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# Polyhedra associated with locating-dominating, open locating-dominating and locating total-dominating sets in graphs

Gabriela Argiroffo<sup>a</sup> Silvia Bianchi<sup>a</sup> Yanina Lucarini<sup>a</sup>  
Annegret Wagler<sup>b</sup>

<sup>a</sup>*Facultad de Ciencias Exactas, Ingeniería y Agrimensura  
Universidad Nacional de Rosario, Rosario, Argentina  
email: {garua, sbianchi, lucarini}@fceia.unr.edu.ar*

<sup>b</sup>*Laboratoire d'Informatique, de Modélisation et d'Optimisation des Systèmes  
(LIMOS), Université Clermont Auvergne, Clermont-Ferrand, France  
email: annegret.wagler@uca.fr*

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## Abstract

The problems of determining locating-dominating, open locating-dominating or locating total-dominating sets of minimum cardinality in a graph  $G$  are variations of the classical minimum dominating set problem in  $G$  and are all known to be hard for general graphs. A typical line of attack is therefore to determine the cardinality of minimum such sets in special graphs.

In this work we study the three problems from a polyhedral point of view. We provide the according linear relaxations, discuss their combinatorial structure, and demonstrate how the associated polyhedra can be entirely described or polyhedral arguments can be applied to find minimum such sets for special graphs.

*Key words:* locating-dominating set problem, open locating-dominating set problem, locating total-dominating set problem, polyhedral approach

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## 1 Introduction

Surveillance problems for facilities where detection devices have to be placed in rooms to locate an intruder (like a fire, a thief or a saboteur) lead to different

location-domination problems in the graph modeling the facility. Depending on the features of the detection devices (to detect an intruder only if it is present in the room where the detector is installed and/or also in any neighbored room), different dominating sets can be used to determine the optimal distribution of the detection devices in the facility. In the following, we study four problems arising in this context which all have been actively studied during the last decade, see e.g. the bibliography maintained by Lobstein [20].

Let  $G = (V, E)$  be a graph. The open neighborhood of a node  $i$  is the set  $N(i)$  of all nodes of  $G$  adjacent to  $i$ , and  $N[i] = \{i\} \cup N(i)$  is the closed neighborhood of  $i$ . A subset  $C \subseteq V$  is *dominating* (resp. *total-dominating*) if  $N[i] \cap C$  (resp.  $N(i) \cap C$ ) are non-empty sets for all  $i \in V$ . A subset  $C \subseteq V$  is

- an *identifying code* (*ID-set*) if it is a dominating set and  $N[i] \cap C \neq N[j] \cap C$ , for distinct  $i, j \in V$  [19];
- a *locating-dominating set* (*LD-set*) if it is a dominating set and  $N(i) \cap C \neq N(j) \cap C$ , for  $i, j \in V - C$  [26].
- an *open locating-dominating set* (*OLD-set*) if it is a total-dominating set and  $N(i) \cap C \neq N(j) \cap C$ , for distinct  $i, j \in V$  [25];
- a *locating total-dominating set* (*LTD-set*) if it is a total-dominating set and  $N(i) \cap C \neq N(j) \cap C$ , for distinct  $i, j \in V - C$  [17].

Figure 1 illustrates the four concepts.

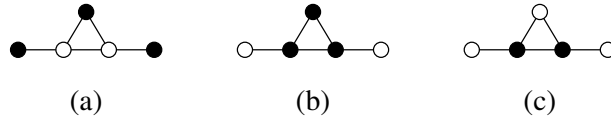


Fig. 1. A graph where the black nodes form a minimum (a) *ID-set*, (b) *OLD-set*, (c) *LD-set* and *LTD-set*.

Note that a graph  $G$  admits an *ID-set* (or is *identifiable*) only if there are no true twins in  $G$ , i.e., there is no pair of distinct nodes  $i, j \in V$  such that  $N[i] = N[j]$ , see [19]. Analogously, a graph  $G$  without isolated nodes admits an *OLD-set* if there are no false twins in  $G$ , i.e., there is no pair of distinct nodes  $i, j \in V$  such that  $N(i) = N(j)$ , see [25].

Given a graph  $G$ , for  $X \in \{ID, LD, OLD, LTD\}$ , the  $X$ -problem on  $G$  is the problem of finding an  $X$ -set of minimum size in  $G$ . The size of such a set is called the  $X$ -number of  $G$  and is denoted by  $\gamma_X(G)$ . From the definitions, the following relations hold for any graph  $G$  (admitting an  $X$ -set):

$$\gamma_{LD}(G) \leq \gamma_{LTD}(G) \leq \gamma_{OLD}(G), \quad (1)$$

and

$$\gamma_{LD}(G) \leq \gamma_{ID}(G), \quad (2)$$

whereas  $\gamma_{ID}(G)$  and  $\gamma_{OLD}(G)$  are not comparable in general as the examples in Figure 2 show.

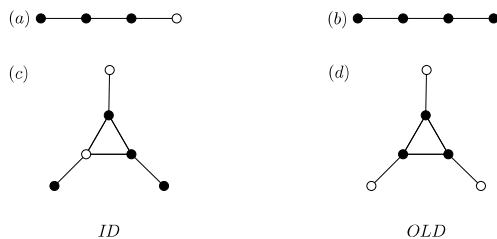


Fig. 2. (a, b)  $\gamma_{ID}(P_4) = 3 < 4 = \gamma_{OLD}(P_4)$ ; (c, d)  $\gamma_{ID}(G) = 4 > 3 = \gamma_{OLD}(G)$

It has been shown that determining  $\gamma_X(G)$  is in general NP-hard for all  $X \in \{ID, LD, LTD, OLD\}$ . Determining  $\gamma_{ID}(G)$  is in general NP-hard [11] and even remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [11], and two classes of chordal graphs, namely split graphs and interval graphs [14]. Determining  $\gamma_{LD}(G)$  is also in general NP-hard [12] and remains hard for bipartite graphs [11]. This result is extended to planar bipartite unit disk graphs in [21] and to interval and intersection graphs in [15]. Also determining  $\gamma_{OLD}(G)$  is in general NP-hard [25] and remains NP-hard for perfect elimination bipartite graphs and APX-complete for chordal graphs with maximum degree 4 [23]. Concerning the *LTD*-problem we observe that it is as hard as the *OLD*-problem by just using the same arguments as in [25].

Typical lines of attack are therefore to determine minimum  $X$ -sets of special graphs. Closed formulas for the exact value of  $\gamma_{ID}(G)$  have been found so far only for restricted graph families (e.g. for paths and cycles by [10], for stars by [16], and for complete multipartite graphs, some suns and split graphs by [2,4,5,7]). Closed formulas for the exact value of  $\gamma_{LD}(G)$  have been found so far for e.g. paths [26] and cycles [10]. Closed formulas for the exact value of  $\gamma_{OLD}(G)$  have been found so far only for cliques and paths [25], some algorithmic aspects are discussed in [23]. Bounds for the *LTD*-number of trees are given in [17,18], whereas the *LTD*-number in special families of graphs, including cubic graphs and grid graphs, is investigated in [18]. We refer the reader to the bibliography maintained by Lobstein [20] for further results.

As polyhedral methods have been already proved to be successful for several other NP-hard combinatorial optimization problems, it was suggested in [2] to apply such techniques to the *ID*-problem. For that, the following reformulation as set covering problem has been proposed.

For a 0/1-matrix  $M$  with  $n$  columns, the set covering polyhedron is  $Q^*(M) = \text{conv} \{ \mathbf{x} \in \mathbb{Z}_+^n : M\mathbf{x} \geq \mathbf{1} \}$  and  $Q(M) = \{ \mathbf{x} \in \mathbb{R}_+^n : M\mathbf{x} \geq \mathbf{1} \}$  is its linear relaxation. A *cover* of  $M$  is a 0/1-vector  $\mathbf{x}$  such that  $M\mathbf{x} \geq \mathbf{1}$ , and the *covering number*  $\tau(M)$  equals  $\min \mathbf{1}^T \mathbf{x}, \mathbf{x} \in Q^*(M)$ .

We obtain such a constraint system  $M\mathbf{x} \geq \mathbf{1}$  for the *ID*-problem as follows. Consider a graph  $G = (V, E)$ . Domination clearly requires that any *ID*-set  $C$  intersects the closed neighborhood  $N[i]$  of each node  $i \in V$ ; separation means that no two intersections  $C \cap N[i]$  and  $C \cap N[j]$  are equal. The latter condition can be reformulated that  $C$  intersects each symmetric difference  $N[i] \Delta N[j]$  for distinct nodes  $i, j \in V$ . It was shown in [2] that only symmetric differences matter if the nodes  $i, j \in V$  have distance  $dist(i, j) = 1$  (i.e., are adjacent) or distance  $dist(i, j) = 2$  (i.e., are non-adjacent but have a common neighbor).

Hence, determining a minimum *ID*-set in a graph  $G = (V, E)$  can be formulated as set covering problem  $\min \mathbf{1}^T \mathbf{x}, M_{ID}(G)\mathbf{x} \geq \mathbf{1}, \mathbf{x} \in \{0, 1\}^{|V|}$  by:

$$\begin{aligned} \min \mathbf{1}^T \mathbf{x} \\ x(N[j]) = \sum_{i \in N[j]} x_i \geq 1 \quad \forall j \in V \quad (\text{domination}) \\ x(N[j] \Delta N[k]) = \sum_{i \in N[j] \Delta N[k]} x_i \geq 1 \quad \forall j, k \in V, j \neq k \quad (\text{separation}) \\ \mathbf{x} \in \{0, 1\}^{|V|} \end{aligned}$$

By [2], the matrix  $M_{ID}(G)$  encoding row-wise the closed neighborhoods of the nodes and their symmetric differences is called the *identifying code matrix* of  $G$ , and the *identifying code polyhedron* of  $G$  is defined as

$$P_{ID}(G) = Q^*(M_{ID}) = \text{conv}\{\mathbf{x} \in \mathbb{Z}_+^{|V|} : M_{ID}(G)\mathbf{x} \geq \mathbf{1}\}.$$

It is clear by construction that a graph is identifiable if and only if none of the symmetric differences results in a zero-row of  $M_{ID}(G)$ , and that  $\gamma_{ID}(G)$  equals the covering number  $\tau(M_{ID}(G))$ .

