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# The continuous collision-induced nonlinear fragmentation equation with non-integrable fragment daughter distributions

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**Abstract.** Existence, non-existence, and uniqueness of mass-conserving weak solutions to the continuous collision-induced nonlinear fragmentation equations are established for the collision kernels  $\Phi$  satisfying  $\Phi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} + x_2^{\lambda_1} x_1^{\lambda_2}$ ,  $(x_1, x_2) \in (0, \infty)^2$ , with  $\lambda_1 \leq \lambda_2 \leq 1$ , and non-integrable fragment daughter distributions. In particular, global existence of mass-conserving weak solutions is shown when  $1 \leq \lambda := \lambda_1 + \lambda_2 \leq 2$  with  $\lambda_1 \geq k_0$ , the parameter  $k_0 \in (0, 1)$  being related to the non-integrability of the fragment daughter distribution. The existence of at least one mass-conserving weak solution is also demonstrated when  $2k_0 \leq \lambda < 1$  with  $\lambda_1 \geq k_0$  but its maximal existence time is shown to be finite. Uniqueness is also established in both cases. The last result deals with the non-existence of mass-conserving weak solutions, even on a small time interval, for power law fragment daughter distribution when  $\lambda_1 < k_0$ . It is worth mentioning that the previous literature on the nonlinear fragmentation equation does not treat non-integrable fragment daughter distribution functions.

**Keywords.** Collision-induced fragmentation, non-integrable fragment daughter distributions, existence, uniqueness, non-existence, mass conservation

**AMS subject classifications.** 35R09, 45K05, 35A01, 35A02, 35D30

## 1. INTRODUCTION

The nonlinear fragmentation equation, sometimes known as collision-induced breakage equation, describes the dynamics of particles which change their sizes in response to collisions with other particles and is given by the following nonlinear integro-differential equation

$$\partial_t u(t, x_1) = \mathcal{F}u(t, x_1) - \mathcal{L}u(t, x_1), \quad (t, x_1) \in (0, \infty)^2, \quad (1.1)$$

$$u(0, x_1) = u^{\text{in}}(x_1) \geq 0, \quad x_1 \in (0, \infty), \quad (1.2)$$

where the formation and loss term of particles with size  $x_1$  are given by

$$\mathcal{F}u(x_1) := \frac{1}{2} \int_{x_1}^{\infty} \int_0^{x_2} \beta(x_1, x_2 - x_3, x_3) \Phi(x_2 - x_3, x_3) u(x_2 - x_3) u(x_3) dx_3 dx_2 \quad (1.3)$$

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and

$$\mathcal{L}u(x_1) := \int_0^\infty \Phi(x_1, x_2)u(x_1)u(x_2)dx_2, \quad (1.4)$$

for  $x_1 \in (0, \infty)$ .

In equations (1.1)-(1.4), the quantity  $u(t, x_1)$  represents the density of particles with size  $x_1 > 0$  at time  $t \geq 0$ . The collision kernel  $\Phi(x_1, x_2) = \Phi(x_2, x_1) \geq 0$  describes the rate at which particles of sizes  $x_1$  and  $x_2$  collide, while the daughter distribution function  $\beta(x_3, x_1, x_2) = \beta(x_3, x_2, x_1) \geq 0$  gives the number of particles of size  $x_3 \in (0, x_1 + x_2)$  that result from the collision between particles of sizes  $x_1$  and  $x_2$ . In equation (1.1), the first term on the right-hand side accounts for the formation of particles with size  $x_1$  due to the collision between particles of respective sizes  $x_2 - x_3$  and  $x_3$ , where  $x_2 > x_1$  and  $x_3 \in (0, x_2)$ . The second term on the right-hand side represents the loss of particles of size  $x_1$  due to their collision with particles of arbitrary size.

Let us assume that local mass remains conserved, i.e.,

$$\int_0^{x_1+x_2} x_3\beta(x_3, x_1, x_2)dx_3 = x_1 + x_2 \quad \text{and} \quad \beta(x_3, x_1, x_2) = 0 \quad \text{for} \quad x_3 > x_1 + x_2, \quad (1.5)$$

for all  $(x_1, x_2) \in (0, \infty)^2$ . This is a fundamental property in the modeling of collisional breakup of particles. It ensures that no matter is created nor destroyed during the collision of particles. The property (1.5) also leads to the global mass conservation principle, which is given by

$$\int_0^\infty x_1u(t, x_1)dx_1 = \int_0^\infty x_1u^{\text{in}}(x_1)dx_1, \quad t \geq 0. \quad (1.6)$$

The identity (1.6) states that the total mass of particles at any time  $t$  is equal to the initial total mass of particles.

Furthermore, it can be inferred from equation (1.5) that the combination of particles with sizes  $x_1$  and  $x_2$  through collision cannot produce fragments that are larger than the sum of their sizes  $x_1 + x_2$ . However, it is reasonable to expect that collisional fragmentation between  $x_1$  and  $x_2$  may generate particles larger than  $x_1$  and  $x_2$  due to mass transfer from the smallest to the largest particles. However, if there exists a function  $\beta_*$  which is non-negative and satisfies

$$\beta(x_3, x_1, x_2) = \beta_*(x_3, x_1, x_2)\mathbf{1}_{(0, x_1)}(x_3) + \beta_*(x_3, x_2, x_1)\mathbf{1}_{(0, x_2)}(x_3), \quad (1.7)$$

for  $(x_1, x_2, x_3) \in (0, \infty)^3$  and

$$\int_0^{x_1} x_3\beta_*(x_3, x_1, x_2)dx_3 = x_1, \quad (1.8)$$

then there will be no mass transfer during the collision.

The integrability of the non-negative function  $\beta_*$  is an important assumption, as it ensures that the number of daughter particles produced during the breakage of a particle of size  $x_1$  is finite, which can be mathematically expressed as

$$\mathbf{N}(x_1, x_2) := \int_0^{x_1} \beta_*(x_3, x_1, x_2)dx_3 < \infty, \quad (x_1, x_2) \in (0, \infty)^2. \quad (1.9)$$

A specific type of breakage function, known as power law breakage, is given by

$$\beta_*(x_3, x_1, x_2) = (\nu + 2)x_3^\nu x_1^{-\nu-1} \mathbf{1}_{(0, x_1)}(x_3), \quad (1.10)$$

with  $\nu \in (-2, 0]$  [25], and satisfies the integrability assumption (1.9) when  $\nu \in (-1, 0]$ , but fails to do so when  $\nu \in (-2, -1]$ .

To the best of our knowledge, the integrability assumption (1.9) is a standing assumption in the studies of the nonlinear fragmentation equation (1.1) performed so far [10, 12, 17] and the main purpose of this work is to relax this assumption, thereby allowing to handle more singular fragment daughter distributions such as the one given by (1.10) with  $\nu \in (-2, -1]$ . We shall thus rather assume below that there are  $k_0 \in (0, 1)$  and  $p_0 \in (1, 1 + k_0)$  such that  $\beta_*$  satisfies

$$\int_0^{x_1} x_3^{k_0} \beta_*(x_3, x_1, x_2)^p dx_3 \leq E_{k_0, p} x_1^{k_0+1-p}, \quad p \in [1, p_0], \quad (1.11)$$

for some positive constant  $E_{k_0, p}$ . It is worth pointing out that the assumption (1.11) is a rather natural generalization of the integrability assumption (1.9) which somehow corresponds to the choice  $k_0 = 0$  in (1.11). Furthermore, the power law distribution (1.10) satisfies (1.11) for any  $k_0 > |\nu| - 1$  and  $p < \frac{k_0+1}{|\nu|}$ .

As for the collision kernel  $\Phi$ , we assume for simplicity that it is given by

$$\Phi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} + x_2^{\lambda_1} x_1^{\lambda_2}, \quad k_0 \leq \lambda_1 \leq \lambda_2 \leq 1, \quad (x_1, x_2) \in (0, \infty)^2, \quad (1.12)$$

for  $k_0 \in (0, 1)$  stated in (1.11) with homogeneity  $\lambda = \lambda_1 + \lambda_2$ .

The nonlinear breakage/fragmentation equation has not been studied as completely as its linear counterpart. Over the past few decades, there has been a lot of interest in the linear fragmentation equation [26], which was first investigated by Filippov [14], Kapur [20], McGrady and Ziff [25, 30] and later studied by functional analytic methods in [1–4, 7, 13] and by stochastic approaches in [5, 19], see also the books [4, 6] for a more detailed account.

In contrast, there are only a few studies available in the physics literature on the collision-induced breakage equation. An asymptotic analysis of the continuous collision-induced breakage events is performed by Cheng and Redner in [9, 10] for a class of models in which a collision of two particles causes both particles to split into two equal halves or either the largest particle, or the smallest one splits in two. In a later study, Krapivsky and Ben-Naim [23] investigate the dynamics of collision-induced fragmentation, using its travelling wave behaviour to calculate the fragment mass distribution analytically. Kostoglou and Karabelas [21, 22] also study analytical solutions to the collision-induced fragmentation and their asymptotic information using the Gamma distribution approximation. Ernst and Pagonabarraga [12] describe the asymptotic behaviour of the collision-induced breakage equation for the product kernel  $\Phi(x_1, x_2) = (x_1 x_2)^{\lambda/2}$  and  $0 \leq \lambda \leq 2$ . In this case, the collision-induced breakage equation can be mapped to a linear breakage equation.

Recently, in [17], existence and uniqueness of mass-conserving weak solutions to (1.1)–(1.2) are discussed for integrable daughter distribution functions and collision kernels of the form  $\Phi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} + x_1^{\lambda_2} x_2^{\lambda_1}$ , where  $\lambda := \lambda_1 + \lambda_2 \in [0, 2]$ . Moreover, the non-existence of weak solutions is also shown in [17] when  $\lambda_1 < 0$ .

The goal of this article is to establish results on the existence, uniqueness and non-existence of mass-conserving weak solutions to (1.1)-(1.2) for a class of non-integrable daughter distribution functions. In particular, the existence of global weak solutions is shown for collision kernels that grow at least linearly while only local weak solutions are achieved for the case when collision kernels grow at most linearly. As in [17], the existence proof relies on the weak  $L^1$ -compactness method introduced in [27] for the coagulation-fragmentation equations and later developed in several papers, see [4] for a detailed account and further references. As for uniqueness, we shall follow the approach developed in [13, 28] by showing a Lipschitz property in a suitably chosen weighted  $L^1$ -space. We finally adapt an argument from [4, 8, 17, 29] to show the non-existence of mass-conserving weak solutions for some collision kernels.

Before stating our results, we introduce the following notation: given a function  $W$  which is non-negative and measurable and defined on the interval  $(0, \infty)$ , we set  $X_W := L^1((0, \infty), W(x_1)dx_1)$  and

$$\|g\|_W := \int_0^\infty |g(x_1)|W(x_1)dx_1, \quad \mu_W(g) := \int_0^\infty g(x_1)W(x_1)dx_1, \quad g \in X_W.$$

The space  $X_W$  endowed with its weak topology is denoted by  $X_{W,w}$ , while its positive cone is denoted by  $X_{W,+}$ . In particular, we denote  $X_k := X_{W_k}$  when  $W(x_1) = W_k(x_1) := x_1^k$ ,  $x_1 \in (0, \infty)$ , for some  $k \in \mathbb{R}$  and

$$\|g\|_k := \int_0^\infty x_1^k |g(x_1)|dx_1, \quad \mu_k(g) := \mu_{W_k}(g) = \int_0^\infty x_1^k g(x_1)dx_1, \quad g \in X_k.$$

## 2. MAIN RESULTS

In order to state the main results of the paper, we first need to make clear what we mean by a weak solution to (1.1)-(1.2). The notion of weak solution used here differs slightly from that used for non-integrable fragment distributions in the classical coagulation-fragmentation equation [4, 24]. It requires the integrability of function  $(x_1^{k_0} + x_2^{k_0})\Phi(x_1, x_2)u(\tau, x_1)u(\tau, x_2)$  rather than integrability of function  $\min\{x_1, x_2\}^{k_0}\Phi(x_1, x_2)u(\tau, x_1)u(\tau, x_2)$  for all  $(\tau, x_1, x_2) \in (0, t) \times (0, \infty)^2$ , where  $t \in (0, T)$  and  $T \in (0, \infty]$ .

