the converse.

Since G is facet-defining, from the results in [38, 39], we have that G is the support graph of a clique family inequality

$$(p-r) \sum_{i \in W} x_i + (p-r-1) \sum_{i \in W_o} x_i \leq (p-r) \left\lfloor \frac{n}{p} \right\rfloor$$

associated with a web subgraph W_n^{p-1} of $G[W]$. Recall that every node in $W \setminus V(W_n^{p-1})$ (resp. in W_o) is adjacent to at least $2p-1$ (resp. to exactly $2p-2$) consecutive nodes of W_n^{p-1} and $r \geq 1$ (see Section 2.4).

By Theorem 12, W_n^{p-1} is LS_+ -perfect if and only if it is an odd cycle or its complement. Since G is LS_+ -perfect we restrict our attention to web subgraphs of $G[W]$ where $n = 2k+1$ and $p = 2$ or $p = k \geq 3$. In both cases, $r = 1$ follows and we have to prove that $G = W_n^{p-1}$.

Observe that if $p = 2$, then $p-r-1 = 0$ and the inequality takes the form

$$\sum_{i \in W} x_i \leq k.$$

Suppose that there exists $v \in W \setminus V(W_{2k+1}^1)$. Then, v is connected to $t \geq 2p-1 = 3$ consecutive nodes in W_{2k+1}^1 .

If $t \geq 5$ and $k = 2$, G has a 5-wheel as induced subgraph, contradicting the fact that G is quasi-line. But, if $t \geq 5$ and $k \geq 3$, G has a claw, again a contradiction. Then, $t \leq 4$.

Now, if $t = 3$ (resp. $t = 4$) G contains an odd-subdivision of an edge of G_{LT} (resp. G_{EMN}).

Since G is LS_+ -perfect then $W \setminus V(W_{2k+1}^1) = \emptyset$, or equivalently, $G = W_{2k+1}^1 = C_{2k+1}$.

Let us now analyze the case $p = k \geq 3$ and $G[W_o \cup W] = G$.

Suppose that there exists $v \in (W_o \cup W) \setminus V(W_{2k+1}^{k-1})$. It holds that v is adjacent to at least $2k-2$ consecutive nodes in W_{2k+1}^{k-1} .

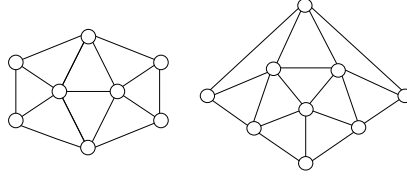


Figure 4: The gear and the 3-gear graphs.

It follows that the subgraph G' of G induced by $V(W_{2k+1}^{k-1}) \cup \{v\}$ has stability number two. Since G' is LS_+ -perfect, from Theorem 8, v is completely joined to $V(W_{2k+1}^{k-1})$, contradicting that G is quasi-line.

Then, we conclude $G[W_o \cup W] = W_{2k+1}^{k-1}$ or, equivalently, the complementary graph of C_{2k+1} and the proof is complete. \square

Clearly the theorem above implies that the LS_+ -Perfect Graph Conjecture is true for fuzzy circular interval graphs. Since the class of quasi-line graphs divides into semi-line graphs and fuzzy circular interval graphs, we obtain, as direct consequence:

Corollary 14. *The LS_+ -Perfect Graph Conjecture is true for quasi-line graphs.*

3.2 Claw-free graphs that are not quasi-line

It is left to treat the case of claw-free graphs that are not quasi-line. When $\alpha(G) \geq 4$, we study LS_+ -perfection by using the strip decomposition given in [31]. Actually, we prove that in this class there is no connected LS_+ -perfect graph G without clique cutset such that $\alpha(G) \geq 4$. Finally, when G has stability number 3, we obtain a decomposition of connected LS_+ -perfect graphs without clique cutset that allows us to conclude that the only graphs with these properties are the gear and the 3-gear graphs depicted in Figure 4.

In order to simplify the writing, we denote by \mathcal{G} the class of connected claw-free not quasi-line graphs without clique cutset.

We first prove the following:

Theorem 15. *Let $G \in \mathcal{G}$. If G is LS_+ -perfect then $\alpha(G) \leq 3$.*

Proof. Assume that G is LS_+ -perfect and $\alpha(G) \geq 4$.