It turned out that studying the *ID*-problem from a polyhedral point of view can lead to interesting results, see e.g. [2,4,5,7]. The aim of this paper is to apply the polyhedral approach to minimum  $X$ -sets for  $X \in \{LD, LTD, OLD\}$ .

In Section 2, we give the according definitions of the matrices  $M_X(G)$  for  $X \in \{LD, LTD, OLD\}$  and of the associated polyhedra, provide some basic properties of the polyhedra  $P_X(G)$  for  $X \in \{LD, LTD, OLD\}$  and introduce their canonical linear relaxations. Afterwards, we discuss several lines to apply polyhedral techniques. In Section 3, we present cases where  $M_{LD}(G)$ ,  $M_{OLD}(G)$  or  $M_{LTD}(G)$  are composed of matrices for which the set covering polyhedron is known and we, thus, immediately can obtain a complete description of  $P_{LD}(G)$ ,  $P_{OLD}(G)$  or  $P_{LTD}(G)$  and the exact value of  $\gamma_{LD}(G)$ ,  $\gamma_{OLD}(G)$  or  $\gamma_{LTD}(G)$ . This demonstrates how polyhedral techniques can be applied in this context. We close with a discussion on future lines of research, including how the here obtained results can be extended to other classes of graphs.

Parts of the here presented results appeared without proofs in [3,6].

## 2 Polyhedra associated to LD-, OLD- and LTD-sets

In order to apply the polyhedral approach to the three studied  $X$ -problems, we first give according reformulations as set covering problem.

**Theorem 1** *Let  $G = (V, E)$  be a graph.*

- (a)  $C \subseteq V$  is an LD-set if and only if  $C$  has a non-empty intersection with
- (LD<sub>1</sub>):  $N[i]$  for all  $i \in V$ ,
  - (LD<sub>2</sub>):  $N(i) \Delta N(j)$  for all distinct  $i, j \in V$  with  $\text{dist}(i, j) = 1$ ,
  - (LD<sub>3</sub>):  $N[i] \Delta N[j]$  for all distinct  $i, j \in V$  with  $\text{dist}(i, j) = 2$ .
- (b) Let  $G$  have neither isolated nodes nor false twins.  $C \subseteq V$  is an OLD-set if and only if  $C$  has a non-empty intersection with
- (OLD<sub>1</sub>):  $N(i)$  for all  $i \in V$ ,
  - (OLD<sub>2</sub>):  $N(i) \Delta N(j)$  for all distinct  $i, j \in V$  with  $\text{dist}(i, j) \in \{1, 2\}$ .
- (c)  $C \subseteq V$  is an LTD-set if and only if  $C$  has a non-empty intersection with
- (LTD<sub>1</sub>):  $N(i)$  for all  $i \in V$ ,
  - (LTD<sub>2</sub>):  $N(i) \Delta N(j)$  for all distinct  $i, j \in V$  with  $\text{dist}(i, j) = 1$ ,
  - (LTD<sub>3</sub>):  $N[i] \Delta N[j]$  for all distinct  $i, j \in V$  with  $\text{dist}(i, j) = 2$ .

*Proof.* Let  $G = (V, E)$  be a graph without isolated nodes. We notice that if  $\text{dist}(i, j) = 1$  for  $i, j \in V$  then  $N(i) \Delta N(j) = \{i, j\} \cup (N[i] \Delta N[j])$ . If  $\text{dist}(i, j) = 2$  for  $i, j \in V$  then  $N[i] \Delta N[j] = \{i, j\} \cup (N(i) \Delta N(j))$ .

- (a) Let  $C$  be an LD-set of  $G$ . It is immediate to see that (LD<sub>1</sub>) holds for all  $i \in V$ . Concerning the symmetric differences, we have  $N(i) \Delta N(j) = N(i) \cup N(j)$  if  $\text{dist}(i, j) \geq 3$ . So, for the remaining conditions we have to take into account nodes  $i, j \in V$  with  $\text{dist}(i, j) \leq 2$ . Now, given  $i, j \in V - C$ , from the definition, we have  $(N(i) \Delta N(j)) \cap C \neq \emptyset$ . If  $i \in C$  or  $j \in C$ , we have  $(N(i) \Delta N(j)) \cap C \neq \emptyset$  when  $\text{dist}(i, j) = 1$  and  $(N[i] \Delta N[j]) \cap C \neq \emptyset$  when  $\text{dist}(i, j) = 2$ . Then conditions (LD<sub>2</sub>) and (LD<sub>3</sub>) hold for every  $i, j \in V$ .

Conversely, let  $C \subseteq V$  such that its nodes satisfy conditions (LD<sub>1</sub>), (LD<sub>2</sub>) and (LD<sub>3</sub>). From (LD<sub>1</sub>),  $C$  is a dominating set of  $G$ . Let  $i, j \in V - C$ . If  $\text{dist}(i, j) = 1$ , from (LD<sub>2</sub>),  $(N(i) \Delta N(j)) \cap C \neq \emptyset$ . If  $\text{dist}(i, j) = 2$ , we have that  $(N[i] \Delta N[j]) \cap C = (N(i) \Delta N(j)) \cap C$ . Hence, from (LD<sub>3</sub>) we get that  $(N(i) \Delta N(j)) \cap C \neq \emptyset$ . Thus,  $C$  is an LD-set of  $G$ .

- (b) Let  $G = (V, E)$  be a graph without isolated nodes and false twins.

Let  $C$  be an OLD-set of  $G$ . As  $G$  has no isolated nodes, every node in  $V$  is total-dominated by  $C$ , and condition (OLD<sub>1</sub>) holds. Now, given  $i, j \in V$ , from the definition, we have  $(N(i) \Delta N(j)) \cap C \neq \emptyset$  for every pair  $i, j \in V$  such that  $\text{dist}(i, j) = 1$  or  $\text{dist}(i, j) = 2$ , and condition (OLD<sub>2</sub>) holds.

Conversely, let  $C \subseteq V$  satisfy (OLD<sub>1</sub>) and (OLD<sub>2</sub>). From (OLD<sub>1</sub>), it follows that  $C$  is a total-dominating set of  $G$ . Let  $i, j \in V$  such that  $\text{dist}(i, j) = 1$  or  $\text{dist}(i, j) = 2$ . As condition (OLD<sub>2</sub>) holds we have that

$(N(i) \Delta N(j)) \cap C \neq \emptyset$ . Besides, if  $dist(i, j) \geq 3$ ,  $N(i) \Delta N(j) = N(i) \cup N(j)$ . As  $G$  has no false twins, all these sets are different, and  $C$  is an *OLD*-set of  $G$ .

- (c) Let  $C$  be an *LTD*-set of  $G = (V, E)$ . As every node in  $V$  is total-dominated by  $C$ , condition (*LTD*<sub>1</sub>) holds. The same arguments as used for *LD*-sets show that conditions (*LTD*<sub>2</sub>) and (*LTD*<sub>3</sub>) hold for every  $i, j \in V$ .

Conversely, let  $C \subseteq V$  such that its nodes satisfy conditions (*LTD*<sub>1</sub>), (*LTD*<sub>2</sub>) and (*LTD*<sub>3</sub>). From (*LTD*<sub>1</sub>),  $C$  is a total-dominating set of  $G$ . The same arguments as used for *LD*-sets show that (*LTD*<sub>2</sub>) and (*LTD*<sub>3</sub>) imply that  $C$  is an *LTD*-set of  $G$ .

This shows that the above conditions properly encode the studied  $X$ -sets.  $\square$

The matrices  $M_X(G)$  encoding row-wise the closed or open neighborhoods and their respective symmetric differences read, therefore, as

$$M_{LD}(G) = \begin{pmatrix} N[G] \\ \Delta_1(G) \\ \Delta_2[G] \end{pmatrix}, \quad M_{OLD}(G) = \begin{pmatrix} N(G) \\ \Delta_1(G) \\ \Delta_2(G) \end{pmatrix}, \quad M_{LTD}(G) = \begin{pmatrix} N(G) \\ \Delta_1(G) \\ \Delta_2[G] \end{pmatrix}$$

where every row in  $N[G]$  (resp.  $N(G)$ ) is the characteristic vector of a closed (resp. an open) neighborhood of a node in  $G$  and  $\Delta_k(G)$  (resp.  $\Delta_k[G]$ ) is composed of the characteristic vectors of the symmetric difference of open (resp. closed) neighborhoods of nodes at distance  $k$  in  $G$ . We define by

$$P_X(G) = Q^*(M_X(G)) = \text{conv}\{\mathbf{x} \in \mathbb{Z}_+^{|V|} : M_X(G) \mathbf{x} \geq \mathbf{1}\}$$

the  $X$ -polyhedron for  $X \in \{LD, OLD, LTD\}$ . We first address the dimension of the three polyhedra. It is known from Balas and Ng [9] that a set covering polyhedron  $Q^*(M)$  is full-dimensional if and only if the matrix  $M$  has at least two ones per row. From the submatrix  $N[G]$  encoding the closed neighborhoods, we see that

$$V_N[G] = \{k \in V : \{k\} = N[k], k \in V\},$$

i.e., isolated nodes are the cases that result in a row with only one 1-entry. From the submatrix  $N(G)$  encoding the open neighborhoods, we see that

$$V_N(G) = \{k \in V : \{k\} = N(i), i \in V\}$$

are the cases that result in a row with only one 1-entry. From the submatrix  $\Delta_1(G)$ , every row has at least two 1-entries (namely  $i$  and  $j$  for  $N(i) \Delta N(j)$ ). From the submatrix  $\Delta_2(G)$ , we see that

$$V_2(G) = \{k \in V : \{k\} = N(i) \Delta N(j), i, j \in V\}$$

are the cases that result in a row with only one 1-entry, whereas every row from the submatrix  $\Delta_2[G]$  has at least two 1-entries (namely  $i$  and  $j$  for  $N[i] \Delta N[j]$ ). Moreover, if  $\{k\} = N(i)$  and  $dist(i, j) = 2$ , then  $k \in N(j)$ . Thus  $V_2(G) \cap V_N(G) = \emptyset$  follows. We conclude:

**Corollary 1** *Let  $G = (V, E)$  be a graph.*

- (a) *We have  $dim(P_{LD}(G)) = |V - V_N[G]|$ .*
- (b) *Let  $G$  have neither isolated nodes nor false twins. We have  $dim(P_{OLD}(G)) = |V - V_N(G) - V_2(G)|$ .*
- (c) *We have  $dim(P_{LTD}(G)) = |V - V_N(G)|$ .*