### 2.1. Weak solution.

**Definition 2.1.** *Let us fix  $k_0 \in (0, 1)$ , a non-negative and symmetric collision kernel  $\Phi$  and a fragment daughter distribution  $\beta$  satisfying (1.5), as well as*

$$\int_0^{x_1+x_2} x_3^{k_0} \beta(x_3, x_1, x_2)dx_3 \leq B (x_1^{k_0} + x_2^{k_0}), \quad (x_1, x_2) \in (0, \infty)^2,$$

for some  $B > 0$ , and consider an initial condition  $u^{\text{in}} \in X_{k_0,+} \cap X_1$  and  $T \in (0, \infty]$ . A weak solution to the nonlinear fragmentation equation (1.1)-(1.2) on  $[0, T)$  is a non-negative function  $u$  such that

$$u \in \mathcal{C}([0, T), X_{k_0,w}) \cap L^\infty((0, T), X_1), \quad (2.1)$$

with  $u(0) = u^{\text{in}}$  in  $(0, \infty)$ ,

$$(\tau, x_1, x_2) \longmapsto (x_1^{k_0} + x_2^{k_0})\Phi(x_1, x_2)u(\tau, x_1)u(\tau, x_2) \in L^1((0, t) \times (0, \infty)^2), \quad (2.2)$$

which also satisfies

$$\begin{aligned} \int_0^\infty \varsigma(x_1)u(t, x_1)dx_1 &= \int_0^\infty \varsigma(x_1)u^{\text{in}}(x_1)dx_1 \\ &+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_\varsigma(x_1, x_2)\Phi(x_1, x_2)u(\tau, x_1)u(\tau, x_2)dx_2dx_1d\tau, \end{aligned} \quad (2.3)$$

for all  $t \in (0, T)$  and  $\varsigma \in \mathcal{T}^{k_0}$ , where

$$\mathcal{T}^{k_0} := \{\varsigma \in \mathcal{C}^{0, k_0}([0, \infty)) \cap L^\infty((0, \infty)) : \varsigma(0) = 0\},$$

and

$$\Upsilon_\varsigma(x_1, x_2) = \Upsilon_\varsigma(x_2, x_1) := \int_0^{x_1+x_2} \varsigma(x_3)\beta(x_3, x_1, x_2)dx_3 - \varsigma(x_1) - \varsigma(x_2). \quad (2.4)$$

In addition,  $u$  is said to be mass-conserving on  $[0, T)$  if it satisfies

$$\boldsymbol{\mu}_1(u(t)) = \boldsymbol{\mu}_1(u^{\text{in}}), \quad t \in [0, T).$$

First, we establish that there exists at least one weak solution to (1.1)-(1.2) on some time interval  $[0, T_*)$  in the sense of Definition 2.1, which is also mass-conserving. The result established here provides an extension of [17] to the class of daughter distributions which are non-integrable.

**Theorem 2.2.** *Suppose that  $\Phi$  is defined according to equation (1.12) and that  $\beta$  fulfills conditions (1.7), (1.8), and (1.11) for some fixed  $k_0 \in (0, 1)$ . Consider an initial condition  $u^{\text{in}} \in X_{k_0, +} \cap X_1$  with a positive mass  $\rho := \boldsymbol{\mu}_1(u^{\text{in}})$  such that*

$$\int_0^\infty x_1^{k_0+1}u^{\text{in}}(x_1)dx_1 < \infty. \quad (2.5)$$

*Then there exists at least one mass-conserving weak solution  $u$  to (1.1)-(1.2) on  $[0, T_*)$  in the sense of Definition 2.1, where*

$$T_* = \begin{cases} T_{k_0} & \text{if } \lambda \in (0, 1), \\ \infty & \text{if } \lambda \in [1, 2], \end{cases}$$

*and  $T_{k_0}$  is defined in Lemma 4.2, see (4.18), and only depends on  $u^{\text{in}}$ ,  $\rho$ , and  $\lambda = \lambda_1 + \lambda_2$ .*

Theorem 2.2 is proved using a weak compactness approach in the space  $X_{k_0} \cap X_{1+k_0}$ , a method pioneered in [27] for the coagulation-fragmentation equation, and adapted in particular to (1.1) in [17] for integrable fragment daughter distributions. For  $u^{\text{in}} \in X_{0,+} \cap X_1$ , it is known from [17] that both global and local mass-conserving weak solutions to (1.1)-(1.2) exist under the assumptions (1.7), (1.8), (1.11), and (1.12) with  $k_0 = 0$ .

In addition to the existence of mass-conserving weak solutions, a result on uniqueness is demonstrated for a more restricted set of initial data  $u^{\text{in}}$ .

**Theorem 2.3.** *Suppose that  $\Phi$  is defined according to equation (1.12) and that  $\beta$  fulfills conditions (1.7), (1.8), and (1.11) for some fixed  $k_0 \in (0, 1)$ . Consider an initial condition*

$u^{\text{in}} \in X_{k_0,+} \cap X_{k_0+1}$  and  $T \in (0, \infty]$ . Then there exists a unique weak solution  $u$  to the equation (1.1)-(1.2) on the time interval  $[0, T)$  satisfying the condition

$$\boldsymbol{\mu}_{1+k_0+\lambda_2}(u) \in L^1(0, t) \text{ for every } t \in (0, T). \quad (2.6)$$

In order to prove uniqueness, similar to [4, 13, 16, 17, 28], it is necessary to control the distance between two solutions in a suitable weighted  $L^1$ -space. The challenging aspect of this proof is to select an appropriate weight function, which in this case is given by  $w(x_1) := \max\{x_1^{k_0}, x_1^{1+k_0}\}$  for  $x_1 > 0$ .

We finally identify a range of the parameters  $(\lambda_1, \lambda_2)$ , along with a class of power law fragment daughter distributions (1.10), for which no non-zero mass-conserving weak solutions to equation (1.1)-(1.2) on  $[0, T)$  exists whatever the value of  $T > 0$ .

**Theorem 2.4.** *Suppose that  $\Phi$  is defined according to equation (1.12) and that  $\beta$  fulfills conditions (1.7), (1.8), and (1.10) for some  $\nu \in (-2, -1]$ . Fix  $k_0 \in (|\nu| - 1, 1)$  and consider an initial condition  $u^{\text{in}} \in X_{k_0,+} \cap X_{1+k_0}$  with  $\rho := \boldsymbol{\mu}_1(u^{\text{in}}) > 0$  and  $T > 0$ . If  $\lambda_1 < |\nu| - 1$  and  $\lambda = \lambda_1 + \lambda_2 < 1$ , then (1.1)-(1.2) does not admit a mass-conserving weak solution on  $[0, T)$ .*

A similar result is established in [17, Theorem 1.8] for integrable fragment daughter distributions (1.10) with  $\nu \in (-1, 0]$  when  $\lambda_1 < 0$ , extending an observation from [12] when  $\lambda_1 = \lambda_2 < 0$ . The novelty here is that the non-existence result is not restricted to collision kernels involving at least a negative exponent. In particular, Theorem 2.4 applies to the constant collision kernel (corresponding to  $\lambda_1 = \lambda_2 = 0$ ) as soon as  $\nu \in (-2, -1)$ . As in [17], the proof of Theorem 2.4 is adapted from [4, 8, 29].

Let us now describe the content of the paper. The following section contains additional properties of weak solutions to (1.1)-(1.2) based on Definition 2.1, including a criterion for mass conservation provided by Proposition 3.2, along with some tail control in Lemma 3.4. In section 4, we provide the proof of Theorem 2.2 by a weak  $L^1$  compactness approach. In order to prove Theorem 2.2, first we prove that (1.1) is well-posed when  $\beta$  satisfies (1.7), (1.8) and (1.11) for suitably truncated collision kernels with the help of Banach fixed point theorem. To avoid concentration at a finite size and prevent the escape of matter, a uniform integrability estimate is derived along with estimates for both small and large sizes. The Dunford-Pettis theorem ensures that the approximating sequence is weakly compact with respect to size. In the next step, we estimate time equicontinuity to obtain the compactness with respect to time. Section 5 is devoted to the proof of Theorem 2.3. We end the paper with the proof of the non-existence result in section 6.

### 3. FUNDAMENTAL PROPERTIES

First, let us recall that Definition 2.1 makes sense due to the regularity properties of  $\beta$  described in Definition 2.1.

**Lemma 3.1.** *Let  $k_0 \in (0, 1)$  and  $\beta$  satisfies (1.7), (1.8), and (1.11). Consider a function  $\varsigma \in C^{0,k_0}([0, \infty))$  satisfying  $\varsigma(0) = 0$ . Then*

$$|\Upsilon_\varsigma(x_1, x_2)| \leq E_{k_0}(x_1^{k_0} + x_2^{k_0}) \|\varsigma\|_{C^{0,k_0}}, \quad (x_1, x_2) \in (0, \infty)^2,$$

where  $E_{k_0} := E_{k_0,1} + 1$ .

*Proof.* The proof directly follows from the definition of  $\Upsilon_\varsigma$ , (1.7), (1.8), and (1.11).  $\square$

**3.1. Conservation of mass.** In the following proposition, we demonstrate the conditions under which a weak solution to equation (1.1)-(1.2) is mass-conserving.

**Proposition 3.2.** *Let  $T \in (0, \infty]$  and  $k_0 \in (0, 1)$ . Suppose that  $u$  is a weak solution to (1.1)-(1.2) on  $[0, T)$  with initial condition  $u^{\text{in}} \in X_{k_0,+} \cap X_1$  in the sense of Definition 2.1 satisfying additionally*

$$\int_0^t \int_0^\infty \int_0^\infty (x_1 + x_2) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_1 dx_2 d\tau < \infty \quad (3.1)$$

for all  $t \in (0, T)$ . Then  $u$  is a mass-conserving weak solution to (1.1)-(1.2) on  $[0, T)$ .

*Proof.* The proof is the same as that of [17, Proposition 2.2], to which we refer.  $\square$

**Lemma 3.3.** *Let  $T \in (0, \infty]$  and  $k_0 \in (0, 1)$ , and consider a weak solution  $u$  to (1.1)-(1.2) on  $[0, T)$  with initial condition  $u^{\text{in}} \in X_{k_0,+} \cap X_1$  in the sense of Definition 2.1 which also satisfies (3.1). Then the validity of the weak formulation (2.3) extends to any function*

$$\varsigma \in \mathcal{C}^{0,k_0}([0, \infty)) \quad \text{with } \varsigma(0) = 0 \text{ and } \sup_{x_1 > 1} \left\{ \frac{|\varsigma(x_1)|}{x_1} \right\} < \infty.$$

*Proof.* Consider  $\varsigma \in \mathcal{C}^{0,k_0}([0, \infty))$  such that  $\varsigma(0) = 0$  which satisfies  $|\varsigma(x_1)| \leq Cx_1$  for  $x_1 > 1$ , where  $C$  is a positive constant. Observe that

$$|\varsigma(x_1)| \leq C(x_1^{k_0} + x_1) \text{ for } x_1 > 0. \quad (3.2)$$

For  $L > 0$ , let us define

$$\varsigma_L(x_1) = \varsigma(x_1) \mathbf{1}_{(0,L)}(x_1) + \varsigma(L) \mathbf{1}_{(L,\infty)}(x_1), \quad x_1 > 0,$$

and note that  $\varsigma_L$  also satisfies (3.2). Also we can easily see that  $\varsigma_L \in \mathcal{F}^{k_0}$ , so that, from (2.3), we can write for  $L > 0$ ,

$$\begin{aligned} \int_0^\infty \varsigma_L(x_1) u(t, x_1) dx_1 &= \int_0^\infty \varsigma_L(x_1) u^{\text{in}}(x_1) dx_1 \\ &+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_{\varsigma_L}(x_1, x_2) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau. \end{aligned} \quad (3.3)$$

Now, using (1.7), (1.8), (1.11), (2.4), and (3.2) (for  $\varsigma_L$ ), we get

$$|\Upsilon_{\varsigma_L}(x_1, x_2)| \leq CE_{k_0}(x_1^{k_0} + x_2^{k_0} + x_1 + x_2), \quad (3.4)$$

since (1.8) and (1.11) imply that  $E_{k_0,1} \geq 1$ . Now, with the help of (2.2), (3.1), and (3.4), we may apply the Lebesgue dominated convergence theorem and pass to the limit as  $L \rightarrow \infty$  in (3.3) to obtain

$$\begin{aligned} \int_0^\infty \varsigma(x_1) u(t, x_1) dx_1 &= \int_0^\infty \varsigma(x_1) u^{\text{in}}(x_1) dx_1 \\ &+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_\varsigma(x_1, x_2) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau \end{aligned} \quad (3.5)$$



and complete the proof.  $\square$

In particular, when  $0 < L_1 < L_2$ , the function

$$\varsigma(x_1) = x_1^k \mathbf{1}_{(L_1, L_2)}(x_1) + x_1 L_2^{k-1} \mathbf{1}_{(L_2, \infty)}(x_1)$$

defined for  $x_1 \in (0, \infty)$  and  $k \geq 1$  can be employed as a test function in equation (2.3).

**3.2. Control on linear and superlinear moments for large sizes.** Control on the first moment, which is nothing but control on mass for large sizes, has been noted in [12] and rigorously proven in [17, Lemma 2.3] by using the test function  $\varsigma(x_1) = x_1 \mathbf{1}_{(0, L_1)}(x_1)$  for  $L_1 > 0$ . Similarly, we establish a control on superlinear moments for large sizes.

**Lemma 3.4.** *Let  $T \in (0, \infty]$ ,  $k_0 \in (0, 1)$ , and a non-negative and symmetric collision kernel  $\bar{\Phi}$  satisfying  $\bar{\Phi} \leq \Phi$ . Consider a mass-conserving weak solution  $u$  to the equation (1.1)-(1.2) on the time interval  $[0, T)$  with  $\bar{\Phi}$  instead of  $\Phi$ . Let the initial condition  $u^{\text{in}} \in X_{k_0, +} \cap X_1$ . Assuming that  $\beta$  satisfies (1.7) and (1.8), and that  $u^{\text{in}} \in X_k$  for some  $k \geq 1$ , we have*

$$\int_{x_1}^{\infty} x_2^k u(t, x_2) dx_2 \leq \int_{x_1}^{\infty} x_2^k u^{\text{in}}(x_2) dx_2, \quad (3.6)$$

for all  $(t, x_1) \in [0, T) \times (0, \infty)$ . In particular,

$$\mu_k(u(t)) \leq \mu_k(u^{\text{in}}), \quad t \in [0, T). \quad (3.7)$$

*Proof.* Let  $0 < L_1 < L_2$  and consider the function

$$\varsigma(x_1) = x_1^k \mathbf{1}_{(L_1, L_2)}(x_1) + x_1 L_2^{k-1} \mathbf{1}_{(L_2, \infty)}(x_1), \quad x_1 \in (0, \infty).$$

First we study the properties of the function  $\Upsilon_\varsigma$  defined in (2.4), using (1.7) and (1.8).