As we have already pointed out in Section 2.4, according to [31], G has a decomposition into strips, where at most one strip is quasi-line and all the remaining ones have stability number at most 3 and contain a 5-wheel each. Since G is not quasi-line, it contains at least one 5-wheel strip G' . Also, recall that there are only three types of 5-wheel strips, Fig. 3 shows the “basic” types, which can be extended by adding nodes belonging to the neighborhood of the 5-wheels.

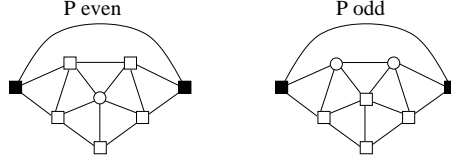


Figure 5: LS_+ -imperfect subgraphs if G' is of type 1.

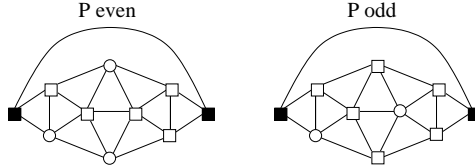


Figure 6: LS_+ -imperfect subgraphs if G' is of type 2.

If G' is of type 3, then G' contains G_{LT} , induced by the squared nodes indicated in Fig. 3, contradicting the assumption that G is LS_+ -perfect. Let us analyze the cases where G' is of type 1 or 2.

Note further that G' is a proper subgraph of G since $\alpha(G') \leq 3$ and $\alpha(G) \geq 4$. Then, G contains as subgraph the composition of G' and another strip graph, say G'' . Then, there are two nodes in G'' playing the role of the two simplicial nodes of G' (the two black nodes in Fig. 3). Since G has no clique cutset, these two nodes are connected by a path P in G'' .

If G' is of type 1 and P has even length, G has a node stretching of G_{EMN} as induced subgraph. Similarly, if P is of odd length it has a node stretching of G_{LT} (see the squared nodes in Fig. 5).

Similarly, if G' is of type 2 and P has even length, G has a node stretching of G_{LT} as induced subgraph. Finally, if P is odd G' has a node stretching of G_{EMN} (see the squared nodes in Fig. 6).

Hence, in all cases we contradict the LS_+ -perfection of G and we conclude that $\alpha(G) \leq 3$. \square

As an immediate consequence, we obtain:

Corollary 16. *Every facet-defining claw-free not quasi-line graph G with $\alpha(G) \geq 4$ is LS_+ -imperfect.*

For claw-free graphs having stability number three, there is no decomposition known yet. Hence, we start with some technical results that will allow us to characterize LS_+ -perfect graphs in \mathcal{G} with stability number three.

In what follows, given $G = (V, E)$ and $v \in V$, we denote by G_v the subgraph of G induced by $N[v]$. It is clear that there is no stable set of size 3 in $N(v)$ for every node v of a claw-free graph. The following remark directly follows:

Remark 17. Let $G = (V, E)$ be a claw-free graph. Then, $\alpha(G_v) \leq 2$ for all $v \in V$. Moreover, given K, L, M three mutually disjoint nonempty subsets of V such that $K \cup L$ is completely joined to M , it holds that $\alpha(G[K \cup L]) \leq 2$. In particular, if no node of K is adjacent to any node in L , then K and L are cliques.

In addition, if $C \subset V(G)$ induces an odd antihole, we assume $C = \{u_i : i \in [2k+1]\}$ with $k \geq 2$ and, for all $i \in [2k+1]$, u_i, u_{i+k} are not adjacent in G . We say that u_i and u_{i+1} are consecutive (in C), for all $i \in [2k+1]$.

We have the following result:

Lemma 18. Let $G = (V, E)$ be a connected claw-free graph and let $C \subset V$ induce an odd antihole in G . If there exists $y \in V \setminus C$ such that $N(y) \cap C \neq \emptyset$ and $\alpha(G[C \cup \{y\}]) = 3$ then $k = 2$. Moreover, y has exactly two neighbors in C which are consecutive.

Proof. Let $C = \{u_i : i \in [2k+1]\}$ for some $k \geq 2$ and $y \in V$ such that $N(y) \cap C \neq \emptyset$ and $\alpha(G[C \cup \{y\}]) = 3$. Then, there exists $i \in [2k+1]$ such that $S = \{y, u_i, u_{i+k}\}$ is a stable set of G .