In addition, the matrices  $M_X(G)$  may contain redundant rows, where we say that  $\mathbf{y}$  is *redundant* if  $\mathbf{x}$  and  $\mathbf{y}$  are two rows of  $M$  and  $\mathbf{x} \leq \mathbf{y}$ . As the covering number of a matrix does not change after removing redundant rows, we define the corresponding clutter matrices  $C_{LD}(G)$ ,  $C_{OLD}(G)$  and  $C_{LTD}(G)$ , obtained by removing redundant rows from  $M_{LD}(G)$ ,  $M_{OLD}(G)$  and  $M_{LTD}(G)$ , respectively. We clearly have

$$P_X(G) = Q^*(C_X(G)) = \text{conv}\{\mathbf{x} \in \mathbb{Z}_+^{|V|} : C_X(G) \mathbf{x} \geq \mathbf{1}\}$$

for  $X \in \{LD, OLD, LTD\}$ . Moreover, also in [9] it is proved that the only facet-defining inequalities of a set covering polyhedron  $Q^*(M)$  with integer coefficients and right hand side equal to 1 are those of the system  $M\mathbf{x} \geq \mathbf{1}$ . Hence we have:

**Corollary 2** *All constraints from  $C_X(G) \mathbf{x} \geq \mathbf{1}$  define facets of  $P_X(G)$  for  $X \in \{LD, OLD, LTD\}$ .*

We obtain the corresponding linear relaxations, the *fractional  $X$ -polyhedron*  $Q_X(G)$ , by considering all vectors satisfying the above inequalities:

$$Q_X(G) = Q(C_X(G)) = \{\mathbf{x} \in \mathbb{R}_+^{|V|} : C_X(G) \mathbf{x} \geq \mathbf{1}\}$$

for  $X \in \{LD, OLD, LTD\}$ . To study the three problems from a polyhedral point of view, we propose to firstly determine the clutter matrices  $C_X(G)$  and then to determine which further constraints have to be added to the linear relaxation  $Q_X(G)$  in order to obtain the integral polyhedron  $P_X(G)$ .

### 3 Complete multi-partite graphs

In this section, we consider complete  $p$ -partite graphs and establish a connection to so-called complete 2-roses of order  $n$ . Given  $n > q \geq 2$ , let  $\mathcal{R}_n^q = (V, \mathcal{E})$  be the hypergraph where  $V = \{1, \dots, n\}$  and  $\mathcal{E}$  contains all  $q$ -element subsets



of  $V$ . Nobili and Sassano [22] called the incidence matrix of  $\mathcal{R}_n^q$  the *complete  $q$ -rose of order  $n$*  and we denote it by  $M(\mathcal{R}_n^q)$ . In [8], it was shown (see [2] for the proof):

**Theorem 2 ([2,8])** *The covering polyhedron  $Q^*(M(\mathcal{R}_n^q))$  is given by the non-negativity constraints and*

$$x(V') \geq |V'| - q + 1$$

for all subsets  $V' \subseteq \{1, \dots, n\}$  with  $|V'| \in \{q + 1, \dots, n\}$ .

### 3.1 Complete bipartite graphs

We start to consider complete bipartite graphs  $K_{m,n}$  with bipartition  $A = \{1, \dots, m\}$  and  $B = \{m + 1, \dots, m + n\}$ . We note that  $K_{m,n}$  has false twins (unless  $m = 1 = n$ ) and, thus, no *OLD*-set, hence we only analyse *LD*- and *LTD*-sets. We begin with the case of stars  $K_{1,n}$ , i.e.,  $A = \{1\}$  and  $n \geq 2$ . Note that  $K_{1,2} = P_3$  and it is easy to see that  $\gamma_{LD}(K_{1,2}) = \gamma_{LTD}(K_{1,2}) = 2$  holds.

**Lemma 1** *For a star  $K_{1,n}$  with  $n \geq 3$ , we have*

$$C_{LD}(K_{1,n}) = M(\mathcal{R}_{n+1}^2) \text{ and } C_{LTD}(K_{1,n}) = \left( \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & M(\mathcal{R}_n^2) & \\ 0 & & & \end{array} \right).$$

*Proof.* Consider a star  $K_{1,n}$  with bipartition  $A = \{1\}$  and  $B = \{2, \dots, n + 1\}$  with  $n \geq 3$ .

From Theorem 1, the rows of the matrix  $M_{LD}(K_{1,n})$  are the characteristic vectors of the sets

$$\begin{aligned} N[1] &= V \text{ and } N[i] = \{1, i\} \text{ for all } i \in B, \\ N(1) \triangle N(i) &= B \triangle \{1\} = V \text{ for all } i \in B, \\ N[i] \triangle N[j] &= \{1, i\} \triangle \{1, j\} = \{i, j\} \text{ for } i, j \in B. \end{aligned}$$

Thus,  $C_{LD}(G)$  is clearly composed by rows corresponding to  $N[i] = \{1, i\}$  for  $i \in B$ , and  $\Delta[i, j] = \{i, j\}$  for  $i, j \in B$ , which implies  $C_{LD}(K_{1,n}) = M(\mathcal{R}_{n+1}^2)$ .

From Theorem 1, the rows of the matrix  $M_{LTD}(K_{1,n})$  are the characteristic vectors of the sets

$N(1) = B$  and  $N(i) = \{1\}$  for all  $i \in B$ ,  
 $N(1) \triangle N(i) = N[1]$  for all  $i \in B$ ,  
 $N[j] \triangle N[k] = \{j, k\}$ ,  $j \neq k$ ,  $j, k \in B$ .

Deleting the redundant rows, we obtain

$$C_{LTD}(K_{1,n}) = \left( \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & M(\mathcal{R}_n^2) & \\ 0 & & & \end{array} \right).$$

□

From the above description of the facets of the covering polyhedron associated with complete  $q$ -roses by [2], we conclude:

**Corollary 3** Consider  $K_{1,n}$  with bipartition  $A = \{1\}$ ,  $B = \{2, \dots, n+1\}$ ,  $n \geq 3$ .

- (a)  $P_{LD}(K_{1,n})$  is described by nonnegativity constraints and the inequalities  $x(V') \geq |V'| - 1$  for all nonempty subsets  $V' \subseteq \{1, \dots, n+1\}$ .
- (b)  $P_{LTD}(K_{1,n})$  is described by nonnegativity constraints,  $x_1 \geq 1$  and the inequalities  $x(B') \geq |B'| - 1$  for all nonempty subsets  $B' \subseteq \{2, \dots, n+1\}$ .

For the  $LD$ -number of  $K_{1,n}$ , the full rank constraint  $x(V) \geq |V| - 1$  of  $P_{LD}(K_{1,n})$  immediately implies that  $\gamma_{LD}(K_{1,n}) = n$ . Furthermore, combining  $x_1 \geq 1$  and  $x(B) \geq |B| - 1$  yields the full rank constraint  $x(V) \geq |B|$  for  $P_{LTD}(K_{1,n})$  which implies  $\gamma_{LTD}(K_{1,n}) = |V| - 1 = n$  (and provides an alternative proof for the result given in [18]).

Observe that for  $K_{2,2}$ , it is easy to see that  $\gamma_{LD}(K_{2,2}) = \gamma_{LTD}(K_{2,2}) = 2$ . For general complete bipartite graphs  $K_{m,n}$  with  $m \geq 2, n \geq 3$ , we obtain:

**Lemma 2** For a complete bipartite graph  $K_{m,n}$  with  $m \geq 2, n \geq 3$ , we have

$$C_X(K_{m,n}) = \begin{pmatrix} M(\mathcal{R}_m^2) & 0 \\ 0 & M(\mathcal{R}_n^2) \end{pmatrix}$$

for  $X \in \{LD, LTD\}$ .

*Proof.* Let  $K_{m,n}$  be a complete bipartite graph with bipartition  $A = \{1, \dots, m\}$ ,  $B = \{m+1, \dots, m+n\}$  and  $m \geq 2, n \geq 3$ .

From Theorem 1, the rows of the matrix  $M_{LD}(K_{m,n})$  are the characteristic vectors of the sets

$$\begin{aligned}
N[i] &= \{i\} \cup B \text{ for all } i \in A \text{ and } N[j] = \{j\} \cup A \text{ for all } j \in B, \\
N(i) \triangle N(j) &= B \cup A \text{ for all } i \in A, j \in B, \\
N[i] \triangle N[j] &= \{i, j\} \text{ for } i, j \in A \text{ or } i, j \in B,
\end{aligned}$$

whereas the rows of the matrix  $M_{LTD}(K_{1,n})$  are the characteristic vectors of the sets

$$\begin{aligned}
N(i) &= B \text{ for all } i \in A \text{ and } N(j) = A \text{ for all } j \in B, \\
N(i) \triangle N(j) &= B \cup A \text{ for all } i \in A, j \in B, \\
N[i] \triangle N[j] &= \{i, j\} \text{ for } i, j \in A \text{ or } i, j \in B.
\end{aligned}$$

Deleting the redundant rows, only the sets from  $\Delta_2[G]$  remain and we obtain in both cases

$$C_X(K_{m,n}) = \begin{pmatrix} M(\mathcal{R}_m^2) & 0 \\ 0 & M(\mathcal{R}_n^2) \end{pmatrix}.$$

□

Note that results from [2] show that

$$C_{ID}(K_{m,n}) = C_{LD}(K_{m,n}) = C_{LTD}(K_{m,n}).$$

Hence, we directly conclude from the facet description of  $P_{ID}(K_{m,n})$  by [2]:

**Corollary 4** For  $X \in \{LD, LTD\}$ ,  $P_X(K_{m,n})$  is given by the inequalities

- (1)  $x(C) \geq |C| - 1$  for all nonempty  $C \subseteq A$ ,
- (2)  $x(C) \geq |C| - 1$  for all nonempty  $C \subseteq B$ .

Moreover,  $\gamma_X(K_{m,n}) = |V| - 2 = m + n - 2$ .

The closed formula for  $\gamma_{LTD}(K_{m,n})$  provides an alternative proof for the result previously given in [18].