*Case 1.* For  $(x_1, x_2) \in (0, L_1)^2$ , we have  $\varsigma(x_1) = 0$  and  $\varsigma(x_2) = 0$ , so that

$$\Upsilon_\varsigma(x_1, x_2) = 0.$$

*Case 2.* For  $(x_1, x_2) \in (0, L_1) \times (L_1, L_2)$ , we have  $\varsigma(x_1) = 0$  and  $\varsigma(x_2) = x_2^k$ , so that

$$\begin{aligned} \Upsilon_\varsigma(x_1, x_2) &= \int_{L_1}^{x_2} x_3^k \beta_*(x_3, x_2, x_1) dx_3 - x_2^k \\ &\leq x_2^{k-1} \int_0^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 - x_2^k = 0. \end{aligned}$$

*Case 3.* For  $(x_1, x_2) \in (L_1, L_2) \times (0, L_1)$ , we have  $\Upsilon_\varsigma(x_1, x_2) = \Upsilon_\varsigma(x_2, x_1)$  and we deduce from *Case 2* that  $\Upsilon_\varsigma(x_1, x_2) \leq 0$ .

*Case 4.* For  $(x_1, x_2) \in (L_1, L_2)^2$ , we have  $\varsigma(x_1) = x_1^k$  and  $\varsigma(x_2) = x_2^k$ , so that

$$\begin{aligned} \Upsilon_\varsigma(x_1, x_2) &= \int_{L_1}^{x_1} x_3^k \beta_*(x_3, x_1, x_2) dx_3 + \int_{L_1}^{x_2} x_3^k \beta_*(x_3, x_2, x_1) dx_3 - x_1^k - x_2^k \\ &\leq 0. \end{aligned}$$

*Case 5.* For  $(x_1, x_2) \in (0, L_1) \times (L_2, \infty)$ , we have  $\varsigma(x_1) = 0$  and  $\varsigma(x_2) = x_2 L_2^{k-1}$ , so that

$$\Upsilon_\varsigma(x_1, x_2) = \int_{L_1}^{L_2} x_3^k \beta_*(x_3, x_2, x_1) dx_3 + \int_{L_2}^{x_2} x_3 L_2^{k-1} \beta_*(x_3, x_2, x_1) dx_3 - x_2 L_2^{k-1}$$

$$\leq L_2^{k-1} \int_{L_1}^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 - x_2 L_2^{k-1} \leq 0.$$

*Case 6.* For  $(x_1, x_2) \in (L_2, \infty) \times (0, L_1)$ , we have  $\Upsilon_\zeta(x_1, x_2) = \Upsilon_\zeta(x_2, x_1)$  and we deduce from *Case 5* that  $\Upsilon_\zeta(x_1, x_2) \leq 0$ .

*Case 7.* For  $(x_1, x_2) \in (L_1, L_2) \times (L_2, \infty)$ , we have  $\zeta(x_1) = x_1^k$  and  $\zeta(x_2) = x_2 L_2^{k-1}$ , so that

$$\begin{aligned} \Upsilon_\zeta(x_1, x_2) &= \int_{L_1}^{x_1} x_3^k \beta_*(x_3, x_1, x_2) dx_3 + \int_{L_1}^{L_2} x_3^k \beta_*(x_3, x_2, x_1) dx_3 \\ &\quad + \int_{L_2}^{x_2} x_3 L_2^{k-1} \beta_*(x_3, x_2, x_1) dx_3 - x_1^k - x_2 L_2^{k-1} \\ &\leq x_1^{k-1} \int_{L_1}^{x_1} x_3 \beta_*(x_3, x_1, x_2) dx_3 + L_2^{k-1} \int_{L_1}^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 \\ &\quad - x_1^k - x_2 L_2^{k-1} \leq 0. \end{aligned}$$

*Case 8.* For  $(x_1, x_2) \in (L_2, \infty) \times (L_1, L_2)$ , we have  $\Upsilon_\zeta(x_1, x_2) = \Upsilon_\zeta(x_2, x_1)$  and we deduce from *Case 7* that  $\Upsilon_\zeta(x_1, x_2) \leq 0$ .

*Case 9.* For  $(x_1, x_2) \in (L_2, \infty)^2$ , we have  $\zeta(x_1) = x_1 L_2^{k-1}$  and  $\zeta(x_2) = x_2 L_2^{k-1}$ , so that

$$\begin{aligned} \Upsilon_\zeta(x_1, x_2) &= \int_{L_1}^{L_2} x_3^k \beta_*(x_3, x_1, x_2) dx_3 + \int_{L_2}^{x_1} x_3 L_2^{k-1} \beta_*(x_3, x_1, x_2) dx_3 - x_1 L_2^{k-1} \\ &\quad + \int_{L_1}^{L_2} x_3^k \beta_*(x_3, x_2, x_1) dx_3 + \int_{L_2}^{x_2} x_3 L_2^{k-1} \beta_*(x_3, x_2, x_1) dx_3 - x_2 L_2^{k-1} \\ &\leq L_2^{k-1} \int_{L_1}^{x_1} x_3 \beta_*(x_3, x_1, x_2) dx_3 + L_2^{k-1} \int_{L_1}^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 \\ &\quad - x_1 L_1^{k-1} - x_2 L_2^{k-1} \leq 0. \end{aligned}$$

Since  $\Upsilon_\zeta(x_1, x_2) \leq 0$  for all  $(x_1, x_2) \in (0, \infty)^2$ , it follows from (2.3) that

$$\int_{L_1}^{L_2} x_1^k (u(t, x_1) - u^{\text{in}}(x_1)) dx_1 + L_2^{k-1} \int_{L_2}^{\infty} x_1 (u(t, x_1) - u^{\text{in}}(x_1)) dx_1 \leq 0$$

for  $t \in (0, T)$ , which can be rewritten as

$$\int_{L_1}^{\infty} \min\{x_1^k, L_2^{k-1} x_1\} u(t, x_1) dx_1 \leq \int_{L_1}^{\infty} x_1^k u^{\text{in}}(x_1) dx_1$$

for  $t \in (0, T)$ . The claimed result follows naturally from allowing  $L_2 \rightarrow \infty$  in the preceding inequality with the help of Fatou's lemma.  $\square$

#### 4. EXISTENCE BY A COMPACTNESS APPROACH

This section is devoted to the proof of Theorem 2.2. We fix

$$k_0 \in (0, 1), \quad p \in (1, 1 + k_0), \quad (4.1)$$

and consider  $\beta$  and  $\Phi$  satisfying (1.7), (1.8), (1.11), and (1.12), respectively. Let  $u^{\text{in}} \in X_{k_0,+} \cap X_1$  be an initial condition satisfying  $\rho = \mu_1(u^{\text{in}}) > 0$  as well as the additional integrability condition (2.5); that is,  $u^{\text{in}} \in X_{1+k_0}$ .

With a properly constructed estimate for the collision kernel  $\Phi$ , we first demonstrate the well-posedness to (1.1) which is based on Picard-Lindelöf theorem. More specifically, the truncated collision kernel  $\Phi_n$  and the initial data  $u_n^{\text{in}}$  for integer values of  $n \geq 1$  are defined as

$$\Phi_n(x_1, x_2) := \Phi(x_1, x_2) \mathbf{1}_{(1/n, n)}(x_1) \mathbf{1}_{(1/n, n)}(x_2), \quad (x_1, x_2) \in (0, \infty)^2, \quad (4.2)$$

and

$$u_n^{\text{in}} := u^{\text{in}} \mathbf{1}_{(0, 2n)}. \quad (4.3)$$

**Proposition 4.1.** *For each  $n \geq 1$ , there is a unique strong solution*

$$u_n \in C^1([0, \infty), X_{k_0,+} \cap X_1)$$

to

$$\partial_t u_n(t, x_1) = \mathcal{F}_n(u_n(t, x_1)) - \mathcal{L}_n(u_n(t, x_1)), \quad (t, x_1) \in (0, \infty)^2, \quad (4.4)$$

where  $\mathcal{F}_n$  and  $\mathcal{L}_n$  are defined by (1.3) and (1.4), respectively, with  $\Phi_n$  instead of  $\Phi$ , and

$$u_n(0, x_1) = u_n^{\text{in}}(x_1) \geq 0, \quad x_1 \in (0, \infty). \quad (4.5)$$

Moreover,  $u_n$  satisfies  $u_n(t, x_1) = 0$  for a.e.  $x_1 > 2n$  and  $t \geq 0$  and it is also a mass-conserving weak solution to (4.4)-(4.5) on  $[0, \infty)$ ; that is,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \varsigma(x_1) u_n(t, x_1) dx_1 \\ = \frac{1}{2} \int_0^\infty \int_0^\infty \Upsilon_\varsigma(x_1, x_2) \Phi_n(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \end{aligned} \quad (4.6)$$

for all  $t > 0$  and  $\varsigma \in \mathcal{T}^{k_0}$ , as well as

$$\mu_1(u_n(t)) = \mu_1(u_n^{\text{in}}), \quad t \geq 0. \quad (4.7)$$

Finally,  $u_n \in L^\infty((0, \infty), X_{1+k_0})$  with

$$\int_{x_1}^\infty x_2^{k_0+1} u_n(t, x_2) dx_2 \leq \int_{x_1}^\infty x_2^{k_0+1} u_n^{\text{in}}(x_2) dx_2 \quad (4.8)$$

for all  $(t, x_1) \in [0, \infty) \times (0, \infty)$ , which implies in particular that

$$\mu_{k_0+1}(u_n(t)) \leq \mu_{k_0+1}(u_n^{\text{in}}), \quad t \geq 0. \quad (4.9)$$

*Proof.* First, we will show that the operator  $\mathcal{F}_n - \mathcal{L}_n$  is locally Lipschitz continuous on  $X_{k_0} \cap X_1$ . Let  $(u, v) \in X_{k_0} \times X_{k_0}$  and observe the following bounds on  $\Phi_n$ :

$$\Phi_n(x_1, x_2) \leq 4n^{\lambda+k_0} \frac{x_1^{k_0} x_2^{k_0}}{x_1^{k_0} + x_2^{k_0}}, \quad (x_1, x_2) \in (0, \infty)^2, \quad (4.10)$$

$$\Phi_n(x_1, x_2) \leq 4n^{\lambda+1} \frac{x_1 x_2}{x_1 + x_2}, \quad (x_1, x_2) \in (0, \infty)^2, \quad (4.11)$$

$$\Phi_n(x_1, x_2) \leq 4n^{\lambda+1} \frac{x_1^{k_0} x_2}{x_1^{k_0} + x_2^{k_0}}, \quad (x_1, x_2) \in (0, \infty)^2. \quad (4.12)$$

It follows from (1.7), (1.8), (1.11), (4.10), and the Fubini-Tonelli theorem that

$$\begin{aligned} \|\mathcal{F}_n u - \mathcal{F}_n v\|_{k_0} &\leq \frac{1}{2} \int_0^\infty \int_{x_1}^\infty \int_0^{x_2} x_1^{k_0} \beta(x_1, x_2 - x_3, x_3) \Phi_n(x_2 - x_3, x_3) \\ &\quad \times |u(x_2 - x_3)u(x_3) - v(x_2 - x_3)v(x_3)| dx_3 dx_2 dx_1 \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^{x_2+x_3} x_1^{k_0} \beta(x_1, x_2, x_3) \Phi_n(x_2, x_3) \\ &\quad \times |u(x_2)u(x_3) - v(x_2)v(x_3)| dx_1 dx_2 dx_3 \\ &\leq \frac{E_{k_0,1}}{2} \int_0^\infty \int_0^\infty (x_2^{k_0} + x_3^{k_0}) \Phi_n(x_2, x_3) \\ &\quad \times (|u(x_2)||u(x_3) - v(x_3)| + |v(x_3)||u(x_2) - v(x_2)|) dx_2 dx_3 \\ &\leq 2E_{k_0,1} n^{\lambda+k_0} \int_0^\infty \int_0^\infty x_2^{k_0} x_3^{k_0} |u(x_2)||u(x_3) - v(x_3)| dx_2 dx_3 \\ &\quad + 2E_{k_0,1} n^{\lambda+k_0} \int_0^\infty \int_0^\infty x_2^{k_0} x_3^{k_0} |v(x_3)||u(x_2) - v(x_2)| dx_2 dx_3 \\ &\leq 2E_{k_0,1} n^{\lambda+k_0} (\|u\|_{k_0} + \|v\|_{k_0}) \|u - v\|_{k_0}. \end{aligned}$$

Similarly, using (4.10), we estimate

$$\begin{aligned} \|\mathcal{L}_n u - \mathcal{L}_n v\|_{k_0} &\leq \int_0^\infty \int_0^\infty x_2^{k_0} \Phi_n(x_2, x_3) |u(x_2)u(x_3) - v(x_2)v(x_3)| dx_2 dx_3 \\ &\leq 4n^{\lambda+k_0} \int_0^\infty \int_0^\infty x_2^{k_0} x_3^{k_0} |u(x_2)||u(x_3) - v(x_3)| dx_2 dx_3 \\ &\quad + 4n^{\lambda+k_0} \int_0^\infty \int_0^\infty x_2^{k_0} x_3^{k_0} |v(x_3)||u(x_2) - v(x_2)| dx_2 dx_3 \\ &\leq 4n^{\lambda+k_0} (\|u\|_{k_0} + \|v\|_{k_0}) \|u - v\|_{k_0}. \end{aligned}$$

We have therefore showed the operator  $\mathcal{F}_n - \mathcal{L}_n$  is locally Lipschitz continuous on  $X_{k_0}$ .