If the nodes in S have a common neighbor z then $S \cup \{z\}$ induces a claw. Therefore, y is not adjacent to any node in $N(u_i) \cap N(u_{i+k})$.

Since $C \setminus N(u_i) = \{u_i, u_{i+k}, u_{i+k+1}\}$, $C \setminus N(u_{i+k}) = \{u_i, u_{i+k}, u_{i+2k}\}$, and y is not adjacent to u_i, u_{i+k} it holds that $\emptyset \neq N(y) \cap C \subset \{u_{i+k+1}, u_{i+2k}\}$.

If $N(y) \cap C = \{u_{i+2k}\}$ then $\{u_{i+2k}, u_{i+1}, y, u_{i+k+1}\}$ induces a claw. Similarly, if $N(y) \cap C = \{u_{i+k+1}\}$ then $\{u_{i+k+1}, u_{i+1}, y, u_{i+2k}\}$ is a set inducing a claw. Therefore, $N(y) \cap C = \{u_{i+k+1}, u_{i+2k}\}$.

Finally, if $k \geq 3$, $u_{i+k+2} \neq u_{i+2k}$ then we have again a claw induced by the nodes $\{u_{i+2k}, u_{i+1}, y, u_{i+k+2}\}$. Then, $k = 2$ implying that the two neighbors of y in C are u_{i+3} and u_{i+4} , which are consecutive. \square

Let us focus on graphs $G \in \mathcal{G}$ with stability number three. By Remark 17, $\alpha(G_v) \leq 2$ for every node v . But, if there is $C \subset N(v)$ inducing an odd antihole in G it holds that $\alpha(G_v) = 2$. If G_v is LS_+ -perfect, using Theorem 8 we obtain the next observation that will be useful in what remains of this section.

Remark 19. Let $v \in V(G)$ such that G_v is LS_+ -perfect. If there exists $C \subset N(v)$ inducing an odd antihole in G , then C is completely joined to $N[v] \setminus C$.

The next result gives a decomposition of graphs in \mathcal{G} with stability number three such that every subgraph with stability number two is LS_+ -perfect.

Lemma 20. Let $G = (V, E)$ be a graph such that $G \in \mathcal{G}$, $\alpha(G) = 3$, and for all $G' \subset G$ with $\alpha(G') = 2$, G' is LS_+ -perfect. Then, V can be partitioned into the sets Q, C, Y_2, Y_3, Z such that C induces C_5 , $|Y_3| \geq 2$, while Q, Y_2 , and Z are cliques. In addition,

- (1) Q is completely joined to C and no node in Q is adjacent to any node in $V \setminus (C \cup Q)$,

- (2) Y_2 is completely joined to $C \cup Y_3$,
- (3) Z is completely joined to Y_3 and no node in Z is adjacent to any node in $Q \cup C$,
- (4) every node in Y_3 has exactly two neighbors in C which are consecutive. Moreover, if two nodes in Y_3 share a neighbor in C then they are adjacent. Also, not all nodes in Y_3 have the same neighbors in C .

Proof. Since G is claw-free and not quasi-line, there exists a node v such that $N(v)$ cannot be partitioned into two cliques. Then, the complementary graph of $G_v \setminus \{v\}$ is not bipartite and it has a set of nodes C inducing an odd cycle. Since $\alpha(G_v) \leq 2$, $|C| \geq 5$. Then C induces an antihole in G and we denote $C = \{u_i : i \in [2k+1]\}$ for some $k \geq 2$.

Let $W = N(v) \setminus C$ and $Q = W \cup \{v\}$. Clearly, C, Q is a partition of $N[v]$. By assumption G_v is LS_+ -perfect. Then, Remark 19 ensures that Q is completely joined to C . Thus, we have partially proved (1).

Since $\alpha(G_v) = 2$ and $\alpha(G) = 3$, $V \setminus N[v] \neq \emptyset$. Now, consider the partition of $V \setminus N[v]$ into the sets X, Y, Z such that:

- $X = \{x \in V \setminus N[v] : N(x) \cap W \neq \emptyset\}$,
- $Y = \{y \in V \setminus N[v] : N(y) \cap W = \emptyset \text{ and } N(y) \cap C \neq \emptyset\}$, and
- $Z = \{z \in V \setminus N[v] : N(z) \cap N[v] = \emptyset\}$.