### 3.2 Complete $p$ -partite graphs

The above results can be further generalized for complete  $p$ -partite graphs. Consider  $K_{n_1, \dots, n_p} = (U_1 \cup \dots \cup U_p, E)$  where each  $U_i$  induces a nonempty stable set and all edges between  $U_i$  and  $U_j$ ,  $i \neq j$  are present. We use  $|U_i| = n_i$  for  $i = 1, \dots, p$ ,  $|V| = n$  and assume  $n_1 \leq n_2 \leq \dots \leq n_p$  as well as  $p \geq 3$ . For illustration, complete 3-partite and 4-partite graphs are depicted in Figure 3.

We note that  $K_{n_1, \dots, n_p}$  has false twins and, thus, no *OLD*-set, unless  $n_1 = \dots = n_p = 1$  and the graph is a clique.

**Lemma 3** Let  $K_{n_1, n_2, \dots, n_p}$  be a complete  $p$ -partite graph.

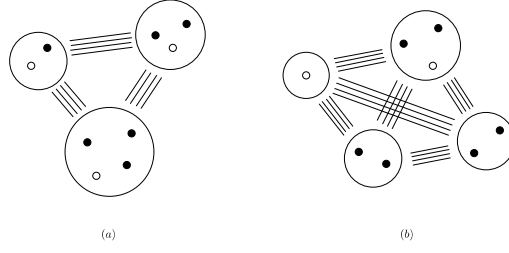


Fig. 3. (a) A complete 3-partite graph with  $n_1 = 2$ ,  $n_2 = 3$  and  $n_3 = 4$ , (b) A complete 4-partite graph with  $n_1 = 1$ ,  $n_2 = n_3 = 2$  and  $n_4 = 3$ .

(a) If  $n_1 = \dots = n_p = 1$ , then  $K_{n_1, n_2, \dots, n_p}$  equals the clique  $K_p$  and

$$C_X(K_{n_1, n_2, \dots, n_p}) = M(\mathcal{R}_p^2)$$

for  $X \in \{LD, OLD, LTD\}$ .

(b) If  $n_1 = \dots = n_r = 1$  with  $r \geq 2$  and  $n_{r+1} \geq 2$ , then

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} M(\mathcal{R}_r^2) & 0 & 0 & \dots & 0 \\ 0 & M(\mathcal{R}_{n_{r+1}}^2) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \dots & M(\mathcal{R}_{n_p}^2) \end{pmatrix}$$

for  $X \in \{LD, LTD\}$ .

(c) If  $n_1 = 1$  and  $n_2 \geq 2$ , then

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} 0 & M(\mathcal{R}_{n_2}^2) & 0 & 0 & \dots & 0 \\ 0 & 0 & M(\mathcal{R}_{n_3}^2) & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & & & M(\mathcal{R}_{n_p}^2) \end{pmatrix}$$

for  $X \in \{LD, LTD\}$ .

(d) If  $n_1 \geq 2$ , then

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} M(\mathcal{R}_{n_1}^2) & 0 & 0 & \dots & 0 \\ 0 & M(\mathcal{R}_{n_2}^2) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \dots & M(\mathcal{R}_{n_p}^2) \end{pmatrix}$$

for  $X \in \{LD, LTD\}$ .

*Proof.* (of Lemma 3) Let  $G = K_{n_1, n_2, \dots, n_p}$  be a complete  $p$ -partite graph, and assume  $n_1 \leq n_2 \leq \dots \leq n_p$ .

(a) If  $n_1 = \dots = n_p = 1$ , let  $\{v_i\} = U_i$  for  $i \in \{1, \dots, p\}$ . From Theorem 1, the rows of the matrix  $M_{LD}(G)$  are the characteristic vectors of the sets

$$N[v_i] = V, \text{ for all } v_i \in V,$$

$$N(v_i) \Delta N(v_j) = (V - \{v_i\}) \Delta (V - \{v_j\}) = \{v_i, v_j\}, \text{ for } v_i, v_j \in V,$$

whereas the rows of  $M_{OLD}(G)$  and  $M_{LTD}(G)$  correspond to the sets

$$N(v_i) = V - U_i, \text{ for all } v_i \in V,$$

$$N(v_i) \Delta N(v_j) = \{v_i, v_j\}, \text{ for all } v_i, v_j \in V.$$

Deleting redundant rows, only the symmetric differences remain and we obtain in all three cases  $C_X(G) = M(\mathcal{R}_p^2)$ .

(b) If  $n_1 = n_2 = \dots = n_r = 1$  with  $r \in \{2, \dots, p-1\}$ , let  $\{v_i\} = U_i$  with  $i \in \{1, \dots, r\}$  and  $\bigcup_{j=1}^{n_i} \{v_{i_j}\} = U_i$  with  $i \in \{r+1, \dots, p\}$ . Theorem 1 implies

that the rows of the matrix  $M_{LD}(G)$  are the characteristic vectors of the sets

$$N[v_i] = V, \text{ for } v_i \in U_i \text{ with } i \in \{1, \dots, r\},$$

$$N[v_{i_j}] = (V - U_i) \cup \{v_{i_j}\}, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{r+1, \dots, p\} \text{ and } j \in \{1, \dots, n_i\},$$

$$N(v_i) \Delta N(v_j) = \{v_i, v_j\}, \text{ for } v_i \in U_i, v_j \in U_j \text{ with } i, j \in \{1, \dots, r\},$$

$$N(v_i) \Delta N(v_{j_k}) = \{v_i\} \cup U_j, \text{ for } v_i \in U_i \text{ with } i \in \{1, \dots, r\} \text{ and } v_{j_k} \in U_j \text{ with } j \in \{r+1, \dots, p\} \text{ and } k \in \{1, \dots, n_j\},$$

$$N(v_{i_k}) \Delta N(v_{j_l}) = U_i \cup U_j, \text{ for } v_{i_k} \in U_i, v_{j_l} \in U_j \text{ with } i, j \in \{r+1, \dots, p\}, k \in \{1, \dots, n_i\} \text{ and } l \in \{1, \dots, n_j\},$$

$$N[v_{i_j}] \Delta N[v_{i_k}] = \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{r+1, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\},$$

whereas the rows of  $M_{LTD}(G)$  are the characteristic vectors of the sets

$$N(v_i) = V - U_i, \text{ for } v_i \in U_i \text{ with } i \in \{1, \dots, r\},$$

$$N(v_i) \Delta N(v_j) = \{v_i, v_j\}, \text{ for } v_i \in U_i, v_j \in U_j \text{ with } i, j \in \{1, \dots, r\},$$

$$N(v_i) \Delta N(v_{j_k}) = \{v_i\} \cup U_j, \text{ for } v_i \in U_i \text{ with } i \in \{1, \dots, r\} \text{ and } v_{j_k} \in U_j \text{ with } j \in \{r+1, \dots, p\} \text{ and } k \in \{1, \dots, n_j\},$$

$$N(v_{i_k}) \Delta N(v_{j_l}) = U_i \cup U_j, \text{ for } v_{i_k} \in U_i, v_{j_l} \in U_j \text{ with } i, j \in \{r+1, \dots, p\}, k \in \{1, \dots, n_i\} \text{ and } l \in \{1, \dots, n_j\},$$

$$N[v_{i_j}] \Delta N[v_{i_k}] = \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{r+1, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\}.$$

After deleting the redundant rows, we have in both cases

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} M(\mathcal{R}_r^2) & 0 & 0 & \dots & 0 \\ 0 & M(\mathcal{R}_{n_{r+1}}^2) & 0 & \dots & 0 \\ & \ddots & \ddots & & \vdots \\ 0 & \dots & & & M(\mathcal{R}_{n_p}^2) \end{pmatrix}.$$

- (c) If  $n_1 = 1$  and  $n_2 \geq 2$ , let  $\{v_1\} = U_1$  and  $\bigcup_{j=1}^{n_i} \{v_{i_j}\} = U_i$  with  $i \in \{2, \dots, p\}$ .

Again from Theorem 1, the rows of the matrix  $M_{LD}(G)$  are the characteristic vectors of the sets

$$\begin{aligned} N[v_1] &= V, \\ N[v_{i_j}] &= (V - U_i) \cup \{v_{i_j}\}, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } j \in \{1, \dots, n_i\}, \\ N(v_1) \Delta N(v_{i_j}) &= (V - \{v_1\}) \Delta (V - U_i) = U_i \cup \{v_1\}, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } j \in \{1, \dots, n_i\}, \\ N(v_{i_k}) \Delta N(v_{j_l}) &= (V - U_i) \Delta (V - U_j) = U_i \cup U_j, \text{ for } v_{i_k} \in U_i, v_{j_l} \in U_j \text{ with } i, j \in \{2, \dots, p\}, k \in \{1, \dots, n_i\} \text{ and } l \in \{1, \dots, n_j\}, \\ N[v_{i_j}] \Delta N[v_{i_k}] &= ((V - U_i) \cup \{v_{i_j}\}) \Delta ((V - U_i) \cup \{v_{i_k}\}) = \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\}, \end{aligned}$$

whereas the rows of  $M_{LTD}(G)$  are the characteristic vectors of the sets

$$\begin{aligned} N(v_1) &= V - U_1, \\ N(v_{i_j}) &= V - U_i, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } j \in \{1, \dots, n_i\}, \\ N(v_1) \Delta N(v_{i_k}) &= U_i \cup \{v_1\} \text{ for } v_{i_k} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } k \in \{1, \dots, n_i\}, \\ N(v_{i_k}) \Delta N(v_{j_l}) &= U_i \cup U_j, \text{ for } v_{i_k} \in U_i, v_{j_l} \in U_j \text{ with } i, j \in \{2, \dots, p\}, k \in \{1, \dots, n_i\} \text{ and } l \in \{1, \dots, n_j\}, \\ N[v_{i_j}] \Delta N[v_{i_k}] &= \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{2, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\}. \end{aligned}$$

After deleting the redundant rows, we have in both cases

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} 0 & M(\mathcal{R}_{n_2}^2) & 0 & 0 & \dots & 0 \\ 0 & 0 & M(\mathcal{R}_{n_3}^2) & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & & & M(\mathcal{R}_{n_p}^2) \end{pmatrix}.$$

- (d) Finally, if  $n_1 \geq 2$ , let  $\bigcup_{j=1}^{n_i} \{v_{i_j}\} = U_i$  with  $i \in \{1, \dots, p\}$ . Again from