Next, let  $(u, v) \in X_1 \times X_1$ . It follows from (1.7), (1.8), (1.11), (4.11), and the Fubini-Tonelli theorem that

$$\begin{aligned} \|\mathcal{F}_n u - \mathcal{F}_n v\|_1 &\leq \frac{1}{2} \int_0^\infty \int_{x_1}^\infty \int_0^{x_2} x_1 \beta(x_1, x_2 - x_3, x_3) \Phi_n(x_2 - x_3, x_3) \\ &\quad \times |u(x_2 - x_3)u(x_3) - v(x_2 - x_3)v(x_3)| dx_3 dx_2 dx_1 \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^{x_2+x_3} x_1 \beta(x_1, x_2, x_3) \Phi_n(x_2, x_3) \\ &\quad \times |u(x_2)u(x_3) - v(x_2)v(x_3)| dx_1 dx_2 dx_3 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \int_0^\infty \int_0^\infty (x_2 + x_3) \Phi_n(x_2, x_3) \left[ |u(x_2)| |(u-v)(x_3)| \right. \\
&\quad \left. + |v(x_3)| |(u-v)(x_2)| \right] dx_2 dx_3 \\
&\leq 2n^{\lambda+1} \int_0^\infty \int_0^\infty x_2 x_3 |u(x_2)| |(u-v)(x_3)| dx_2 dx_3 \\
&\quad + 2n^{\lambda+1} \int_0^\infty \int_0^\infty x_2 x_3 |v(x_3)| |(u-v)(x_2)| dx_2 dx_3 \\
&\leq 2n^{\lambda+1} (\|u\|_1 + \|v\|_1) \|u-v\|_1.
\end{aligned}$$

Similarly, using (4.11), we have

$$\begin{aligned}
\|\mathcal{L}_n u - \mathcal{L}_n v\|_1 &\leq \int_0^\infty \int_0^\infty x_2 \Phi_n(x_2, x_3) |u(x_2)u(x_3) - v(x_2)v(x_3)| dx_2 dx_3 \\
&\leq 4n^{\lambda+1} \int_0^\infty \int_0^\infty x_2 x_3 |u(x_2)| |(u-v)(x_3)| dx_2 dx_3 \\
&\quad + 4n^{\lambda+1} \int_0^\infty \int_0^\infty x_2 x_3 |v(x_3)| |(u-v)(x_2)| dx_2 dx_3 \\
&\leq 4n^{\lambda+1} (\|u\|_1 + \|v\|_1) \|u-v\|_1.
\end{aligned}$$

Thus, we have shown that  $\mathcal{F}_n - \mathcal{L}_n$  is locally Lipschitz continuous on  $X_1$ .

Introducing the operator

$$\bar{\mathcal{F}}_n u = (\mathcal{F}_n u)_+ = \max \{ \mathcal{F}_n u, 0 \}, \quad u \in X_{k_0} \cap X_1, \quad (4.13)$$

it is also a locally Lipschitz continuous map from  $X_{k_0} \cap X_1$  to  $X_{k_0} \cap X_1$ . Consequently, the Picard-Lindelöf theorem guarantees that there are  $T_n \in (0, \infty]$  and a unique function  $u_n \in C^1([0, T_n], X_{k_0} \cap X_1)$  such that  $u_n$  solves

$$\partial_t u_n(t, x_1) = \bar{\mathcal{F}}_n(u_n(t, x_1)) - \mathcal{L}_n(u_n(t, x_1)) \text{ in } (0, T_n) \times (0, \infty) \quad (4.14)$$

with initial condition  $u_n(0, \cdot) = u_n^{\text{in}}$ , recalling that the latter is defined in (4.3). In addition, there are two possibilities: either  $T_n = \infty$  or  $T_n < \infty$  and  $(\|u(t)\|_{k_0} + \|u(t)\|_1) \rightarrow \infty$  as  $t \rightarrow T_n$ .

Now, it follows from (4.10), (4.13), and (4.14) that, for  $t \in (0, T_n)$ ,

$$\begin{aligned}
&\frac{d}{dt} \int_0^\infty x_1^{k_0} (-u_n(t, x_1))_+ dx_1 \\
&= - \int_0^\infty x_1^{k_0} \text{sign}_+(-u_n(t, x_1)) \partial_t u_n(t, x_1) dx_1 \\
&\leq \int_0^\infty x_1^{k_0} \text{sign}_+(-u_n(t, x_1)) u_n(t, x_1) \int_0^\infty \Phi_n(x_1, x_2) u_n(t, x_2) dx_2 dx_1 \\
&\leq \int_0^\infty x_1^{k_0} (-u_n(t, x_1))_+ \int_0^\infty \Phi_n(x_1, x_2) |u_n(t, x_2)| dx_2 dx_1 \\
&\leq 4n^{\lambda+k_0} \|u_n(t)\|_{k_0} \|(-u_n(t))_+\|_{k_0}.
\end{aligned}$$

Hence, after integrating with respect to time,

$$\|(-u_n(t))_+\|_{k_0} \leq \|(-u_n^{\text{in}})_+\|_{k_0} \exp\left\{4n^{\lambda+k_0} \int_0^t \|u_n(s)\|_{k_0} ds\right\} = 0,$$

which shows that  $u_n(t) \in X_{k_0,+}$  for all  $t \in (0, T_n)$ . The non-negativity of  $u_n$  readily implies that  $\bar{\mathcal{F}}_n u_n = \mathcal{F}_n u_n$  and we deduce from (4.14) that  $u_n \in C^1([0, T_n], X_{k_0} \cap X_1)$  solves (4.4)-(4.5).

Next, it follows from (4.10) that

$$\begin{aligned} \int_0^\infty \int_0^\infty (x_1^{k_0} + x_2^{k_0}) \Phi_n(x_1, x_2) u_n(\tau, x) u_n(\tau, y) dx_2 dx_1 &\leq 4n^{\lambda+k_0} \|u_n(\tau)\|_{k_0}^2 \\ &\leq 4n^{\lambda+k_0} \sup_{\tau \in [0, t]} \{ \|u_n(\tau)\|_{k_0}^2 \} \end{aligned}$$

for  $\tau \in (0, t)$  and  $t \in (0, T_n)$ , so that  $u_n$  satisfies (2.2). Thus,  $u_n$  is a weak solution to (4.4)-(4.5) on  $[0, T_n)$ .

Next, it follows from (4.11) that, for any  $t \in (0, T_n)$  and  $\tau \in (0, t)$ , the following inequality holds:

$$\int_0^\infty \int_0^\infty (x_1 + x_2) \Phi_n(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_1 dx_2 \leq 4n^{\lambda+1} \sup_{\tau \in [0, t]} \|u_n(\tau)\|_1^2.$$

Thus, based on Proposition 3.2 applied with  $\bar{\Phi} = \Phi_n \leq \Phi$ , it follows that  $u_n$  is a mass-conserving weak solution to (4.4)-(4.5) on the interval  $[0, T_n)$ . Therefore,

$$\|u_n(t)\|_1 = \boldsymbol{\mu}_1(u_n(t)) = \boldsymbol{\mu}_1(u_n^{\text{in}}) = \|u_n^{\text{in}}(t)\|_1 \leq \rho, \quad t \in [0, T_n). \quad (4.15)$$

Also, let  $t \in (0, T_n)$ . Substituting  $\varsigma(x_1) = W_{k_0}(x_1) = x_1^{k_0}$  into (4.6) and using (1.11), (4.12), and (4.15), we get

$$\begin{aligned} \frac{d}{dt} \|u_n(t)\|_{k_0} &\leq \frac{1}{2} \int_0^\infty \int_0^\infty \Upsilon_{W_{k_0}}(x_1, x_2) \Phi_n(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq \frac{E_{k_0,1}}{2} \int_0^\infty \int_0^\infty (x_1^{k_0} + x_2^{k_0}) \Phi_n(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq 2E_{k_0,1} n^{\lambda+2k_0+1} \int_0^\infty \int_0^\infty x_1^{k_0} x_2 u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq 2E_{k_0,1} n^{\lambda+1} \|u_n(t)\|_{k_0} \|u_n(t)\|_1 \\ &\leq 2E_{k_0,1} n^{\lambda+1} \rho \|u_n(t)\|_{k_0}. \end{aligned}$$

Consequently,

$$\|u_n(t)\|_{k_0} \leq \|u_n^{\text{in}}\|_{k_0} \exp\{2E_{k_0,1} n^{\lambda+1} \rho t\}, \quad t \in [0, T_n),$$

which implies, together with the mass conservation property (4.15), that  $(\|u(t)\|_{k_0} + \|u(t)\|_1) \rightarrow \infty$  as  $t \rightarrow T_n$ . Hence  $T_n = \infty$ .

Finally, since  $u^{\text{in}}$  satisfies (2.5), by Lemma 3.4, we have

$$\int_{x_1}^\infty x_2^{k_0+1} u_n(t, x_2) dx_2 \leq \int_{x_1}^\infty x_2^{k_0+1} u_n^{\text{in}}(x_2) dx_2 \leq \int_{x_1}^\infty x_2^{k_0+1} u^{\text{in}}(x_2) dx_2$$

for all  $(t, x_1) \in [0, \infty) \times (0, \infty)$ . In particular, choosing  $x_1 = 0$  leads to the estimate

$$\boldsymbol{\mu}_{k_0+1}(u_n(t)) \leq \boldsymbol{\mu}_{k_0+1}(u^{\text{in}}), \quad t \in [0, \infty),$$

while the choice  $x_1 = 2n$  and (4.3) implies that

$$\int_{2n}^{\infty} x_2^{k_0+1} u_n(t, x_2) dx_2 \leq \int_{2n}^{\infty} x_2^{k_0+1} u^{\text{in}}(x_2) dx_2 = 0, \quad t \in [0, \infty).$$

Hence, owing to the non-negativity of  $u_n$ ,

$$u_n(t, x_1) = 0 \quad \text{for a.e. } x_1 > 2n \text{ and } t \in [0, \infty),$$

and the proof of Proposition 4.1 is complete.  $\square$

Having proved the well-posedness of (4.4)-(4.5) for each  $n \geq 1$ , we now turn to the derivation of estimates which do not depend on  $n \geq 1$ , in order to be able to find cluster points of the sequence  $(u_n)_{n \geq 1}$  as  $n \rightarrow \infty$ , which are natural candidates as solutions to (1.1)-(1.2) according to the choice (4.2) of  $\Phi_n$ . Recalling (4.7) and (4.9), we have already obtained estimates on  $(u_n)_{n \geq 1}$  for large size particles which do not depend on  $n \geq 1$ . The next step is to look for a similar estimate for small size particles.

**4.1. Control on small size particles.** We now study the behaviour for small sizes. To this end, we need the following lemma.

**Lemma 4.2.** *Let  $T_*$  be defined in Theorem 2.2,  $T \in (0, T_*)$  and  $C_0 > 0$  such that  $\boldsymbol{\mu}_{k_0}(u^{\text{in}}) \leq C_0$ . There exists a positive constant  $C_1(T)$  which is independent of  $n \geq 1$  and depends solely on  $\Phi$ ,  $\beta$ ,  $u^{\text{in}}$ ,  $C_0$ , and  $T$ , such that the following inequality holds for all  $t \in [0, T]$ ,*

$$\boldsymbol{\mu}_{k_0}(u_n(t)) \leq C_1(T).$$

*Proof.* For  $t \geq 0$  and  $\varsigma(x_1) = W_{k_0}(x_1) = x_1^{k_0}$ , we deduce from (1.11), (1.12), and (4.6) that

$$\begin{aligned} \frac{d}{dt} \boldsymbol{\mu}_{k_0}(u_n(t)) &\leq \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \Upsilon_{\varsigma}(x_1, x_2) \Phi(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq E_{k_0, 1} \int_0^{\infty} \int_0^{\infty} (x_1^{k_0} + x_2^{k_0}) x_1^{\lambda_1} x_2^{\lambda_2} u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq E_{k_0, 1} [\boldsymbol{\mu}_{k_0+\lambda_1}(u_n(t)) \boldsymbol{\mu}_{\lambda_2}(u_n(t)) + \boldsymbol{\mu}_{k_0+\lambda_2}(u_n(t)) \boldsymbol{\mu}_{\lambda_1}(u_n(t))]. \end{aligned} \quad (4.16)$$

As  $(\lambda_1, \lambda_2) \in [k_0, 1]^2$ , we obtain the following moment estimates by means of Hölder's inequality and (4.7)

$$\boldsymbol{\mu}_{\lambda_1}(u_n) \leq \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_1}{1-k_0}} \boldsymbol{\mu}_1(u_n)^{\frac{\lambda_1-k_0}{1-k_0}} \leq \rho^{\frac{\lambda_1-k_0}{1-k_0}} \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_1}{1-k_0}},$$

and

$$\boldsymbol{\mu}_{\lambda_2}(u_n) \leq \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_2}{1-k_0}} \boldsymbol{\mu}_1(u_n)^{\frac{\lambda_2-k_0}{1-k_0}} \leq \rho^{\frac{\lambda_2-k_0}{1-k_0}} \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_2}{1-k_0}}.$$

In addition, when  $k_0 + \lambda_1 \in [k_0, 1]$ , we have

$$\boldsymbol{\mu}_{k_0+\lambda_1}(u_n) \leq \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-k_0-\lambda_1}{1-k_0}} \boldsymbol{\mu}_1(u_n)^{\frac{\lambda_1}{1-k_0}} \leq \rho^{\frac{\lambda_1}{1-k_0}} \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-k_0-\lambda_1}{1-k_0}},$$

and when  $k_0 + 1 \geq k_0 + \lambda_1 \geq 1$ , we get

$$\begin{aligned} \boldsymbol{\mu}_{k_0+\lambda_1}(u_n) &\leq \boldsymbol{\mu}_1(u_n)^{\frac{1-\lambda_1}{k_0}} \boldsymbol{\mu}_{k_0+1}(u_n)^{\frac{k_0+\lambda_1-1}{k_0}} \leq \rho^{\frac{1-\lambda_1}{k_0}} \boldsymbol{\mu}_{k_0+1}(u_n)^{\frac{k_0+\lambda_1-1}{k_0}} \\ &\leq \rho^{\frac{1-\lambda_1}{k_0}} \boldsymbol{\mu}_{k_0+1}(u^{\text{in}})^{\frac{k_0+\lambda_1-1}{k_0}}, \end{aligned}$$

where we have used (4.9) to obtain the last estimate.