Observe that, since $\alpha(G) = 3$, $\alpha(G_v) = 2$, and no node in Z is connected with any node in $N[v]$, we have that $\alpha(G[Z]) \leq 1$ and then Z is a (possibly empty) clique.

In order to complete the proof of (1) we need to show that $X = \emptyset$.

Let us assume that $X \neq \emptyset$.

Let $x \in X$ and $w \in N(x) \cap W$. Since $C \subset N(w)$ and we assume that G_w is LS_+ -perfect, Remark 19 implies that C is completely joined to $N[w] \setminus C$ and, in particular, C is completely joined to $\{x\}$. Then X is completely joined to C . Now, by Remark 17 considering $K = \{v\}$, $L = X$, and $M = C$ it holds that X is a clique.

Since $Q \cup X$ is completely joined to C , by Remark 17 with $M = C$, $K = X$, and $L = Q$ we obtain that $\alpha(G[Q \cup X]) \leq 2$. Then $\alpha(G[N[v] \cup X]) = 2$. Since $\alpha(G) = 3$, $Y \cup Z \neq \emptyset$. If $Y = \emptyset$, X is a clique cutset, contradicting the hypothesis. Then, $Y \neq \emptyset$.

Let $y \in Y$ and $u \in C$ such that $y \in N(u)$. Again, considering Remark 17 with $K = \{v\}$, $L = X \cup \{y\}$, and $M = \{u\}$, we have that L is a clique. Therefore Y is completely joined to X .

So we have that $C \cup Y \subset N(x)$ for every $x \in X$ and by Remark 19 (using that G_x is LS_+ -perfect) we get that C is completely joined to Y . Applying again Remark 17 for $K = \{v\}$, $L = X \cup Y$ and $M = C$ we obtain that $X \cup Y$ is a clique and then $\alpha(N[v] \cup X \cup Y) = 2$. This implies $Z \neq \emptyset$ and $X \cup Y$ is a clique cutset, a contradiction to the assumption on G .

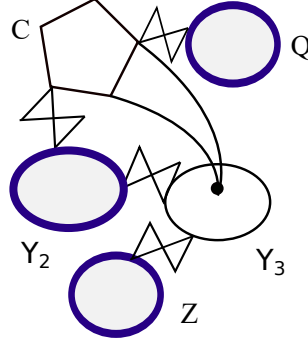


Figure 7: A scheme of a graph with a partition as in Lemma 20 (sets with bold lines stand for complete graphs, the \bowtie symbol indicates a complete join between two sets).

Then, $X = \emptyset$ and (1) holds.

Since G is connected, (1) implies that $Y \neq \emptyset$. Moreover, $|Y| \geq 2$ otherwise Y is a clique cutset.

Let $y \in Y$ and $u \in C$ such that $y \in N(u)$. Since C is completely joined to Q , Remark 17 with $K = Q$, $L = \{y\}$ and $M = \{u\}$ implies that Q is a clique.

Let us show that any two nodes in Y sharing a node in U are adjacent. Indeed, consider $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $\{y_1, y_2\} \subset N(u)$ for some $u \in C$. It follows from Remark 17 with $K = Q$, $L = \{y_1, y_2\}$ and $M = \{u\}$ that L is a clique.

Now, we split Y into the sets $Y_i = \{y \in Y : \alpha(G[\{y\} \cup C]) = i\}$ for $i = 2, 3$. In this way, V is partitioned into C , Q , Y_2 , Y_3 , and Z . Recall that Q is a nonempty clique ($v \in Q$).

In order to prove (2), observe that Theorem 8 ensures that Y_2 is completely joined to C . Since two nodes in Y sharing a neighbor in C are adjacent, it follows that Y_2 is a clique. Moreover, Y_2 is completely joined to $C \cup Y_3$.

Suppose now that $Y_3 = \emptyset$. Then $\alpha(G[N[v] \cup Y]) = 2$ and $Z \neq \emptyset$ follows. Thus, Y_2 is a clique cutset and we arrive to a contradiction. Then, $Y_3 \neq \emptyset$. If $|Y_3| = 1$ and using (2) we have Y is a clique cutset, a contradiction. Thus, $|Y_3| \geq 2$. Moreover, by Lemma 18, C induces C_5 and every node in Y_3 is adjacent to two consecutive nodes in C .