Theorem 1, the rows of the matrix  $M_{LD}(G)$  are the characteristic vectors of the sets

$$\begin{aligned} N[v_{i_j}] &= (V - U_i) \cup \{v_{i_j}\}, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{1, \dots, p\} \text{ and } j \in \{1, \dots, n_i\}, \\ N(v_{i_k}) \Delta N(v_{j_l}) &= (V - U_i) \Delta (V - U_j) = U_i \cup U_j, \text{ for } v_{i_k} \in U_i, v_{j_l} \in U_j \text{ with } i, j \in \{1, \dots, p\}, k \in \{1, \dots, n_i\} \text{ and } l \in \{1, \dots, n_j\}, \\ N[v_{i_j}] \Delta N[v_{i_k}] &= ((V - U_i) \cup \{v_{i_j}\}) \Delta ((V - U_i) \cup \{v_{i_k}\}) = \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{1, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\}, \end{aligned}$$

whereas the rows of  $M_{LTD}(G)$  are the characteristic vectors of the sets

$$\begin{aligned} N(v_{i_j}) &= V - U_i, \text{ for } v_{i_j} \in U_i \text{ with } i \in \{1, \dots, p\} \text{ and } j \in \{1, \dots, n_i\}, \\ N(v_{i_j}) \Delta N(v_{k_h}) &= U_i \cup U_k, \text{ for } v_{i_j} \in U_i, v_{k_h} \in U_k \text{ with } i, k \in \{1, \dots, p\}, j \in \{1, \dots, n_i\} \text{ and } h \in \{1, \dots, n_k\}, \\ N[v_{i_j}] \Delta N[v_{i_k}] &= \{v_{i_j}, v_{i_k}\}, \text{ for } v_{i_j}, v_{i_k} \in U_i \text{ with } i \in \{1, \dots, p\} \text{ and } j, k \in \{1, \dots, n_i\}. \end{aligned}$$

In this case, only the last sets from  $\Delta_2[G]$  remain and we have in both cases

$$C_X(K_{n_1, n_2, \dots, n_p}) = \begin{pmatrix} M(\mathcal{R}_{n_1}^2) & 0 & 0 & \dots & 0 \\ 0 & M(\mathcal{R}_{n_2}^2) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & M(\mathcal{R}_{n_p}^2) \end{pmatrix}.$$

□

From the description of the facets of the covering polyhedron associated with complete  $q$ -roses by [2] and taking the block structure of the matrices into account, we conclude:

**Corollary 5** *Let  $K_{n_1, n_2, \dots, n_p}$  be a complete  $p$ -partite graph.*

- (a) *If  $n_1 = \dots = n_p = 1$ , then  $P_X(K_{n_1, n_2, \dots, n_p})$  is given by the inequalities*
- $x(V') \geq |V'| - 1$  *for all nonempty subsets  $V' \subseteq V$*
  - and  $\gamma_X(K_{n_1, n_2, \dots, n_p}) = n - 1$  *for  $X \in \{LD, OLD, LTD\}$ .*
- (b) *If  $n_1 = \dots = n_r = 1$  with  $r \geq 2$  and  $n_{r+1} \geq 2$ , then  $P_X(K_{n_1, n_2, \dots, n_p})$  is given by the inequalities*
- $x(V') \geq |V'| - 1$  *for all nonempty subsets  $V' \subseteq U_1 \cup \dots \cup U_r$ ,*
  - $x(V') \geq |V'| - 1$  *for all nonempty subsets  $V' \subseteq U_i$  for  $i \in \{r+1, \dots, p\}$*
  - and  $\gamma_X(K_{n_1, n_2, \dots, n_p}) = n - p + r - 1$  *for  $X \in \{LD, LTD\}$ .*
- (c) *If  $n_1 = 1$  and  $n_2 \geq 2$ , then  $P_X(K_{n_1, n_2, \dots, n_p})$  is given by the inequalities*
- $x(U_1) \geq 0$ ,
  - $x(V') \geq |V'| - 1$  *for all nonempty subsets  $V' \subseteq U_i$  for  $i \in \{2, \dots, p\}$ ,*
  - and  $\gamma_X(K_{n_1, n_2, \dots, n_p}) = n - p$  *for  $X \in \{LD, LTD\}$ .*
- (d) *If  $n_1 \geq 2$ , then  $P_X(K_{n_1, n_2, \dots, n_p})$  is given by the inequalities*
- $x(V') \geq |V'| - 1$  *for all nonempty subsets  $V' \subseteq U_i$  for  $i \in \{1, \dots, p\}$*
  - and  $\gamma_X(K_{n_1, n_2, \dots, n_p}) = n - p$  *for  $X \in \{LD, LTD\}$ .*

Corollary 5(a) provides an alternative proof for the result on *OLD*-sets in cliques given in [25].

#### 4 Some families of split graphs

A graph  $G = (C \cup S, E)$  is a *split graph* if its node set can be partitioned into a clique  $C$  and a stable set  $S$ . Split graphs are closed under taking complements and form the complementary core of chordal graphs since  $G$  is a split graph if and only if  $G$  and  $\overline{G}$  are chordal or if and only if  $G$  is  $(C_4, \overline{C}_4, C_5)$ -free [13]. Our aim is to study  $X$ -sets in some families of split graphs having a regular structure from a polyhedral point of view.

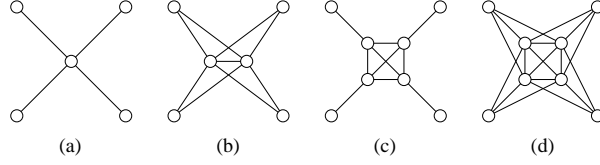


Fig. 4. (a) star, (b) crown, (c) thin headless spider, (d) thick headless spider.

#### 4.1 Complete split graphs

A *complete split graph* is a split graph where all edges between  $C$  and  $S$  are present. Complete split graphs can be seen as special case of complete multipartite graphs studied in Section 3. In fact, a complete split graph is a clique if  $|S| = 1$ , a star if  $|C| = 1$ , and a crown if  $|C| = 2$ , see Fig. 4(a),(b). Otherwise, the graph can be seen as a complete multi-partite graph where all parts but one have size 1, i.e. as  $K_{n_1, n_2, \dots, n_p}$  with  $n_1 = \dots = n_{p-1} = 1$  and  $n_p \geq 2$  such that  $U_1 \cup \dots \cup U_{p-1}$  induce the clique  $C$  and  $U_p$  the stable set  $S$ . Hence, we directly conclude from Lemma 3 and Corollary 5:

**Corollary 6** *Let  $G = (C \cup S, E)$  be a complete split graph.*

(a) *If  $|S| = 1$ , then  $G$  is a clique,*

$$C_X(G) = M(\mathcal{R}_{|C|+1}^2)$$

*and  $\gamma_X(G) = |C|$  for  $X \in \{LD, OLD, LTD\}$ .*

(b) *If  $|C| = 1$ , then  $G$  is a star, we have  $C_{LD}(G) = M(\mathcal{R}_{|S|+1}^2)$ ,*

$$C_{LTD}(G) = \left( \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & M(\mathcal{R}_{|S|}^2) & \\ 0 & & & \end{array} \right).$$

*and  $\gamma_X(G) = |S|$  for  $X \in \{LD, LTD\}$ .*

(c) *Otherwise, we have*

$$C_X(G) = \begin{pmatrix} M(\mathcal{R}_{|C|}^2) & 0 \\ 0 & M(\mathcal{R}_{|S|}^2) \end{pmatrix}$$

*and  $\gamma_X(G) = |S| + |C| - 2$  for  $X \in \{LD, LTD\}$ .*



## 4.2 Headless spiders

A *headless spider* is a split graph with  $C = \{c_1, \dots, c_k\}$  and  $S = \{s_1, \dots, s_k\}$ ; it is *thin* (resp. *thick*) if  $s_i$  is adjacent to  $c_j$  if and only if  $i = j$  (resp.  $i \neq j$ ), see Figure 4(c),(d) for illustration. Clearly, the complement of a thin spider is a thick spider, and vice-versa. It is easy to see that for  $k = 2$ , the path  $P_4$  equals the thin and thick headless spider. Moreover, it is easy to check that headless spiders are twin-free, so *OLD*-sets exist.

We start our analysis with thick headless spiders. A thick headless spider with  $k = 3$  equals the 3-sun  $S_3$  and it is easy to see that  $\gamma_{LD}(S_3) = 3$ ,  $\gamma_{OLD}(S_3) = 4$  and  $\gamma_{LTD}(S_3) = 3$  holds. To describe the clutters for  $k \geq 4$ , we use the following notations. Let  $J_n$  denote the  $n \times n$  matrix having 1-entries only and  $I_n$  the  $n \times n$  identity matrix. Furthermore, let  $J_{n-1,n}(i)$  denote a matrix s.t. its  $i$ -th column has 0-entries only and removing the  $i$ -th column results in  $J_{n-1}$ , and  $I_{n-1,n}(j)$  denote a matrix s.t. its  $j$ -th column has 1-entries only and removing the  $j$ -th column results in  $I_{n-1}$ .

**Lemma 4** *For a thick headless spider  $G = (C \cup S, E)$  with  $k \geq 4$ , we have*

$$C_{OLD}(G) = \begin{pmatrix} M(\mathcal{R}_{|S|}^{|S|-1}) & 0 \\ 0 & M(\mathcal{R}_{|C|}^2) \end{pmatrix},$$

whereas

$$C_{LD}(G) = \begin{pmatrix} I_{|S|} & M(\mathcal{R}_{|C|}^{|C|-1}) \\ J_{k-1,k}(k) & I_{k-1,k}(k) \\ \vdots & \vdots \\ J_{k-1,k}(1) & I_{k-1,k}(1) \\ M(\mathcal{R}_{|S|}^2) & M(\mathcal{R}_{|C|}^2) \\ J_{|S|} & I_{|C|} \end{pmatrix} \quad \text{and} \quad C_{LTD}(G) = \begin{pmatrix} 0 & M(\mathcal{R}_{|C|}^{|C|-1}) \\ J_{k-1,k}(k) & I_{k-1,k}(k) \\ \vdots & \vdots \\ J_{k-1,k}(1) & I_{k-1,k}(1) \\ M(\mathcal{R}_{|S|}^2) & M(\mathcal{R}_{|C|}^2) \\ J_{|S|} & I_{|C|} \end{pmatrix}.$$

The lines in the rows of the matrix  $M(\mathcal{R}_{|C|}^\epsilon)$  are ordered according to the rows of  $M(\mathcal{R}_{|S|}^\epsilon)$  (that is for the same pairs  $(i, j) \in \{1, \dots, k\}^2$ ).