Thus

$$\boldsymbol{\mu}_{k_0+\lambda_1}(u_n) \leq c_1 \boldsymbol{\mu}_{k_0}(u_n)^{\frac{(1-k_0-\lambda_1)_+}{1-k_0}},$$

with

$$c_1 := \max \left\{ \rho^{\frac{\lambda_1}{1-k_0}}, \rho^{\frac{1-\lambda_1}{k_0}} \boldsymbol{\mu}_{k_0+1}(u^{\text{in}})^{\frac{k_0+\lambda_1-1}{k_0}} \right\} > 0.$$

Similarly,

$$\boldsymbol{\mu}_{k_0+\lambda_2}(u_n) \leq c_2 \boldsymbol{\mu}_{k_0}(u_n)^{\frac{(1-k_0-\lambda_2)_+}{1-k_0}},$$

with

$$c_2 := \max \left\{ \rho^{\frac{\lambda_2}{1-k_0}}, \rho^{\frac{1-\lambda_2}{k_0}} \boldsymbol{\mu}_{k_0+1}(u^{\text{in}})^{\frac{k_0+\lambda_2-1}{k_0}} \right\} > 0.$$

Inserting the above estimates on  $\boldsymbol{\mu}_{\lambda_i}$  and  $\boldsymbol{\mu}_{k_0+\lambda_i}$  for  $i = \{1, 2\}$  into the inequality (4.16), we get

$$\frac{d}{dt} \boldsymbol{\mu}_{k_0}(u_n(t)) \leq c_3 \left[ \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_2+(1-k_0-\lambda_1)_+}{1-k_0}} + \boldsymbol{\mu}_{k_0}(u_n)^{\frac{1-\lambda_1+(1-k_0-\lambda_2)_+}{1-k_0}} \right], \quad (4.17)$$

with

$$c_3 := \max \left\{ c_1 \rho^{\frac{\lambda_2-k_0}{1-k_0}}, c_2 \rho^{\frac{\lambda_1-k_0}{1-k_0}} \right\} > 0.$$

Now, the estimate to be deduced from (4.17) obviously depends on the range of the powers of  $\boldsymbol{\mu}_{k_0}(u_n)$  in the right-hand side of (4.17), which requires to split the analysis.

(1) If  $\lambda = \lambda_1 + \lambda_2 \in [2k_0, 1)$ , then

$$k_0 + \lambda_1 \leq k_0 + \lambda_2 \leq \lambda_1 + \lambda_2 < 1,$$

which implies that

$$\frac{1 - \lambda_1 + (1 - k_0 - \lambda_2)_+}{1 - k_0} = \frac{1 - k_0 + 1 - \lambda}{1 - k_0} > 1$$

and

$$\frac{1 - \lambda_2 + (1 - k_0 - \lambda_1)_+}{1 - k_0} = \frac{1 - k_0 + 1 - \lambda}{1 - k_0} > 1.$$

In that case, the right-hand side of the differential inequality (4.17) is a superlinear function of  $\boldsymbol{\mu}_{k_0}(u_n)$  and integrating (4.17) gives

$$\begin{aligned} \boldsymbol{\mu}_{k_0}(u_n(t)) &\leq \left[ \boldsymbol{\mu}_{k_0}(u_n^{\text{in}})^{-\frac{1-\lambda}{1-k_0}} - \frac{2(1-\lambda)c_3}{1-k_0} t \right]^{-\frac{1-k_0}{1-\lambda}} \\ &\leq \left[ \boldsymbol{\mu}_{k_0}(u^{\text{in}})^{-\frac{1-\lambda}{1-k_0}} - \frac{2(1-\lambda)c_3}{1-k_0} t \right]^{-\frac{1-k_0}{1-\lambda}}, \end{aligned}$$



provided  $t \in [0, T_{k_0})$  with

$$T_{k_0} := \frac{1 - k_0}{2(1 - \lambda)c_3} \boldsymbol{\mu}_{k_0}(u^{\text{in}})^{-\frac{1-\lambda}{1-k_0}}. \quad (4.18)$$

We have thus shown that, for  $T \in (0, T_{k_0})$  and  $t \in [0, T]$ ,

$$\boldsymbol{\mu}_{k_0}(u_n(t)) \leq C_1(T) := \left[ \boldsymbol{\mu}_{k_0}(u^{\text{in}})^{-\frac{1-\lambda}{1-k_0}} - \frac{2(1-\lambda)c_3}{1-k_0} T \right]^{-\frac{1-k_0}{1-\lambda}},$$

which completes the proof of Lemma 4.2 for  $\lambda \in [2k_0, 1)$ .

(2) If  $\lambda = \lambda_1 + \lambda_2 \in [1, 2]$ , then

$$\frac{1 - \lambda_1}{1 - k_0} + \frac{(1 - k_0 - \lambda_2)_+}{1 - k_0} = \begin{cases} \frac{1 - \lambda_1}{1 - k_0} \leq 1 & \text{if } k_0 + \lambda_2 \geq 1, \\ 1 + \frac{1 - \lambda}{1 - k_0} \leq 1 & \text{if } k_0 + \lambda_2 < 1, \end{cases}$$

and, similarly,

$$\frac{1 - \lambda_2 + (1 - k_0 - \lambda_1)_+}{1 - k_0} \leq 1.$$

Consequently, by Young's inequality and (4.17),

$$\frac{d}{dt} \boldsymbol{\mu}_{k_0}(u_n(t)) \leq \frac{2c_3}{1 - k_0} [1 + \boldsymbol{\mu}_{k_0}(u_n(t))],$$

from which we readily obtain that

$$\boldsymbol{\mu}_{k_0}(u_n(t)) \leq \left[ 1 + \boldsymbol{\mu}_{k_0}(u_n^{\text{in}}) \right] e^{\frac{2c_3 t}{1 - k_0}} \leq \left[ 1 + \boldsymbol{\mu}_{k_0}(u^{\text{in}}) \right] e^{\frac{2c_3 t}{1 - k_0}}$$

for  $t \geq 0$ . Introducing

$$C_1(T) := \left[ 1 + \boldsymbol{\mu}_{k_0}(u^{\text{in}}) \right] e^{\frac{2c_3 T}{1 - k_0}}$$

for  $T > 0$ , we obtain the estimate stated in Lemma 4.2. □

Next, we derive a more precise approximation for small sizes. For that purpose, we recall the following variant of the de la Vallée-Poussin theorem [11] which we apply to  $u^{\text{in}} \in X_{k_0}$  and can be established as [18, Lemma A.1].

**Proposition 4.3.** *Recalling that  $k_0 \in (0, 1)$  and  $p \in (1, 1 + k_0)$  are defined in (4.1), we pick  $\epsilon \in (-k_0, (p - k_0 - 1)/p)$ . There exists a non-negative, convex, and non-increasing function  $\psi_0 \in C^1((0, \infty))$  such that*

$$\boldsymbol{\mu}_{\psi_0}(u^{\text{in}}) := \int_0^\infty \psi_0(x_1) |u^{\text{in}}(x_1)| dx_1 < \infty \quad (4.19)$$

and

$$\lim_{x \rightarrow 0} \frac{\psi_0(x_1)}{x_1^{k_0}} = \infty, \quad \lim_{x \rightarrow 0} x_1^\epsilon \psi_0(x_1) = 0, \quad x_1 \mapsto x_1^\epsilon \psi_0(x_1) \text{ is non-decreasing.} \quad (4.20)$$

**Lemma 4.4.** For  $T \in (0, T_*)$ , there exists a positive constant  $C_2(T)$  which is independent of  $n \geq 1$  and depends solely on  $\Phi$ ,  $\beta$ ,  $u^{\text{in}}$ , and  $T$ , such that the following inequality holds for all  $t \in [0, T]$ ,

$$\boldsymbol{\mu}_{\psi_0}(u_n(t)) := \int_0^\infty \psi_0(x_1)u_n(t, x_1)dx_1 \leq C_2(T). \quad (4.21)$$

*Proof.* We first observe that, by Young's inequality and (1.12),

$$\Phi(x_1, x_2) \leq 2(x_1^{k_0} + x_1)(x_2^{k_0} + x_2), \quad (x_1, x_2) \in (0, \infty)^2. \quad (4.22)$$

Now, let  $t \in [0, T]$ . By (1.7), (1.8), (2.4), (4.22), and the non-negativity of  $\psi_0$ ,

$$\begin{aligned} \frac{d}{dt} \boldsymbol{\mu}_{\psi_0}(u_n(t)) &\leq \frac{1}{2} \int_0^\infty \int_0^\infty \Upsilon_{\psi_0}(x_1, x_2) \Phi(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq \int_0^\infty \int_0^\infty \int_0^{x_1} \psi_0(x_3) \beta_*(x_3, x_1, x_2) (x_1^{k_0} + x_1) (x_2^{k_0} + x_2) \\ &\quad \times u_n(t, x_1) u_n(t, x_2) dx_3 dx_2 dx_1 \\ &\quad + \int_0^\infty \int_0^\infty \int_0^{x_2} \psi_0(x_3) \beta_*(x_3, x_2, x_1) (x_1^{k_0} + x_1) (x_2^{k_0} + x_2) \\ &\quad \times u_n(t, x_1) u_n(t, x_2) dx_3 dx_2 dx_1. \end{aligned} \quad (4.23)$$

Next, from (1.11), (4.20) (with the above choice of  $p$  and  $\epsilon$ ), and Hölder's inequality,

$$\begin{aligned} \int_0^{x_1} \psi_0(x_3) \beta_*(x_3, x_1, x_2) dx_3 &= \int_0^{x_1} x_3^\epsilon \psi_0(x_3) x_3^{(-\epsilon - \frac{k_0}{p})} x_3^{\frac{k_0}{p}} \beta_*(x_3, x_1, x_2) dx_3 \\ &\leq x_1^\epsilon \psi_0(x_1) \left( \int_0^{x_1} x_3^{\frac{(-k_0 - p\epsilon)}{p-1}} dx_3 \right)^{\frac{p-1}{p}} \\ &\quad \times \left( \int_0^{x_1} x_3^{k_0} \beta_*(x_3, x_1, x_2)^p dx_3 \right)^{\frac{1}{p}} \\ &\leq B_{k_0, p} \psi_0(x_1), \end{aligned}$$

where

$$B_{k_0, p} := \left[ \left( \frac{p-1}{p-1-\epsilon p - k_0} \right)^{(p-1)} E_{k_0, p} \right]^{\frac{1}{p}}.$$

Now we deduce from the above inequality and (4.23) that

$$\frac{d}{dt} \boldsymbol{\mu}_{\psi_0}(u_n(t)) \leq 2B_{k_0, p} \int_0^\infty \int_0^\infty \psi_0(x_1) (x_2^{k_0} + x_2) (x_1^{k_0} + x_1) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1.$$

Since

$$\begin{aligned} \int_0^\infty \psi_0(x_1) (x_1^{k_0} + x_1) u_n(t, x_1) dx_1 &\leq 2 \int_0^1 \psi_0(x_1) u_n(t, x_1) dx_1 \\ &\quad + 2\psi_0(1) \int_1^\infty x_1 u_n(t, x_1) dx_1 \\ &\leq 2\boldsymbol{\mu}_{\psi_0}(u_n(t)) + 2\psi_0(1)\rho \end{aligned}$$

by (4.7) and Proposition 4.3, we further obtain, using again (4.7), along with Lemma 4.2,

$$\begin{aligned} \frac{d}{dt} \boldsymbol{\mu}_{\psi_0}(u_n(t)) &\leq 4B_{k_0,p}[\boldsymbol{\mu}_{\psi_0}(u_n(t)) + \psi_0(1)\rho] [\rho + \boldsymbol{\mu}_{k_0}(u_n(t))] \\ &\leq 2B_{k_0,p}[\boldsymbol{\mu}_{\psi_0}(u_n(t)) + \psi(1)\rho][\rho + C_1(T)]. \end{aligned}$$

Integrating the above inequality with respect to time gives the desired result.  $\square$

**4.2. Uniform integrability.** Having derived estimates on the sequence  $(u_n)_{n \geq 1}$  for large and small sizes, we shall now discuss the uniform integrability of this sequence in a suitably chosen weighted  $L^1$ -space. In that direction the main tool is another variant of the de la Vallée Poussin theorem [4, Theorem 7.1.6] which we apply here to  $u^{\text{in}} \in X_{k_0}$ . It guarantees that there exists a function  $\psi \in C^1([0, \infty))$  satisfying the following properties:  $\psi$  is a convex function with  $\psi(0) = \psi'(0) = 0$  and  $\psi'$  is concave and positive on  $(0, \infty)$ ,

$$\int_0^\infty x_1^{k_0} \psi(u^{\text{in}}(x_1)) dx_1 < \infty, \quad (4.24)$$

$$\lim_{s \rightarrow \infty} \psi'(s) = \lim_{s \rightarrow \infty} \frac{\psi(s)}{s} = \infty, \quad (4.25)$$

and, for all  $r \in (1, 2]$ ,

$$\lim_{s \rightarrow \infty} \frac{\psi'(s)}{s^{r-1}} = \lim_{s \rightarrow \infty} \frac{\psi(s)}{s^r} = 0, \quad (4.26)$$

which also guarantees that

$$B_r := \sup_{s \geq 0} \left\{ \frac{\psi(s)}{s^r} \right\} < \infty. \quad (4.27)$$

Also, we recall the following properties of the convex function  $\psi$  (see [4, Proposition 7.1.9 (a) & (b)]),

$$\psi(s) \leq s\psi'(s) \leq 2\psi(s), \quad s \geq 0, \quad (4.28)$$

$$r\psi'(s) \leq \psi(s) + \psi(r), \quad (r, s) \in [0, \infty)^2. \quad (4.29)$$

In particular, it follows from (4.29) and the non-negativity of  $u_n$  and  $\beta$  that

$$\psi'(u_n(t, x_3))\beta(x_3, x_1, x_2) \leq \psi(u_n(t, x_3)) + \psi(\beta(x_3, x_1, x_2)), \quad x_3 \in (0, x_1 + x_2), \quad (4.30)$$

for all  $(x_1, x_2) \in (0, \infty)^2$ .