To complete the proof of (3), suppose that there are $y \in Y_3$ and $z \in Z$ which are not adjacent. It holds that there exists $i \in [5]$ such that $\{y, u_i, u_{i+2}, z\}$ is a stable set of size 4, which is a contradiction on $\alpha(G) = 3$. Then, Z is completely joined to Y_3 .

Finally, not all nodes in Y_3 have the same two neighbors in C , say u_i, u_{i+1} for some $i \in [5]$. Otherwise, $Y_2 \cup \{u_i, u_{i+1}\}$ is a clique cutset. This completes the proof of (4). \square

It is not hard to check that the support graphs of facet-defining inequalities

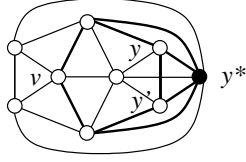


Figure 8: Subgraph induced by $C_5 \cup \{v, y, y', y^*\}$, the bold edges indicate the G_{EMN} . Removing y_* yields the gear.

of the gear and the 3-gear graphs are complete and 5-wheel graphs. Then, the gear and the 3-gear graphs are LS_+ -perfect. Also, since any node of these graphs belongs to a 5-hole (see Figure 4), by Remark 5 we have the following observation:

Remark 21. *The graph obtained after the replication of any node in the gear or the 3-gear graph is LS_+ -imperfect.*

Finally, we obtain:

Theorem 22. *Let $G \in \mathcal{G}$ an LS_+ -perfect graph with $\alpha(G) \geq 3$. Then G is the gear or the 3-gear graph.*

Proof. By Theorem 15, $\alpha(G) = 3$. Since G is LS_+ -perfect, by the previous lemma we can consider the partition of $V(G)$ into the sets Q , C , Y_2 , Y_3 , and Z .

Recall that $Q \neq \emptyset$ and let $v \in Q$. Moreover, we know that $|Y_3| \geq 2$, not all the nodes in Y_3 have the same two consecutive neighbors in C , and two nodes in Y_3 having a common neighbor in C are adjacent. Let us now prove that adjacent nodes in Y_3 share a common neighbor in C .

Let $y \neq y' \in Y_3$ such that y is adjacent to y' and assume that they do not share a common neighbor in C . W.l.o.g. we can assume that $N(y) \cap C = \{u_1, u_2\}$ and $N(y') \cap C = \{u_3, u_4\}$. Then, $\{v, u_1, y, y', u_4, u_5\}$ induces the graph G_{LT} contradicting the fact that G is LS_+ -perfect.

Let us prove that if there is a node in C completely joined to Y_3 then G is the gear graph.

W.l.o.g. assume that $u_2 \in N(y)$ for all $y \in Y_3$. In this case Y_3 is a clique. Then $Z = \emptyset$, otherwise, $Y_2 \cup Y_3$ is a clique cutset. Since not all the nodes in Y_3 have the same neighbors in C , there exist $y, y' \in Y_3$ such that $N(y) \cap C = \{u_1, u_2\}$ and $N(y') \cap C = \{u_2, u_3\}$.

If there is $y^* \in Y_2$ then, $\{v, u_1, u_3, y, y', y^*\}$ induces G_{EMN} (see Figure 8), again a contradiction to the fact that G is LS_+ -perfect. Then, $Y_2 = \emptyset$.

It is easy to see that every node in Q is a replication of v in G . Due to Remark 21, the LS_+ -perfection of G forbids the replications of v and then G is the gear graph.

Let us consider the case where not all the nodes in Y_3 have a common neighbor in C . W.l.o.g. we can assume that there are two nonadjacent nodes y, y' in Y_3 such that $N(y) \cap C = \{u_1, u_2\}$ and $N(y') \cap C = \{u_3, u_4\}$.

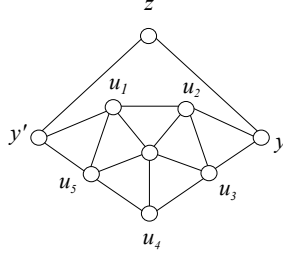


Figure 9: A node stretching of G_{EMN}

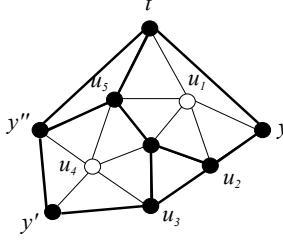


Figure 10: The graph \hat{H} . Black nodes induce a stretching of G_{EMN} .