*Proof.* (of Lemma 4) Let  $G = (C \cup S, E)$  be a thick headless spider with  $k \geq 4$ . From Theorem 1, the rows of the matrices  $M_{LD}(G)$ ,  $M_{OLD}(G)$  and  $M_{LTD}(G)$  are composed of characteristic vectors of the following sets: the closed neighborhoods (for  $M_{LD}(G)$ )

$$\begin{aligned} N[c_i] &= C \cup S - \{s_i\} \text{ for all } i \in \{1, \dots, k\}, \\ N[s_i] &= (C - \{c_i\}) \cup \{s_i\} \text{ for all } i \in \{1, \dots, k\}, \end{aligned}$$

the open neighborhoods (for  $M_{OLD}(G)$  and  $M_{LTD}(G)$ )

$$\begin{aligned} N(s_i) &= C - \{c_i\} \text{ for all } i \in \{1, \dots, k\}, \\ N(c_i) &= (S - \{s_i\}) \cup (C - \{c_i\}) \text{ for all } i \in \{1, \dots, k\}, \end{aligned}$$

the symmetric differences of open neighborhoods of nodes at distance 1 (in all three cases)

$$\begin{aligned} N(s_i) \triangle N(c_j) &= (S - \{s_j\}) \cup \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \\ N(c_i) \triangle N(c_j) &= \{s_i, s_j\} \cup \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \end{aligned}$$

as well as the symmetric differences of open neighborhoods of nodes at distance 2 (for  $M_{OLD}(G)$ )

$$\begin{aligned} N(s_i) \triangle N(s_j) &= \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \\ N(s_i) \triangle N(c_i) &= S - \{s_i\} \text{ for all } i \in \{1, \dots, k\} \end{aligned}$$

and the symmetric differences of closed neighborhoods of nodes at distance 2 (for  $M_{LD}(G)$  and  $M_{LTD}(G)$ )

$$\begin{aligned} N[s_i] \triangle N[s_j] &= \{s_i, s_j\} \cup \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \\ N[s_i] \triangle N[c_i] &= S \cup \{c_i\} \text{ for all } i \in \{1, \dots, k\}. \end{aligned}$$

Deleting the redundant rows from  $M_{OLD}(G)$ , only the submatrix  $\Delta_2(G)$  remains and we have

$$C_{OLD}(G) = \begin{pmatrix} M(\mathcal{R}_{|S|}^{|S|-1}) & 0 \\ 0 & M(\mathcal{R}_{|C|}^2) \end{pmatrix}.$$

From the neighborhoods, only  $N[s_i]$  remain for  $C_{LD}(G)$  (resp.  $N(s_i)$  for  $C_{LTD}(G)$ ). From the symmetric differences,  $N[s_i] \triangle N[s_j]$  equals  $N(c_i) \triangle N(c_j)$  for all  $i, j \in \{1, \dots, k\}$ , which results in

$$C_{LD}(G) = \begin{pmatrix} I_{|S|} & M(\mathcal{R}_{|C|}^{|C|-1}) \\ J_{k-1,k}(k) & I_{k-1,k}(k) \\ \vdots & \vdots \\ J_{k-1,k}(1) & I_{k-1,k}(1) \\ M(\mathcal{R}_{|S|}^2) & M(\mathcal{R}_{|C|}^2) \\ J_{|S|} & I_{|C|} \end{pmatrix} \text{ and } C_{LTD}(G) = \begin{pmatrix} 0 & M(\mathcal{R}_{|C|}^{|C|-1}) \\ J_{k-1,k}(k) & I_{k-1,k}(k) \\ \vdots & \vdots \\ J_{k-1,k}(1) & I_{k-1,k}(1) \\ M(\mathcal{R}_{|S|}^2) & M(\mathcal{R}_{|C|}^2) \\ J_{|S|} & I_{|C|} \end{pmatrix}.$$

Note that  $N(s_i) \triangle N(c_j) = (S - \{s_j\}) \cup \{c_i, c_j\}$  does not contain  $\{s_i, s_j\} \cup \{c_i, c_j\}$  (as  $s_j \notin ((S - \{s_j\}) \cup \{c_i, c_j\})$ ).  $\square$

From the description of the polyhedron associated with complete  $q$ -roses by [2] and taking the block structure of  $C_{OLD}(G)$  into account, we conclude:

**Corollary 7** *For a thick headless spider  $G = (C \cup S, E)$  with  $k \geq 4$ ,  $P_{OLD}(G)$  is given by the inequalities*

- $x_i \geq 0$  for all  $i \in C \cup S$ ,
- $x(S') \geq |S'| - k + 2$  for all  $S' \subseteq S$  with  $|S'| \geq k - 1$ ,
- $x(C') \geq |C'| - 1$  for all  $C' \subseteq C$  with  $|C'| \geq 2$ ,

and  $\gamma_{OLD}(G) = |C| + 1$ .

On the other hand, from the clutter matrices  $C_{LD}(G)$  and  $C_{LTD}(G)$ , we immediately see that  $C$  is an  $X$ -set for  $X \in \{LD, LTD\}$ . However,  $C$  is a minimum  $X$ -set only if  $k = 4$ . For thick headless spiders with  $k \geq 5$ , we can show, using polyhedral arguments, that  $k - 1$  is a lower bound for the cardinality of any  $X$ -set.

For that, we use the following result. For any polyhedron  $P \subset \mathbb{R}^n$ , let  $P_I := \text{conv}(P \cap \mathbb{Z}^n)$  denote the convex hull of all integer points in  $P$ . Given  $a \in \mathbb{Z}^n$  and  $b \notin \mathbb{Z}$ , if the inequality  $a^T x \geq b$  is valid for  $P$  and tight for some  $x^* \in P$ , then the inequality  $a^T x \geq \lceil b \rceil$ , called a *Chvátal-Gomory cut*, is valid for  $P_I$ , but violated by  $x^*$ .

Applying the Chvátal-Gomory procedure to common facets from  $Q_{LD}(G)$  and  $Q_{LTD}(G)$ , we obtain a valid inequality implying the studied lower bound. Exhibiting an  $X$ -set of size  $k - 1$  thus ensures minimality:

**Theorem 3** *For a thick headless spider  $G = (C \cup S, E)$  with  $k \geq 5$ , we have  $\gamma_X(G) = k - 1$  for  $X \in \{LD, LTD\}$ .*

*Proof.* Let  $G = (C \cup S, E)$  be a thick headless spider with  $k \geq 5$  and  $X \in \{LD, LTD\}$ .

**Claim 1** *The constraints*

$$x(S_I) + x(C_I) \geq |I| - 1$$

*with  $S_I = \{s_i \in S : i \in I\}$  and  $C_I = \{c_i \in C : i \in I\}$  are valid for  $P_X(G)$  for all  $I \subseteq \{1, \dots, k\}$  with  $|I| \geq 2$ .*

For  $|I| = 2$ , the constraints correspond to the facets of  $Q_X(G)$  from the submatrix

$$\begin{pmatrix} M(\mathcal{R}_{|S_I}^2) & M(\mathcal{R}_{|C_I}^2) \end{pmatrix}$$

of  $C_X(G)$  associated to the symmetric differences  $N(c_i) \Delta N(c_j) = \{s_i, s_j\} \cup \{c_i, c_j\} = N[s_i] \Delta N[s_j]$  for all  $i, j \in \{1, \dots, k\}$ .

For  $|I| \geq 3$ , the constraints can be derived by the Chvátal-Gomory procedure from all constraints associated to  $I' \subset I$  with  $|I'| = |I| - 1$  in a similar way as in the proof from [2] for 2-roses by applying the procedure to the subsets  $I' \subseteq \{1, \dots, k\}$  (and, thus, simultaneously to both parts  $S$  and  $C$  of  $G$ ).

In order to arrive at the constraint associated to a subset  $I$ , we sum up the constraints

$$x(S_{I'}) + x(C_{I'}) \geq |I'| - 1$$

for all  $I' \subset I$  with  $|I'| = |I| - 1$  and obtain

$$|I'|x(S_I) + |I'|x(C_I) \geq |I'|(|I'| - 1).$$

Dividing this inequality by  $|I'|$  and rounding up the rhs yields

$$x(S_I) + x(C_I) \geq |I'| = |I| - 1$$

by

$$\frac{|I'|(|I'| - 1)}{|I'|} = \frac{(|I'| + 1)(|I'| - 1)}{|I'|} = \frac{|I'|^2 - 1}{|I'|} = |I'| - \frac{1}{|I'|}.$$

That way, we obtain all the constraints

$$x(S_I) + x(C_I) \geq |I| - 1$$

for all  $I \subseteq \{1, \dots, k\}$  with  $|I| \geq 3$ . As all these constraints are derived by the Chvátal-Gomory procedure, they are all valid for  $P_{LTD}(G)$ , and the constraint

$$x(S) + x(C) \geq k - 1$$

associated to  $I = \{1, \dots, k\}$  yields  $k - 1$  as lower bound for the size of any  $X$ -set of  $G$ .

**Claim 2** *Any subset  $V'$  of nodes with*

$$V' \cap S = \{s_i, s_j\} \text{ and } V' \cap C = C - \{c_i, c_j, c_l\} \text{ with distinct } \{i, j, l\} \subset \{1, \dots, k\}$$

*is an LTD-set of  $G$ .*

For that, we show that  $V'$  has a non-empty intersection with all relevant open neighborhoods and symmetric differences. Indeed,

$$\begin{aligned} V' \cap C &\text{ contains (for } k \geq 5) \text{ at least 2 nodes and, thus, intersects } N(s_i) = \\ &C - \{c_i\} \text{ for all } i \in \{1, \dots, k\}, \\ N(c_i) &= (S - \{s_i\}) \cup (C - \{c_i\}) \text{ is redundant for all } i \in \{1, \dots, k\}, \\ V' \cap S &= \{s_i, s_j\} \text{ intersects the symmetric differences } N(s_i) \Delta N(c_j) = S - \\ &\{s_j\} \cup \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \\ V' \cap S &= \{s_i, s_j\} \text{ intersects the symmetric differences } N(c_i) \Delta N(c_j) = \\ &\{s_i, s_j\} \cup \{c_i, c_j\} \text{ containing } s_i \text{ or } s_j, \text{ whereas } V' \cap C = C - \{c_i, c_j, c_l\} \text{ inter-} \\ &\text{sects the remaining symmetric differences of this type,} \end{aligned}$$

this holds also for  $N[s_i] \Delta N[s_j] = \{s_i, s_j\} \cup \{c_i, c_j\}$  for all  $i, j \in \{1, \dots, k\}$ ,  
 $V' \cap S = \{s_i, s_j\}$  clearly intersects  $N[s_i] \Delta N[c_i] = S \cup \{c_i\}$  for all  $i \in \{1, \dots, k\}$ .