**Lemma 4.5.** *For  $T \in (0, T_*)$ , there exists a positive constant  $C_3(T)$ , which is independent of  $n \geq 1$  and depends solely on  $\Phi$ ,  $b$ ,  $u^{\text{in}}$ , and  $T$ , such that the following inequality holds for all  $t \in [0, T]$ ,*

$$\boldsymbol{\mu}_\xi(\psi(u_n(t))) \leq C_3(T),$$

where  $\xi(x_1) = \min\{x_1, x_1^{k_0}\}$  for  $x_1 \in (0, \infty)$ .

*Proof.* Let  $t \in [0, T]$ . Thanks to (1.12), (4.6), (4.22), (4.30), and the non-negativity of both  $\xi$ ,  $u_n$  and  $\psi'$ , we find

$$\frac{d}{dt} \boldsymbol{\mu}_\xi(\psi(u_n(t))) \leq \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^{x_1+x_2} \xi(x_3) \psi'(u_n(t, x_3)) \beta(x_3, x_1, x_2) \Phi(x_1, x_2)$$

$$\begin{aligned}
& \times u_n(t, x_1)u_n(t, x_2)dx_3dx_2dx_1 \\
& \leq \int_0^\infty \int_0^\infty \int_0^{x_1+x_2} \xi(x_3)\psi(u_n(t, x_3))(x_1 + x_1^{k_0})(x_2 + x_2^{k_0}) \\
& \quad \times u_n(t, x_1)u_n(t, x_2)dx_3dx_2dx_1 \\
& \quad + \int_0^\infty \int_0^\infty \int_0^{x_1+x_2} \xi(x_3)\psi(\beta(x_3, x_1, x_2))x_1^{\lambda_1}x_2^{\lambda_2} \\
& \quad \times u_n(t, x_1)u_n(t, x_2)dx_3dx_2dx_1. \quad (4.31)
\end{aligned}$$

We now estimate the following integral with the help of (1.7), (1.8), (1.11), (4.1), and (4.27)

$$\begin{aligned}
\int_0^{x_1+x_2} \xi(x_3)\psi(\beta(x_3, x_1, x_2))dx_3 &= \int_0^{x_1+x_2} \xi(x_3)\frac{\psi(\beta(x_3, x_1, x_2))}{\beta(x_3, x_1, x_2)^p}\beta(x_3, x_1, x_2)^pdx_3 \\
&\leq B_p \int_0^{x_1+x_2} \xi(x_3)\beta(x_3, x_1, x_2)^pdx_3 \\
&\leq B_p \int_0^{x_1} \xi(x_3)\beta_*(x_3, x_1, x_2)^pdx_3 \\
&\quad + B_p \int_0^{x_2} \xi(x_3)\beta_*(x_3, x_2, x_1)^pdx_3.
\end{aligned}$$

Now, either  $x_1 \in (0, 1)$  and

$$\begin{aligned}
\int_0^{x_1} \xi(x_3)\beta_*(x_3, x_1, x_2)^pdx_3 &\leq \int_0^{x_1} x_3\beta_*(x_3, x_1, x_2)^pdx_3 \\
&\leq \int_0^{x_1} x_3^{k_0}\beta_*(x_3, x_1, x_2)^pdx_3 \\
&\leq E_{k_0,p}x_1^{k_0+1-p},
\end{aligned}$$

or  $x_1 \in (1, \infty)$  and

$$\int_0^{x_1} \xi(x_3)\beta_*(x_3, x_1, x_2)^pdx_3 \leq \int_0^{x_1} x_3^{k_0}\beta_*(x_3, x_1, x_2)^pdx_3 \leq E_{k_0,p}x_1^{k_0+1-p}.$$

Therefore, for all  $x_1 \in (0, \infty)$ , we have

$$\int_0^{x_1} \xi(x_3)\beta_*(x_3, x_1, x_2)^pdx_3 \leq E_{k_0,p}x_1^{k_0+1-p}.$$

Similarly, for all  $x_2 \in (0, \infty)$ , we have

$$\int_0^{x_2} \xi(x_3)\beta_*(x_3, x_2, x_1)^pdx_3 \leq E_{k_0,p}x_2^{k_0+1-p}.$$

We next deduce from (4.1), (4.7), (4.31), Lemma 4.2, and the above estimates that

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\mu}_\xi(\psi(u_n(t))) &\leq (\rho + C_1(T))^2\boldsymbol{\mu}_\xi(\psi(u_n(t))) \\
&\quad + B_p E_{k_0,p} \int_0^\infty \int_0^\infty \left[ x_1^{k_0+1-p} + x_2^{k_0+1-p} \right] x_1^{\lambda_1} x_2^{\lambda_2} u_n(t, x_1) u_n(t, x_2) dx_2 dx_1
\end{aligned}$$

$$\begin{aligned} &\leq (\rho + C_1(T))^2 \boldsymbol{\mu}_\xi(\psi(u_n(t))) + B_p E_{k_0,p} \boldsymbol{\mu}_{k_0+1-p+\lambda_1}(u_n(t)) \boldsymbol{\mu}_{\lambda_2}(u_n(t)) \\ &\quad + B_p E_{k_0,p} \boldsymbol{\mu}_{\lambda_1}(u_n(t)) \boldsymbol{\mu}_{k_0+1-p+\lambda_2}(u_n(t)). \end{aligned}$$

Since  $\lambda_i \in [k_0, 1]$  and  $k_0 + 1 - p + \lambda_i \in [k_0, k_0 + 1]$  for  $i \in \{1, 2\}$ , we further use Young's inequality, along with (4.7), (4.8), and Lemma 4.2, to find

$$\begin{aligned} \frac{d}{dt} \boldsymbol{\mu}_\xi(\psi(u_n(t))) &\leq (\rho + C_1(T))^2 \boldsymbol{\mu}_\xi(\psi(u_n(t))) \\ &\quad + 2B_p E_{k_0,p} (\boldsymbol{\mu}_{k_0}(u_n(t)) + \boldsymbol{\mu}_{k_0+1}(u_n(t))) (\boldsymbol{\mu}_{k_0}(u_n(t)) + \boldsymbol{\mu}_1(u_n(t))) \\ &\leq (\rho + C_1(T))^2 \boldsymbol{\mu}_\xi(\psi(u_n(t))) + 2B_p E_{k_0,p} (C_1(T) + \boldsymbol{\mu}_{k_0+1}(u^{\text{in}})) (\rho + C_1(T)). \end{aligned}$$

Therefore, by integrating the above inequality with respect to time, we complete the proof of Lemma 4.5.  $\square$

**4.3. Time equicontinuity.** Finally, we prove the time equicontinuity of the sequence  $(u_n)_{n \geq 1}$ .

**Lemma 4.6.** *Let  $T \in (0, T_*)$ . There exists a positive constant  $C_4(T)$  which is independent of  $n \geq 1$  and depends only on  $\Phi$ ,  $b$ ,  $u^{\text{in}}$ , and  $T$ , such that the following inequality holds for all  $0 \leq t_1 < t_2 \leq T$ ,*

$$\int_0^\infty x_1^{k_0} |u_n(t_2, x_1) - u_n(t_1, x_1)| dx_1 \leq C_4(T)(t_2 - t_1). \quad (4.32)$$

*Proof.* For  $t \in [0, T]$ , we infer from Lemma 3.1, Lemma 4.2, (4.6), (4.7), (4.8), and (4.22) that

$$\begin{aligned} &\int_0^\infty x_1^{k_0} |\partial_t u_n(t, x_1)| dx_1 \\ &\leq \frac{1}{2} \int_0^\infty \int_0^\infty |\Upsilon_{W_{k_0}}(x_1, x_2)| \Phi(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq \frac{E_{k_0}}{2} \int_0^\infty \int_0^\infty (x_1^{k_0} + x_2^{k_0}) \Phi(x_1, x_2) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq E_{k_0} \int_0^\infty \int_0^\infty (x_1^{k_0} + x_2^{k_0}) (x_1 + x_1^{k_0}) (x_2 + x_2^{k_0}) \\ &\quad \times u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq 2E_{k_0} \int_0^\infty \int_0^\infty x_1^{k_0} (x_1 + x_1^{k_0}) (x_2 + x_2^{k_0}) u_n(t, x_1) u_n(t, x_2) dx_2 dx_1 \\ &\leq 2E_{k_0} (\rho + C_1(T)) \left( \boldsymbol{\mu}_{1+k_0}(u^{\text{in}}) + \boldsymbol{\mu}_{2k_0}(u_n(t)) \right). \end{aligned}$$

Since  $k_0 < 2k_0 < k_0 + 1$ , by Hölder's inequality, we get, using once more Lemma 4.2 and (4.8),

$$\boldsymbol{\mu}_{2k_0}(u_n(t)) \leq \boldsymbol{\mu}_{k_0}(u_n(t))^{1-k_0} \boldsymbol{\mu}_{1+k_0}(u_n(t))^{k_0} \leq C_1(T)^{1-k_0} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{k_0}.$$

Consequently,

$$\int_0^\infty x_1^{k_0} |\partial_t u_n(t, x_1)| dx_1 \leq 2E_{k_0}(\rho + C_1(T)) \left( \mu_{1+k_0}(u^{\text{in}}) + C_1(T)^{1-k_0} \mu_{1+k_0}(u^{\text{in}})^{k_0} \right).$$

Since

$$\int_0^\infty x_1^{k_0} |u_n(t_2, x_1) - u_n(t_1, x_1)| dx_1 \leq \int_{t_1}^{t_2} \int_0^\infty x_1^{k_0} |\partial_t u_n(t, x_1)| dx_1 dt$$

for  $0 \leq t_1 < t_2 \leq T$ , integrating the above inequality with respect to time over  $(t_1, t_2)$  provides the result of Lemma 4.6.  $\square$

The results obtained so far can now be utilized to establish Theorem 2.2, which we now proceed to prove.

*Proof of Theorem 2.2 . Compactness:* Recall that  $\xi(x_1) = \min\{x_1, x_1^{k_0}\}$  for  $x_1 \geq 0$ . Let  $T \in (0, T_*)$  be fixed, and let us define

$$\mathcal{E}(T) := \{u_n(t) : t \in [0, T], n \geq 1\}.$$

Consider a measurable subset  $A$  of  $(0, \infty)$  with finite Lebesgue measure, and let  $M > 1$ . Then, due to (4.7), Lemma 4.5, and the monotonicity of  $s \mapsto \psi(s)/s$ , we have, for any  $t \in [0, T]$  and  $n \geq 1$ ,

$$\begin{aligned} \int_A \xi(x_1) u_n(t, x_1) dx_1 &\leq \int_{A \cap (0, M)} \xi(x_1) u_n(t, x_1) dx_1 + \int_M^\infty \xi(x_1) u_n(t, x_1) dx_1 \\ &\leq \int_{A \cap (0, M)} \xi(x_1) u_n(t, x_1) \mathbf{1}_{(0, M)}(u_n(t, x_1)) dx_1 \\ &\quad + \int_{A \cap (0, M)} \xi(x_1) u_n(t, x_1) \mathbf{1}_{[M, \infty)}(u_n(t, x_1)) dx_1 + \frac{\rho}{M^{1-k_0}} \\ &\leq M^2 |A| + \frac{M}{\psi(M)} \int_0^M \xi(x_1) \psi(u_n(t, x_1)) dx_1 + \frac{\rho}{M^{1-k_0}} \\ &\leq M^2 |A| + \frac{M}{\psi(M)} C_3(T) + \frac{\rho}{M^{1-k_0}}. \end{aligned}$$