Observe that $Y_2 = Z = \emptyset$. Indeed, if there is $y^* \in Y_2$ the set $\{y^*, y, y', u_5\}$ induces a claw. If $z \in Z$, from Lemma 20 (3), z is adjacent to every node in Y_3 and then $\{z, y, y'\} \cup C$ induces a node stretching of G_{EMN} (see Figure 9).

Then, $V(G)$ is partitioned into Q , C and Y_3 . Also, $\{v\} \cup C$ induces a 5-wheel for all $v \in Q$, $N(y) \cap C = \{u_1, u_2\}$, and $N(y') \cap C = \{u_3, u_4\}$. Since G has no clique cutset, there must be another node $t \in Y_3$ which is not a replication either of y or y' .

Let us first analyze the case when no node of Y_3 is adjacent to $\{u_2, u_3\}$. In this case, w.l.o.g. we can assume that $N(t) \cap C = \{u_1, u_5\}$. Again, since G has no clique cutset, there must exist another node y'' which is not a replication of any node in $\{y, t, y'\}$. Then, $N(y'') \cup C = \{u_4, u_5\}$. Figure 10 shows the graph \hat{H} induced by $C \cup \{v, y, y', t, y''\}$. Observe that the black nodes in \hat{H} induces a stretching of G_{EMN} , a contradiction to the fact that G is LS_+ -perfect.

It only remains to analyze the case when $N(t) \cap C = \{u_2, u_3\}$.

If there is another node $y'' \in Y_3$, by Remark 21, it is not a replication of any node in $\{y, t, y'\}$ and w.l.o.g. we can assume that $N(y'') = \{u_1, u_5\}$. It is easy to see that the subgraph induced by $C \cup \{v, y, y', t, y''\}$ is the graph \hat{H} after relabeling its nodes (see Figure 11), a contradiction.

Then, there is no other node in Y_3 and G is the graph induced by $C \cup \{v, y, y', t\}$ which is the 3-gear graph (see Figure 12). \square

Since the gear and the 3-gear graphs are not facet-defining, from Theorem

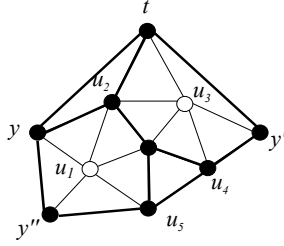


Figure 11: The graph \hat{H} with a relabeling of its nodes.

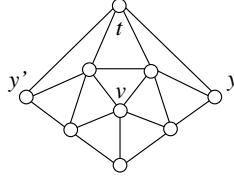


Figure 12: Subgraph induced by $C_5 \cup \{v, y, y', t\}$, called a 3-gear.

22 we conclude:

Corollary 23. *Every facet-defining claw-free not quasi-line graph G with $\alpha(G) \geq 3$ is LS_+ -imperfect.*

Combining corollaries 9, 14, and 23, we obtain our main contribution:

Theorem 24. *The LS_+ -Perfect Graph Conjecture is true for claw-free graphs.*

4 On minimally LS_+ -imperfect claw-free graphs

In this section we address our attention to minimally LS_+ -imperfect graphs that are claw-free. These graphs are facet-defining and therefore the results in Section 3 will be reused in order to describe all minimally LS_+ -imperfect claw-free graphs.

There are some known results on this matter. In [2], we consider for each $k \geq 2$, a family of graphs with $2k+2$ nodes, having a $2k+1$ -antihole as subgraph. Theorem 4.3 in the paper, states that the only minimally LS_+ -imperfect graph with stability number two in the family is the graph for which the node outside the antihole has degree $2k$. This graph is called H_k .

Lemma 25. *Let $G = (V, E)$ be a minimally LS_+ -imperfect graph with $\alpha(G) = 2$. Then, G is either G_{LT} , G_{EMN} , or H_k , for some $k \geq 3$.*

Proof. Since G is imperfect, minimally LS_+ -imperfect and $\alpha(G) = 2$, G has a $(2k+1)$ -antihole induced by $C \subset V$ and a node $v \in V \setminus C$ outside C . By Theorem

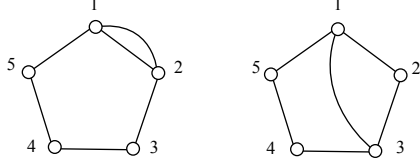


Figure 13: The graphs H^1 and H^2 , for $k = 2$, $t = 1$, $i = 1$ and $j = 2, 3$, respectively.