Hence, any such  $V'$  is indeed an *LTD*-set of  $G$ .

Comparing  $C_{LD}(G)$  with  $C_{LTD}(G)$ , we see that the two matrices differ only in the part related to the neighborhoods. The same set  $V'$  has also a non-empty intersection with all relevant closed neighborhoods as it clearly intersects  $N[s_i] = \{s_i\} \cup (C - \{c_i\})$  for all  $i \in \{1, \dots, k\}$ . Thus, any such set  $V'$  is also an *LD*-set of  $G$ .

This shows that  $\gamma_X(G) \leq k - 1$  holds in both cases, and combining this with the lower bound established above finally shows  $\gamma_X(G) = k - 1$  for  $X \in \{LD, LTD\}$ .  $\square$

The situation is different for thin headless spiders:

**Lemma 5** *For a thin headless spider  $G = (C \cup S, E)$  with  $k \geq 3$ , we have*

$$C_{LD}(G) = \begin{pmatrix} I_{|S|} & I_{|C|} \end{pmatrix} \text{ and } C_{OLD}(G) = C_{LTD}(G) = \begin{pmatrix} 0 & I_{|C|} \end{pmatrix}.$$

*Proof.* Let  $G = (C \cup S, E)$  be a thin headless spider with  $k \geq 4$ . From Theorem 1, the rows of the matrices  $M_{LD}(G)$ ,  $M_{OLD}(G)$  and  $M_{LTD}(G)$  are composed of characteristic vectors of the closed neighborhoods (for  $M_{LD}(G)$ )

$$\begin{aligned} N[s_i] &= \{s_i\} \cup \{c_i\} \text{ for all } s_i \in S, \\ N[c_i] &= \{s_i\} \cup C \text{ for all } c_i \in C, \end{aligned}$$

the open neighborhoods (for  $M_{OLD}(G)$  and  $M_{LTD}(G)$ )

$$\begin{aligned} N(s_i) &= \{c_i\} \text{ for all } i \in \{1, \dots, k\}, \\ N(c_i) &= \{s_i\} \cup (C - \{c_i\}) \text{ for all } i \in \{1, \dots, k\}, \end{aligned}$$

the symmetric differences of open neighborhoods of nodes at distance 1 (in all three cases)

$$\begin{aligned} N(s_i) \Delta N(c_i) &= \{s_i\} \cup C \text{ for all } i \in \{1, \dots, k\}, \\ N(c_i) \Delta N(c_j) &= \{s_i, s_j\} \cup \{c_i, c_j\} \text{ for all } i, j \in \{1, \dots, k\}, \end{aligned}$$

the symmetric differences of open neighborhoods of nodes at distance 2 (for  $M_{OLD}(G)$ )

$$N(s_i) \Delta N(c_j) = \{s_j\} \cup (C - \{c_i, c_j\}) \text{ for all } i, j \in \{1, \dots, k\},$$

and the symmetric differences of closed neighborhoods of nodes at distance 2

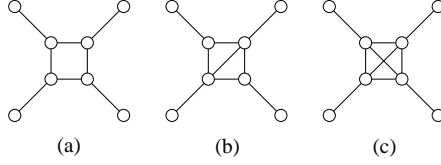


Fig. 5. The three thin suns  $T_4$  where (a) is a sunlet and (c) a thin headless spider.

(for  $M_{LD}(G)$  and  $M_{LTD}(G)$ )

$$N[s_i] \Delta N[c_j] = \{s_i, s_j\} \cup (C - \{c_i\}) \text{ for all } i, j \in \{1, \dots, k\}.$$

Deleting the redundant rows, only the closed resp. open neighborhoods of  $s_i \in S$  remain so that we obtain

$$C_{LD}(G) = \begin{pmatrix} I_{|S|} & I_{|C|} \end{pmatrix} \text{ and } C_{OLD}(G) = C_{LTD}(G) = \begin{pmatrix} 0 & I_{|C|} \end{pmatrix}.$$

□

We immediately conclude:

**Corollary 8** For a thin headless spider  $G = (C \cup S, E)$  with  $k \geq 3$ ,

- (a)  $P_{LD}(G)$  is given by the inequalities
- $x_{s_i} + x_{c_i} \geq 1$  for all  $i \in \{1, \dots, k\}$  and non-negativity constraints and  $\gamma_{LD}(G) = k$  follows;
- (b)  $P_X(G)$  is given by the inequalities
- $x_i \geq 1$  for all  $c_i \in C$  and  $x_i \geq 0$  for all  $s_i \in S$ ,  
 $C$  is the unique  $X$ -set of minimum size and  $\gamma_X(G) = |C| = k$  follows for  $X \in \{OLD, LTD\}$ .

## 5 Thin suns

The latter result on thin headless spiders can be further generalized to thin suns. A *sun* is a graph  $G = (C \cup S, E)$  whose node set can be partitioned into  $S$  and  $C$ , where  $S = \{s_1, \dots, s_k\}$  is a stable set and  $C = \{c_1, \dots, c_k\}$  a (not necessarily chordless) cycle. A *thin sun*  $T_k = (C \cup S, E)$  is a sun where  $s_i$  is adjacent to  $c_j$  if and only if  $i = j$ .

That is, thin headless spiders are special thin suns where all chords of the cycle  $C$  are present (such that  $C$  induces a clique). Another special case are *sunlets* where no chords of the cycle  $C$  are present (such that  $C$  induces a hole). Clearly, the (only) thin sun  $T_3$  equals the thin headless spider; the three possible thin suns  $T_4$  are depicted in Fig. 5.

We call two nodes  $c_i$  and  $c_j$  of a thin sun  $T_k = (C \cup S, E)$  *false C-twins* if  $c_i$  and  $c_j$  are non-adjacent and  $N_C(c_i) = N_C(c_j)$  holds (where  $N_C(v) = N(v) \cap C$ ).

**Lemma 6** *For a thin sun  $T_k(C \cup S, E)$  with  $k \geq 4$ , we have*

$$C_{LD}(T_k) = \begin{pmatrix} I_{|S|} & I_{|C|} \end{pmatrix}, \quad C_{LTD}(T_k) = \begin{pmatrix} 0 & I_{|C|} \end{pmatrix}, \quad C_{OLD}(T_k) = \begin{pmatrix} 0 & I_{|C|} \\ S(\mathcal{R}_{|S|}^2) & 0 \end{pmatrix}$$

where  $S(\mathcal{R}_{|S|}^2)$  is the row-submatrix of  $M(\mathcal{R}_{|S|}^2)$  containing the characteristic vectors of all symmetric differences  $N(c_i) \Delta N(c_j)$  with  $c_i$  and  $c_j$  false C-twins.

*Proof.* Let  $T_k = (C \cup S, E)$  be a thin sun with  $k \geq 4$ . From Theorem 1, the rows of the matrices  $M_{LD}(T_k)$ ,  $M_{OLD}(T_k)$  and  $M_{LTD}(T_k)$  are composed of characteristic vectors of the closed neighborhoods (for  $M_{LD}(T_k)$ )

$$\begin{aligned} N[s_i] &= \{s_i\} \cup \{c_i\} \text{ for all } s_i \in S, \\ N[c_i] &= \{s_i\} \cup N_C[c_i] \text{ for all } c_i \in C, \end{aligned}$$

the open neighborhoods (for  $M_{OLD}(T_k)$  and  $M_{LTD}(T_k)$ )

$$\begin{aligned} N(s_i) &= \{c_i\} \text{ for all } s_i \in S, \\ N(c_i) &= \{s_i\} \cup N_C(c_i) \text{ for all } c_i \in C, \end{aligned}$$

the symmetric differences of open neighborhoods of nodes at distance 1 (in all three cases)

$$\begin{aligned} N(s_i) \Delta N(c_i) &= \{s_i\} \cup N_C[c_i] \text{ for all } i \in \{1, \dots, k\}, \\ N(c_i) \Delta N(c_j) &= \{s_i, s_j\} \cup (N_C[c_i] \Delta N_C[c_j]) \supseteq \{s_i, s_j, c_i, c_j\} \text{ for all } i, j \in \\ &\quad \{1, \dots, k\} \text{ (as } c_i \text{ and } c_j \text{ are adjacent),} \end{aligned}$$

the symmetric differences of open neighborhoods of nodes at distance 2 (for  $M_{OLD}(T_k)$ )

$$\begin{aligned} N(s_i) \Delta N(c_j) &= \{s_j\} \cup (N_C(c_j) - \{c_i\}) \text{ for all } i, j \in \{1, \dots, k\} \text{ (as } c_i \text{ is a} \\ &\quad \text{common neighbor of } s_i \text{ and } c_j), \\ N(c_i) \Delta N(c_j) &= \{s_i, s_j\} \cup (N_C(c_i) \Delta N_C(c_j)) \text{ for all } i, j \in \{1, \dots, k\}, \end{aligned}$$

and the symmetric differences of closed neighborhoods of nodes at distance 2 (for  $M_{LD}(T_k)$  and  $M_{LTD}(T_k)$ )

$$\begin{aligned} N[s_i] \Delta N[c_j] &= \{s_i, s_j\} \cup (N_C[c_j] - \{c_i\}) \supseteq \{s_i, s_j, c_j\} \text{ for all } i, j \in \{1, \dots, k\} \\ &\quad \text{(as } c_i \text{ is a common neighbor of } s_i \text{ and } c_j), \\ N[c_i] \Delta N[c_j] &= \{s_i, s_j\} \cup (N_C[c_i] \Delta N_C[c_j]) \supseteq \{s_i, s_j, c_i, c_j\} \text{ for all } i, j \in \\ &\quad \{1, \dots, k\} \text{ (as } c_i \text{ and } c_j \text{ are non-adjacent).} \end{aligned}$$