Therefore the modulus of uniform integrability

$$\eta\{\mathcal{E}(T); X_\xi\} := \limsup_{\varepsilon \rightarrow 0} \left\{ \int_A \xi(x_1) v(x_1) dx_1 : v \in \mathcal{E}(T), A \subset (0, \infty), |A| \leq \varepsilon \right\}$$

of  $\mathcal{E}(T)$  in  $X_\xi$  satisfies

$$\eta\{\mathcal{E}(T); X_\xi\} \leq \frac{M}{\psi(M)} C_3(T) + \frac{\rho}{M^{1-k_0}}$$

for all  $M > 1$ . Recalling that  $\psi$  satisfies (4.25) and  $k_0 \in (0, 1)$ , we may let  $M \rightarrow \infty$  in the previous inequality to obtain that

$$\eta\{\mathcal{E}(T); X_\xi\} = 0. \tag{4.33}$$

Also, by (4.7),

$$\int_M^\infty \xi(x_1) u_n(t, x_1) dx_1 \leq \frac{\rho}{M^{1-k_0}}$$

for  $n \geq 1$ ,  $t \in [0, T]$ , and  $M > 1$ , so that

$$\lim_{M \rightarrow \infty} \sup_{v \in \mathcal{E}(T)} \left\{ \int_M^\infty \xi(x_1) v(x_1) dx_1 \right\} = 0. \quad (4.34)$$

According to (4.33), (4.34), and the Dunford-Pettis theorem [4, Theorem 7.1.3],  $\mathcal{E}(T)$  is a relatively sequentially weakly compact subset of  $X_\xi$ . Applying a version of the Arzelà-Ascoli theorem [4, Theorem 7.1.16], we can deduce from Lemma 4.6 that  $(u_n)_{n \geq 1}$  is relatively sequentially compact in  $\mathcal{C}([0, T], X_{\xi, w})$ . Since  $T$  is arbitrary in  $(0, T_*)$ , we can use a diagonal process to obtain a subsequence of  $(u_n)_{n \geq 1}$  (not relabeled) and  $u \in \mathcal{C}([0, T_*], X_{\xi, w})$  such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty \xi(x_1) (u_n - u)(t, x_1) \varsigma(x_1) dx_1 \right| = 0 \quad (4.35)$$

for all  $\varsigma \in L^\infty((0, \infty))$  and  $T \in (0, T_*)$ . Thanks to (4.35), we extend the validity of (4.8) and Lemma 4.4 from  $u_n$  to  $u$  as shown in [17] and obtain that, for  $T \in (0, T_*)$ ,

$$\int_{x_1}^\infty x_2^{k_0+1} u(t, x_2) dx_2 \leq \int_{x_1}^\infty x_2^{k_0+1} u^{\text{in}}(x_2) dx_2, \quad (t, x_1) \in [0, T] \times (0, \infty), \quad (4.36)$$

and

$$\boldsymbol{\mu}_{\psi_0}(u(t)) \leq C_2(T), \quad t \in [0, T]. \quad (4.37)$$

We now extend (4.35) to the weak topology of  $X_{k_0} \cap X_{k_0+1}$ , i.e.,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty (x_1^{k_0} + x_1^{k_0+1}) (u_n - u)(t, x_1) \varsigma(x_1) dx_1 \right| = 0 \quad (4.38)$$

for all  $\varsigma \in L^\infty((0, \infty))$  and  $T \in (0, T_*)$ . Indeed, let  $T \in (0, T_*)$ ,  $t \in [0, T]$ ,  $\varsigma \in L^\infty((0, \infty))$ , and  $M > 1$ . Then,

$$\begin{aligned} & \left| \int_0^\infty (x_1^{k_0} + x_1^{k_0+1}) (u_n - u)(t, x_1) \varsigma(x_1) dx_1 \right| \\ & \leq \underbrace{2 \|\varsigma\|_{L^\infty} \int_0^{1/M} x_1^{k_0} (u_n + u)(t, x_1) dx_1}_{I_{1,n}(t)} \\ & \quad + \underbrace{\left| \int_{1/M}^M (x_1^{k_0} + x_1^{k_0+1}) (u_n - u)(t, x_1) \varsigma(x_1) dx_1 \right|}_{I_{2,n}(t)} \\ & \quad + \underbrace{2 \|\varsigma\|_{L^\infty} \int_M^\infty x_1^{k_0+1} (u_n + u)(t, x_1) dx_1}_{I_{3,n}(t)}. \end{aligned}$$

First, thanks to (4.21), (4.37), and the monotonicity of  $\psi_0$ ,

$$\begin{aligned} I_{1,n}(t) & := 2 \|\varsigma\|_{L^\infty} \int_0^{1/M} x_1^{k_0} (u_n + u)(t, x_1) dx_1 \\ & \leq \frac{2 \|\varsigma\|_{L^\infty}}{M^{k_0} \psi_0(1/M)} \int_0^{1/M} \psi_0(x_1) (u_n + u)(t, x_1) dx_1 \end{aligned}$$

$$\leq \frac{2\|\varsigma\|_{L^\infty}}{M^{k_0}\psi_0(1/M)}[\boldsymbol{\mu}_{\psi_0}(u_n(t)) + \boldsymbol{\mu}_{\psi_0}(u(t))] \leq \frac{4\|\varsigma\|_{L^\infty}}{M^{k_0}\psi_0(1/M)}C_2(T). \quad (4.39)$$

Next, by (4.8) and (4.36),

$$I_{3,n}(t) := 2\|\varsigma\|_{L^\infty} \int_M^\infty x_1^{k_0+1}(u_n + u)(t, x_1)dx_1 \leq 4\|\varsigma\|_{L^\infty} \int_M^\infty x_1^{k_0+1}u^{\text{in}}(x_1)dx_1. \quad (4.40)$$

Finally, observe that

$$\left| \frac{x_1^{k_0} + x_1^{k_0+1}}{\xi(x_1)} \mathbf{1}_{(1/M, M)}(x_1)\varsigma(x_1) \right| \leq 2M\|\varsigma\|_{L^\infty},$$

which implies that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} I_{2,n}(t) = 0, \quad (4.41)$$

according to (4.35). Gathering (4.39), (4.40), and (4.41), we obtain that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty (x_1^{k_0} + x_1^{k_0+1})(u_n - u)(t, x_1)\varsigma(x_1)dx_1 \right| \\ \leq 4\|\varsigma\|_{L^\infty} \left[ \frac{C_2(T)}{M^{k_0}\psi_0(1/M)} + \int_M^\infty x_1^{k_0+1}u^{\text{in}}(x_1)dx_1 \right]. \end{aligned}$$

As  $u^{\text{in}}$  satisfies (2.5) and  $\psi_0$  satisfies (4.20), the right-hand side of the above inequality tends to zero as  $M \rightarrow \infty$ , thereby concluding the proof of (4.38).

**Mass conservation:** In particular, the convergence established in (4.38) implies that  $u \in \mathcal{C}([0, T_*], X_{1,w})$  and a straightforward consequence of (4.7) is that

$$\boldsymbol{\mu}_1(u(t)) = \lim_{n \rightarrow \infty} \boldsymbol{\mu}_1(u_n(t)) = \lim_{n \rightarrow \infty} \boldsymbol{\mu}_1(u_n^{\text{in}}) = \boldsymbol{\mu}_1(u^{\text{in}}(t)) = \rho, \quad t \in [0, T_*]. \quad (4.42)$$

**Regularity and integrability properties:** According to (4.38) and (4.42),  $u$  belongs to  $C([0, T_*], X_{k_0, w})$  and to  $L^\infty((0, T_*), X_1)$  and thus satisfies (2.1). In addition, it follows from (1.12), (4.22), and (4.36) that  $u$  satisfies (2.2).

**Limit Equation:** Now, we show that the limit function  $u$  satisfies the weak formulation (2.3). Let  $t \in (0, T_*)$  and  $\varsigma \in \mathcal{F}^{k_0}$ . On the one hand, it follows from (4.3) and (4.38) that

$$\lim_{n \rightarrow \infty} \int_0^\infty \varsigma(x_1)(u_n(t, x_1) - u_n^{\text{in}}(x_1))dx_1 = \int_0^\infty \varsigma(x_1)(u(t, x_1) - u^{\text{in}}(x_1))dx_1.$$

On the other hand, we observe that

$$\lim_{n \rightarrow \infty} \frac{\Upsilon_\varsigma(x_1, x_2)\Phi_n(x_1, x_2)}{(x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})} = \frac{\Upsilon_\varsigma(x_1, x_2)\Phi(x_1, x_2)}{(x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})},$$

for  $(x_1, x_2) \in (0, \infty)^2$  and

$$\frac{(x_1^{k_0} + x_2^{k_0})(x_1 + x_1^{k_0})(x_2 + x_2^{k_0})}{(x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})} \leq 8 \quad \text{for } (x_1, x_2) \in (0, \infty)^2,$$



which implies that, by Lemma 3.1 and (4.22),

$$\begin{aligned} \left| \frac{\Upsilon_\varsigma(x_1, x_2)\Phi_n(x_1, x_2)}{(x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})} \right| &\leq 2E_{k_0} \|\varsigma\|_{C^{0,k_0}} \frac{(x_1^{k_0} + x_2^{k_0})(x_1 + x_1^{k_0})(x_2 + x_2^{k_0})}{(x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})} \\ &\leq 16E_{k_0} \|\varsigma\|_{C^{0,k_0}}. \end{aligned}$$

Since the convergence obtained in (4.38) implies that

$$[(\tau, x_1, x_2) \mapsto u_n(\tau, x_1)u_n(\tau, x_2)] \rightharpoonup [(\tau, x_1, x_2) \mapsto u(\tau, x_1)u(\tau, x_2)]$$

in  $L^1((0, t) \times (0, \infty)^2, (x_1^{k_0} + x_1^{k_0+1})(x_2^{k_0} + x_2^{k_0+1})dx_2dx_1d\tau)$ , we infer from [15, Proposition 2.61] that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_\varsigma(x_1, x_2)\Phi_n(x_1, x_2)u_n(\tau, x_1)u_n(\tau, x_2)dx_2dx_1d\tau \\ = \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_\varsigma(x_1, x_2)\Phi(x_1, x_2)u(\tau, x_1)u(\tau, x_2)dx_2dx_1d\tau. \end{aligned}$$

We have thus verified that  $u$  satisfies all the conditions for being a weak solution to (1.1)-(1.2) on  $[0, T_*)$ , and it is also mass-conserving on  $[0, T_*)$ , according to (4.42).  $\square$

## 5. UNIQUENESS

*Proof of Theorem 2.3.* Let  $u_1$  and  $u_2$  be two weak solutions to (1.1)-(1.2) in the sense of Definition 2.1 on  $[0, T_1)$  and  $[0, T_2)$ , respectively, both satisfying (2.6) for each  $t \in (0, \min\{T_1, T_2\})$ . We set  $e_1 := u_1 - u_2$ ,  $e_2 := u_1 + u_2$ ,  $\Sigma := \text{sign}(u_1 - u_2)$ ,  $w(x_1) := \max\{x_1^{k_0}, x_1^{1+k_0}\}$ ,  $x_1 > 0$ , and we infer from (2.3) that, for  $t \in (0, \min\{T_1, T_2\})$ ,

$$\frac{d}{dt} \int_0^\infty w(x_1)|e_1(t, x_1)| \leq \frac{1}{2} \int_0^\infty \int_0^\infty Q(t, x_1, x_2)e_2(t, x_1)|e_1(t, x_2)|dx_2dx_1, \quad (5.1)$$

where

$$\begin{aligned} Q(t, x_1, x_2) &:= \Phi(x_1, x_2)\Upsilon_{w\Sigma(t)}(x_1, x_2)\Sigma(t, x_2) \\ &\leq \Phi(x_1, x_2) \left[ \int_0^{x_1+x_2} w(x_3)\beta(x_3, x_1, x_2)dx_3 + w(x_1) - w(x_2) \right]. \end{aligned}$$

We now estimate  $Q(t, x_1, x_2)$  with the help of (1.7), (1.11), and (1.12) and split the analysis according to the range of  $(x_1, x_2)$ .

*Case 1.* If  $(x_1, x_2) \in (0, 1)^2$ , then

$$\begin{aligned} Q(t, x_1, x_2) &\leq \Phi(x_1, x_2) \left[ \int_0^{x_1+x_2} x_3^{k_0}\beta(x_3, x_1, x_2)dx_3 + x_1^{k_0} - x_2^{k_0} \right] \\ &\leq 2(x_1x_2)^{k_0} \left[ (E_{k_0,1} + 1)x_1^{k_0} + (E_{k_0,1} - 1)x_2^{k_0} \right] \\ &\leq 4E_{k_0,1}x_1^{k_0}w(x_2), \end{aligned}$$

recalling that  $E_{k_0,1} \geq 1$ .