7, v is not completely joined to C . Since G is minimally LS_+ -imperfect it holds that $V = C \cup \{v\}$ and either $k = 2$ implying that $G = G_{LT}$ or $G = G_{EMN}$, or $k \geq 3$ and from the discussion above we get $G = H_k$. \square

Remind that the facet-defining semi-line graphs are line graphs (of 2-connected hypomatchable graphs). In addition, Corollary 10 in [17] establishes that the line graph of graph H is minimally LS_+ -imperfect if and only if H has a particular structure that can be described as follows: H is an odd cycle C_{2k+1} , $k \geq 2$, with a path of odd length P_t , $t \geq 1$ attached to nodes i and j in C_{2k+1} (observe that the path can have length one and H can have a double edge). It is not hard to see that every such graph H is an odd-subdivision of one of the graphs depicted in Figure 13.

Observe that the line graphs of H^1 and H^2 are G_{LT} and G_{EMN} , respectively. From Lemma 2 in [17] we can easily see that if a graph G is an odd subdivision of an edge e in a graph G' , $L(G)$ can be obtained from $L(G')$ by stretching of its node e .

This allows us to conclude:

Lemma 26. *Let G be a minimally LS_+ -imperfect semi-line graph. Then G is a stretching of G_{LT} or a stretching of G_{EMN} .*

Let us now consider fuzzy circular interval graphs.

Theorem 27. *Let G be a minimally LS_+ -imperfect fuzzy circular interval graph. Then, G is either W_{10}^2 , G_{LT} , G_{EMN} , $G = H_k$ for some $k \geq 3$, a stretching of G_{LT} , or a stretching of G_{EMN} .*

Proof. Let assume that G is neither W_{10}^2 , G_{LT} nor G_{EMN} . Since G is facet-defining, we can follow the reasoning in the proof of Theorem 13.

Remind that G is the support graph of a clique family inequality (2) associated with a web subgraph W_n^{p-1} of G . Since G is minimally LS_+ -imperfect then W_n^{p-1} is LS_+ -perfect and there exists a node v of G outside the web. Again Theorem 12 implies that the web W_n^{p-1} in question is an odd cycle or its complement and we obtain that $n = 2k + 1$ with $k \geq 3$ and $p = 2$ or $p = k \geq 3$. In both cases, $r = 1$.

As we have already observed in the proof of Theorem 13, when $p = 2$ it holds that $p - r - 1 = 0$ and v is connected to $t \geq 2p - 1 = 3$ consecutive nodes in

W_{2k+1}^1 . As G_{LT} and G_{EMN} cannot be subgraphs of a minimally LS_+ -imperfect graph then $k \geq 3$ and G has an stretching of G_{LT} or of G_{EMN} as a subgraph. Since the latter graphs are minimally LS_+ -imperfect then G is one of them.

In the case $p = k \geq 3$, we have also observed that v is adjacent to at least $2k - 2$ consecutive nodes in W_{2k+1}^{k-1} . Thus, the subgraph of G induced by $V(W_{2k+1}^{k-1}) \cup \{v\}$ has stability number two. Then, Lemma 25 implies that the only possible choice is that $G = H_k$ for some $k \geq 3$. \square

Concerning claw-free not quasi-line graphs we have seen in Theorem 15 that if the graph has stability number at least 4, it has a G_{LT} or a G_{EMN} as node induced subgraph.

Remark 28. *There is no minimally LS_+ -imperfect graph in the class of claw-free not quasi-line graphs with stability number at least four.*

A similar result holds when the stability number is equal to three. In this case we make use of Theorem 22.

Theorem 29. *There is no minimally LS_+ -imperfect graph in the class of claw-free not quasi-line graphs with stability number three.*

Proof. Let $G \in \mathcal{G}$ be a minimally LS_+ -imperfect graph with stability number three. It is clear that G cannot have a minimally LS_+ -imperfect graph with stability number two as induced subgraph, then we can consider the partition Q, C, Y_2, Y_3 , and Z of $V(G)$ given by Lemma 20. By doing so, we can follow the reasoning in the proof of Theorem 22 and conclude that if G is LS_+ -imperfect and has stability number three, then it has a node stretching of G_{LT} or G_{EMN} as a subgraph. But, since G is minimally LS_+ -imperfect then G is one of these graphs itself. Then G is a line graph and we arrive to a contradiction. \square