Deleting the redundant rows, from  $M_{LD}(T_k)$  and  $M_{LTD}(T_k)$  only the closed

resp. open neighborhoods of  $s_i \in S$  remain so that we obtain

$$C_{LD}(T_k) = \begin{pmatrix} I_{|S|} & I_{|C|} \end{pmatrix} \text{ and } C_{LTD}(T_k) = \begin{pmatrix} 0 & I_{|C|} \end{pmatrix}.$$

From  $M_{OLD}(T_k)$ , the open neighborhoods of  $s_i \in S$  remain as well as the characteristic vectors of all symmetric differences  $N(c_i) \Delta N(c_j) = \{s_i, s_j\}$  for all pairs  $c_i$  and  $c_j$  that are false  $C$ -twins (and, thus,  $N_C(c_i) \Delta N_C(c_j) = \emptyset$  holds) so that we obtain

$$C_{OLD}(T_k) = \begin{pmatrix} 0 & I_{|C|} \\ S(\mathcal{R}_{|S|}^2) & 0 \end{pmatrix}.$$

□

Therefore, independent from the subgraph induced by  $C$ , the clutters  $C_{LD}(T_k)$  and  $C_{LTD}(T_k)$  are the same as the corresponding clutters for thin headless spiders and we immediately conclude:

**Corollary 9** *For a thin sun  $T_k = (C \cup S, E)$  with  $k \geq 4$ ,*

- (a)  $P_{LD}(T_k)$  is given by the inequalities  $x_{s_i} + x_{c_i} \geq 1$  for all  $i \in \{1, \dots, k\}$  and non-negativity constraints and  $\gamma_{LD}(T_k) = k$  follows;
- (b)  $P_{LTD}(T_k)$  is given by the inequalities  $x_i \geq 1$  for all  $c_i \in C$  and  $x_i \geq 0$  for all  $s_i \in S$ ,  $C$  is the unique LTD-set of minimum size and  $\gamma_{LTD}(T_k) = |C| = k$ .

It is left to study the polyhedron associated with  $OLD$ -sets in thin suns. As false  $C$ -twins play a crucial role, we define the following two set families. Let  $FCT(T_k)$  be the set of all subsets  $C' \subseteq C$  such that all nodes in  $C'$  are pairwise false  $C$ -twins, and let  $MFCT(T_k)$  contain the sets  $C' \in FCT(T_k)$  that are inclusion-wise maximal and have at least two nodes. Furthermore, let  $S(C') = \{s_i \in S : c_i \in C'\}$ .

**Theorem 4** *For a thin sun  $T_k = (C \cup S, E)$  with  $k \geq 4$ ,  $P_{OLD}(T_k)$  is given by the inequalities*

- $x_{s_i} \geq 0$  for all  $s_i \in S$ ,  $x_{c_i} \geq 1$  for all  $c_i \in C$ ,
- $x(S(C')) \geq |S(C')| - 1$  for all  $C' \in MFCT(T_k)$ ,

and we have  $\gamma_{OLD}(T_k) = |C| + \sum_{C' \in MFCT(T_k)} |S(C')| - 1$ .

*Proof.* Let  $T_k = (C \cup S, E)$  be a thin sun with  $k \geq 4$ . The row-submatrix  $(0I_{|C|})$  of  $C_{OLD}(T_k)$  clearly results in the constraints  $x_{c_i} \geq 1$  for all  $c_i \in C$ . Now, consider the row-submatrix  $(S(\mathcal{R}_{|S|}^2)0)$  of  $C_{OLD}(T_k)$ . By the definition of false  $C$ -twins, all sets  $C' \in MFCT(T_k)$  are pairwise disjoint. Hence,  $S(\mathcal{R}_{|S|}^2)$  has a block-structure, where each block equals  $M(\mathcal{R}_{|S(C')|}^2)$  for some  $C' \in$



$MFCT(T_k)$ . From the description of the facets of the covering polyhedron associated with complete  $q$ -roses by [2] and taking the block structure of  $S(\mathcal{R}_{|S|}^2)$  into account, we conclude that  $P_{OLD}(T_k)$  has a constraint

$$x(S(C')) \geq |S(C')| - 1 \quad \forall C' \in FCT(T_k)$$

(where the constraint associated with a set  $C' \in FCT(T_k)$  of size one corresponds to a non-negativity constraint). Hence,  $P_{OLD}(T_k)$  is given by the above mentioned inequalities.

Moreover, adding up all constraints  $x_{c_i} \geq 1$  for all  $c_i \in C$  and

$$x(S(C')) \geq |S(C')| - 1 \quad \forall C' \in MFCT(T_k)$$

results in the valid inequality

$$x(C) + \sum_{C' \in MFCT(T_k)} x(S(C')) \geq |C| + \sum_{C' \in MFCT(T_k)} |S(C')| - 1$$

which provides a lower bound for  $\gamma_{OLD}(T_k)$ . On the other hand, choosing all but one node from each subset  $S(C')$  for all  $C' \in MFCT(T_k)$  and all nodes from  $C$  provides an  $OLD$ -set of the same size, which finally implies

$$\gamma_{OLD}(T_k) = |C| + \sum_{C' \in MFCT(T_k)} |S(C')| - 1.$$

□

The situation clearly simplifies for thin suns  $T_k = (C \cup S, E)$  without false  $C$ -twins:

**Corollary 10** *For a thin sun  $T_k = (C \cup S, E)$  with  $k \geq 4$  and without false  $C$ -twins, we have  $C_{OLD}(T_k) = (0 \ I_{|C|})$  so that  $P_{OLD}(T_k)$  is given by the inequalities  $x_i \geq 1$  for all  $c_i \in C$  and  $x_i \geq 0$  for all  $s_i \in S$ ,  $C$  is the unique  $OLD$ -set of minimum size and  $\gamma_{OLD}(T_k) = |C| = k$  follows.*

This applies in particular to sunlets  $T_k$  with  $k \geq 5$ .

## 6 Concluding remarks

In this paper, we proposed to study the  $LD$ -,  $OLD$ - and  $LTD$ -problem from a polyhedral point of view, motivated by promising polyhedral results for the  $ID$ -problem [2,4,5,7]. That way, we were able to provide closed formulas for the  $LD$ - and  $LTD$ -numbers of all kinds of complete  $p$ -partite graphs, for the studied families of split graphs and thin suns as well as the  $OLD$ -numbers of thin and thick headless spiders and thin suns.

In particular, if we have the same clutter matrix for two different  $X$ -problems, then we can conclude that every solution of one problem is also a solution for the other problem, and vice versa, such that the two  $X$ -polyhedra coincide and the two  $X$ -numbers are equal. This turned out to be the case for

- complete bipartite graphs as  $C_{ID}(K_{m,n}) = C_{LD}(K_{m,n}) = C_{LTD}(K_{m,n})$  holds by Lemma 2 and results from [2],
- cliques as  $C_{LD}(K_p) = C_{OLD}(K_p) = C_{LTD}(K_p)$  holds by Lemma 3(a),
- complete  $p$ -partite graphs as  $C_{LD}(K_{n_1, n_2, \dots, n_p}) = C_{LTD}(K_{n_1, n_2, \dots, n_p})$  holds by Lemma 3,
- thin headless spiders  $G$  as  $C_{OLD}(G) = C_{LTD}(G)$  holds by Lemma 5.

Furthermore, we were able to provide the complete facet descriptions of

- the  $LD$ - and  $LTD$ -polyhedra for all complete  $p$ -partite graphs (including complete split graphs), for thin headless spiders and thin suns (see Section 3, Corollary 8 and Corollary 9),
- the  $OLD$ -polyhedra of cliques, thin and thick headless spiders and thin suns (see Corollary 5, Section 4) and Theorem 4.

The complete descriptions of some  $X$ -polyhedra also provide us with information about the relation between  $Q^*(C_X(G))$  and its linear relaxation  $Q(C_X(G))$ . A matrix  $M$  is *ideal* if  $Q^*(M) = Q(M)$ . From the complete descriptions obtained in Section 3 and Section 4, we conclude:

**Corollary 11** *The  $LD$ -,  $LTD$ - and  $OLD$ -clutters of thin headless spiders are ideal for all  $k \geq 3$ .*

**Corollary 12** *The  $LD$ - and  $LTD$ -clutters of thin suns as well as the  $OLD$ -clutters of thin suns without false  $C$ -twins are ideal for all  $k \geq 3$ .*

For any nonideal matrix, we can evaluate how far  $M$  is from being ideal by considering the inequalities that have to be added to  $Q(M)$  in order to obtain  $Q^*(M)$ . With this purpose, in [1], a matrix  $M$  is called *rank-ideal* if only 0/1-valued constraints have to be added to  $Q(M)$  to obtain  $Q^*(M)$ . We further conclude:

**Corollary 13** *The  $LD$ - and  $LTD$ -clutters of all complete  $p$ -partite graphs and the  $OLD$ -clutters of cliques, thick headless spiders and thin suns are rank-ideal.*

Finally, the  $LD$ - and  $LTD$ -clutters of thick headless spiders have a more complex structure such that also a facet description of the  $LD$ - and  $LTD$ -polyhedra is more involved in this case. However, based on the common part of the clutters and using polyhedral arguments, it was possible to establish that  $k - 1$  is a lower bound for the cardinality of any  $LD$ - and  $LTD$ -set.

Exhibiting an  $LD$ - and  $LTD$ -set of size  $k - 1$  thus allowed us to deduce the exact value of the  $LD$ - and  $LTD$ -number of thick headless spiders (Theorem 3).

This demonstrates how the polyhedral approach can be applied to find  $X$ -sets of minimum size for special graphs  $G$ , by determining and analyzing the  $X$ -clutters  $C_X(G)$ , even in cases where no complete description of  $P_X(G)$  is known yet.

As future lines of research, we plan to work on a complete description of the  $LD$ - and  $LTD$ -polyhedra of thick headless spiders and to apply similar and more advanced techniques for other graphs in order to obtain either  $X$ -sets of minimum size or strong lower bounds stemming from linear relaxations of the  $X$ -polyhedra, enhanced by suitable cutting planes.

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