From the previous results we have that minimally LS_+ -imperfect claw free graphs are fuzzy circular interval graphs, that is:

Corollary 30. *Let G be a minimally LS_+ -imperfect claw-free graph. Then, G is W_{10}^2 , G_{LT} , G_{EMN} , $G = H_k$ for some $k \geq 3$, a stretching of G_{LT} , or a stretching of G_{EMN} .*

5 Conclusion and future research

The context of this work was the study of LS_+ -perfect graphs, i.e., graphs where a single application of the Lovász-Schrijver PSD-operator LS_+ to the edge relaxation yields the stable set polytope. Hereby, we are particularly interested in finding an appropriate polyhedral relaxation $P(G)$ of $STAB(G)$ that coincides with $LS_+(G)$ and $STAB(G)$ if and only if G is LS_+ -perfect. An according conjecture has been recently formulated (LS_+ -Perfect Graph Conjecture); here we verified it for the well-studied class of claw-free graphs (Theorem 24).

Note further that, besides verifying the LS_+ -Perfect Graph Conjecture for claw-free graphs, we obtained a description of all minimally LS_+ -imperfect claw-free graphs (Corollary 30).

For that, it surprisingly turned out that it was not necessary to make use of the description of $STAB(G)$ for claw-free not quasi-line graphs G

- with $\alpha(G) = 2$ (by Cook, see [36]),
- with $\alpha(G) = 3$ (by Pêcher, Wagler [33]),
- with $\alpha(G) \geq 4$ (by Galluccio, Gentile, Ventura [18, 19, 20]).

From the presented results and proofs, we can draw some further conclusions.

First of all, we can determine the subclass of joined a -perfect graphs to which all LS_+ -perfect claw-free graphs belong to. In [25], it is suggested to call a graph G *m-perfect* if the only facets of $STAB(G)$ are associated with cliques and minimally imperfect graphs. According to [11], G is *joined m-perfect* if its facet defined graphs are complete graphs, minimally imperfect graphs and their complete joins. The results from Section 3 provide the complete list of all facet-defining LS_+ -perfect claw-free graphs:

- complete graphs,
- odd holes and odd antiholes,
- complete joins of odd antihole(s) and a (possibly empty) complete graph.

Hence, we conclude:

Corollary 31. *All LS_+ -perfect claw-free graphs are joined m-perfect.*

Finally, the subject of the present work has parallels to the well-developed research area of perfect graph theory also in terms of polynomial time computability. In fact, it has the potential of reaching even stronger results due the following reasons. Actually, in [29], the authors prove that calculating the value

$$\eta_+(G) = \max \mathbf{1}x, x \in LS_+(G)$$

can be obtained with arbitrary precision in polynomial time for every graph G , even in the weighted case. Thus, the stable set problem can be solved in polynomial time for a strict superset of perfect graphs, the LS_+ -perfect graphs, by $\alpha(G) = \eta_+(G)$. Hence, our future lines of research include to find

- new families of graphs where the conjecture holds (e.g., by characterizing the minimally LS_+ -imperfect graphs within the class),
- new subclasses of LS_+ -perfect or joined a -perfect graphs,
- classes of graphs G where $STAB(G)$ and $LS_+(G)$ are “close enough” to have $\alpha(G) = \lfloor \eta_+(G) \rfloor$.

In particular, the class of graphs G with $\alpha(G) = \lfloor \eta_+(G) \rfloor$ can be expected to be large since $\text{LS}_+(G)$ is a much stronger relaxation of $\text{STAB}(G)$ than $\text{TH}(G)$. In all cases, the stable set problem could be approximated with arbitrary precision in polynomial time in these graph classes by optimizing over $\text{LS}_+(G)$. Finally, note that $\text{LS}_+(P(G))$ with

$$\text{STAB}(G) \subseteq P(G) \subseteq \text{ESTAB}(G)$$

clearly gives an even stronger relaxation of $\text{STAB}(G)$ than $\text{LS}_+(G)$. However, already approximating with arbitrary precision over $\text{LS}_+(\text{QSTAB}(G))$ cannot be done in polynomial time anymore for all graphs G by [29]. Hence, LS_+ -perfect graphs or their generalizations satisfying $\alpha(G) = \lfloor \eta_+(G) \rfloor$ are the most promising cases in this context.

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