*Case 2.* If  $(x_1, x_2) \in (0, 1) \times (1, \infty)$ , then

$$Q(t, x_1, x_2) \leq \Phi(x_1, x_2) \left[ \int_0^{x_1} x_3^{k_0}\beta_*(x_3, x_1, x_2)dx_3 + x_1^{k_0} \right]$$

$$\begin{aligned}
& + \Phi(x_1, x_2) \left[ \int_0^1 x_3^{k_0} \beta_*(x_3, x_2, x_1) dx_3 + \int_1^{x_2} x_3^{1+k_0} \beta_*(x_3, x_1, x_2) dx_3 - x_2^{k_0+1} \right] \\
& \leq \Phi(x_1, x_2) \left[ (E_{k_0,1} + 1)x_1^{k_0} + E_{k_0,1}x_2^{k_0} + x_2^{k_0} \int_1^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 - x_2^{k_0+1} \right] \\
& \leq 3E_{k_0,1}x_1^{k_0} (x_2^{\lambda_1} + x_2^{\lambda_2}) x_2^{k_0} \\
& \leq 6E_{k_0,1}x_1^{k_0} w(x_2).
\end{aligned}$$

*Case 3.* If  $(x_1, x_2) \in (1, \infty) \times (0, 1)$ , then

$$\begin{aligned}
Q(t, x_1, x_2) & \leq \Phi(x_1, x_2) \left[ \int_0^{x_2} x_3^{k_0} \beta_*(x_3, x_2, x_1) dx_3 - x_2^{k_0} \right] \\
& + \Phi(x_1, x_2) \left[ \int_0^1 x_3^{k_0} \beta_*(x_3, x_1, x_2) dx_3 + \int_1^{x_1} x_3^{1+k_0} \beta_*(x_3, x_2, x_1) dx_3 + x_1^{k_0+1} \right] \\
& \leq \Phi(x_1, x_2) \left[ (E_{k_0,1} - 1)x_2^{k_0} + E_{k_0,1}x_1^{k_0} + x_1^{k_0} \int_1^{x_1} x_3 \beta_*(x_3, x_1, x_2) dx_3 + x_1^{k_0+1} \right] \\
& \leq x_2^{k_0} (x_1^{\lambda_1} + x_1^{\lambda_2}) [E_{k_0,1} - 1 + E_{k_0,1} + 2] x_1^{k_0+1} \\
& \leq 6E_{k_0,1}x_1^{1+k_0+\lambda_2} w(x_2).
\end{aligned}$$

*Case 4.* If  $(x_1, x_2) \in (1, \infty)^2$ , then

$$\begin{aligned}
Q(t, x_1, x_2) & \leq \Phi(x_1, x_2) \left[ E_{k_0,1}x_1^{k_0} + x_1^{k_0} \int_1^{x_1} x_3 \beta_*(x_3, x_1, x_2) dx_3 + x_1^{k_0+1} \right] \\
& + \Phi(x_1, x_2) \left[ E_{k_0,1}x_2^{k_0} + x_2^{k_0} \int_1^{x_2} x_3 \beta_*(x_3, x_2, x_1) dx_3 - x_2^{k_0+1} \right] \\
& \leq 2(x_1 x_2)^{\lambda_2} [(E_{k_0,1} + 2)x_1^{k_0+1} + E_{k_0,1}x_2^{k_0}] \\
& \leq 8E_{k_0,1}(x_1 x_2)^{\lambda_2} x_1^{k_0+1} x_2^{k_0} \\
& \leq 8E_{k_0,1}x_1^{1+k_0+\lambda_2} w(x_2).
\end{aligned}$$

Using these bounds in (5.1), we get the following differential inequality

$$\begin{aligned}
& \frac{d}{dt} \int_0^\infty w(x_1) |e_1(t, x_1)| dx_1 \\
& \leq 12E_{k_0,1} \int_0^\infty \int_0^\infty (x_1^{k_0} + x_1^{1+k_0+\lambda_2}) w(x_2) e_2(t, x_1) |e_1(t, x_2)| dx_2 dx_1 \\
& = 12E_{k_0,1} [\boldsymbol{\mu}_{k_0}(e_2(t)) + \boldsymbol{\mu}_{1+k_0+\lambda_2}(e_2(t))] \int_0^\infty w(x_2) |e_1(t, x_2)| dx_2.
\end{aligned}$$

The proof can be completed by applying Gronwall's lemma, since both  $\boldsymbol{\mu}_{k_0}(e_2)$  and  $\boldsymbol{\mu}_{1+k_0+\lambda_2}(e_2)$  are in  $L^1(0, t)$ , and  $u_1(0) = u_2(0) = u^{\text{in}}$ .  $\square$

## 6. NON-EXISTENCE

In this section,  $\beta$  and  $\Phi$  are given by (1.7), (1.10), and (1.12) with the parameters  $(\lambda_1, \lambda_2, \nu)$  satisfying

$$\nu \in (-2, -1], \quad \lambda_1 \leq \lambda_2 \leq 1, \quad \lambda_1 < |\nu| - 1, \quad \lambda = \lambda_1 + \lambda_2 < 1. \quad (6.1)$$

We fix  $k_0 \in (|\nu| - 1, 1)$  and consider a mass-conserving weak solution  $u$  to (1.1)-(1.2) in the sense of Definition 2.1 on  $[0, T_1)$  for some  $T_1 > 0$  with initial condition  $u^{\text{in}} \in X_{k_0, +} \cap X_{k_0+1}$  and  $\rho = \mu_1(u^{\text{in}}) > 0$ . Then

$$\mu_1(u(t)) = \rho = \mu_1(u^{\text{in}}), \quad t \in [0, T_1), \quad (6.2)$$

and, by Lemma 3.4,

$$\mu_{k_0+1}(u(t)) \leq \mu_{k_0+1}(u^{\text{in}}), \quad t \in [0, T_1). \quad (6.3)$$

As already mentioned, the proof of Theorem 2.4 is adapted from [8, 29] and the first step is to show that all moments of  $u$  of negative order, as well as sublinear moments, have to be finite.

**Lemma 6.1.** *For any  $T \in (0, T_1)$  and  $k \in (-\infty, 1)$ ,*

$$\sup_{t \in [0, T]} \mu_k(u(t)) < \infty.$$

*Particularly,  $u^{\text{in}} \in X_k$  for all  $k \in (-\infty, 1 + k_0]$ .*

*Proof.* The proof is the same as that of [17, Lemma 7.1] to which we refer.  $\square$

Now, we are going to prove Theorem 2.4.

*Proof of Theorem 2.4.* Consider  $k \in (|\nu| - 1, k_0)$ ,  $\theta \in (0, 1)$ , and define the function

$$\varsigma_{k, \theta}(x_1) = (x_1 + \theta)^k - \theta^k \text{ for } x_1 \in (0, 1/\theta) \text{ and } \varsigma_{k, \theta}(x_1) = \varsigma_{k, \theta}(1/\theta) \text{ otherwise.}$$

Clearly,  $\varsigma_{k, \theta} \in L^\infty((0, \infty))$  and straightforward computations show that

$$|\varsigma_{k, \theta}(x_1) - \varsigma_{k, \theta}(x_2)| \leq \theta^{k-k_0} |x_1 - x_2|^{k_0}, \quad (x_1, x_2) \in (0, \infty)^2.$$

Consequently,  $\varsigma_{k, \theta} \in \mathcal{S}^{k_0}$  and it follows from (2.3) that, for  $t \in (0, T_1)$ ,

$$\begin{aligned} \int_0^\infty \varsigma_{k, \theta}(x_1) u(t, x_1) dx_1 &= \int_0^\infty \varsigma_{k, \theta}(x_1) u^{\text{in}}(x_1) dx_1 \\ &+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_{\varsigma_{k, \theta}}(x_1, x_2) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau. \end{aligned} \quad (6.4)$$

On the one hand, since

$$\varsigma_{k, \theta}(x_1) \leq x_1^k \quad \text{and} \quad \lim_{\theta \rightarrow 0} \varsigma_{k, \theta}(x_1) = x_1^k, \quad x_1 \in (0, \infty),$$

we infer from Lemma 6.1 and the Lebesgue dominated convergence theorem that

$$\begin{aligned} \lim_{\theta \rightarrow 0} \int_0^\infty \varsigma_{k, \theta}(x_1) u(t, x_1) dx_1 &= \mu_k(u(t)), \\ \lim_{\theta \rightarrow 0} \int_0^\infty \varsigma_{k, \theta}(x_1) u^{\text{in}}(x_1) dx_1 &= \mu_k(u^{\text{in}}). \end{aligned} \quad (6.5)$$

On the other hand, since  $\nu + k + 1 > 0$ ,

$$\Upsilon_{\varsigma_{k,\theta}}(x_1, x_2) = \frac{\nu + 2}{x_1^{\nu+1}} \int_0^{x_1} \varsigma_{k,\theta}(x_3) x_3^\nu dx_3 + \frac{\nu + 2}{x_2^{\nu+1}} \int_0^{x_2} \varsigma_{k,\theta}(x_3) x_3^\nu dx_3 - \varsigma_{k,\theta}(x_1) - \varsigma_{k,\theta}(x_2)$$

satisfies

$$|\Upsilon_{\varsigma_{k,\theta}}(x_1, x_2)| \leq \left(1 + \frac{\nu + 2}{k + \nu + 1}\right) (x_1^k + x_2^k), \quad (x_1, x_2) \in (0, \infty)^2,$$

so that

$$\begin{aligned} |\Upsilon_{\varsigma_{k,\theta}}(x_1, x_2)| \Phi(x_1, x_2) &\leq \left(1 + \frac{\nu + 2}{k + \nu + 1}\right) (x_1^{k+\lambda_1} x_2^{\lambda_2} + x_1^{k+\lambda_2} x_2^{\lambda_1}) \\ &\quad + \left(1 + \frac{\nu + 2}{k + \nu + 1}\right) (x_1^{\lambda_1} x_2^{k+\lambda_2} + x_1^{\lambda_2} x_2^{k+\lambda_1}) \end{aligned}$$

for  $(x_1, x_2) \in (0, \infty)^2$ . Since  $u \in L^\infty((0, t), X_m)$  for  $m \in \{\lambda_1, \lambda_2, k + \lambda_1, k + \lambda_2\}$  by (6.1), (6.3), and Lemma 6.1, and

$$\lim_{\theta \rightarrow 0} \Upsilon_{\varsigma_{k,\theta}}(x_1, x_2) = \frac{1 - k}{k + \nu + 1} (x_1^k + x_2^k), \quad (x_1, x_2) \in (0, \infty)^2,$$

we use once more the Lebesgue dominated convergence theorem to conclude that

$$\begin{aligned} \lim_{\theta \rightarrow 0} \int_0^t \int_0^\infty \int_0^\infty \Upsilon_{\varsigma_{k,\theta}}(x_1, x_2) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau \\ = \frac{1 - k}{k + \nu + 1} \int_0^t \int_0^\infty \int_0^\infty (x_1^k + x_2^k) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau. \end{aligned} \quad (6.6)$$

Collecting (6.4), (6.5), and (6.6), we have shown that, for  $t \in (0, T_1)$ ,

$$\begin{aligned} \boldsymbol{\mu}_k(u(t)) &= \boldsymbol{\mu}_k(u^{\text{in}}) \\ &\quad + \frac{1 - k}{2(k + \nu + 1)} \int_0^t \int_0^\infty \int_0^\infty (x_1^k + x_2^k) \Phi(x_1, x_2) u(\tau, x_1) u(\tau, x_2) dx_2 dx_1 d\tau. \end{aligned}$$

Hence, after using the symmetry of  $\Phi$  and (1.12),

$$\boldsymbol{\mu}_k(u(t)) \geq \boldsymbol{\mu}_k(u^{\text{in}}) + \frac{1 - k}{k + \nu + 1} \int_0^t \boldsymbol{\mu}_{k+\lambda_2}(u(\tau)) \boldsymbol{\mu}_{\lambda_1}(u(\tau)) d\tau, \quad t \in [0, T_1]. \quad (6.7)$$

Since  $\lambda_1 < |\nu| - 1 < k < 1$ , we can use Hölder's inequality and (6.2) to obtain, for  $\tau \in [0, t)$ ,

$$\begin{aligned} \boldsymbol{\mu}_k(u(\tau)) &\leq \boldsymbol{\mu}_1(u(\tau))^{(k-\lambda_1)/(1-\lambda_1)} \boldsymbol{\mu}_{\lambda_1}(u(\tau))^{(1-k)/(1-\lambda_1)} \\ &\leq \rho^{(k-\lambda_1)/(1-\lambda_1)} \boldsymbol{\mu}_{\lambda_1}(u(\tau))^{(1-k)/(1-\lambda_1)}. \end{aligned}$$

Next, either  $k < 1 < k + \lambda_2$  and we have

$$\rho = \boldsymbol{\mu}_1(u(\tau)) \leq \boldsymbol{\mu}_k(u(\tau))^{(k+\lambda_2-1)/\lambda_2} \boldsymbol{\mu}_{k+\lambda_2}(u(\tau))^{(1-k)/\lambda_2}.$$

Or  $k + \lambda_2 < 1 < 1 + k_0$  and we have

$$\rho = \boldsymbol{\mu}_1(u(\tau)) \leq \boldsymbol{\mu}_{1+k_0}(u(\tau))^{(1-k-\lambda_2)/(1+k_0-k-\lambda_2)} \boldsymbol{\mu}_{k+\lambda_2}(u(\tau))^{k_0/(1+k_0-k-\lambda_2)}$$

$$\leq \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{(1-k-\lambda_2)/(1+k_0-k-\lambda_2)} \boldsymbol{\mu}_{k+\lambda_2}(u(\tau))^{k_0/(1+k_0-k-\lambda_2)}.$$

Consequently,

$$\boldsymbol{\mu}_{k+\lambda_2}(u(\tau)) \boldsymbol{\mu}_{\lambda_1}(u(\tau)) \geq \ell_2(k) \boldsymbol{\mu}_k(u(\tau))^{(1-k-\ell_1(k))/(1-k)}, \quad \tau \in [0, t] \quad (6.8)$$

with  $\ell_1(k) := \lambda_1 - k + (k + \lambda_2 - 1)_+ < 0$  and

$$\ell_2(k) := \rho^{(\lambda_1-k)/(1-k)} \min\{\rho^{\lambda_2/(1-k)}, \rho^{(1+k_0-k-\lambda_2)/k_0} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{(k+\lambda_2-1)/k_0}\} > 0.$$

Combining (6.7) and (6.8), we derive an integral inequality for  $\boldsymbol{\mu}_k(u)$  expressed as

$$\boldsymbol{\mu}_k(u(t)) \geq \boldsymbol{\eta}_k(t) := \boldsymbol{\mu}_k(u^{\text{in}}) + \frac{(1-k)\ell_2(k)}{k+\nu+1} \int_0^t \boldsymbol{\mu}_k(u(\tau))^{(1-k-\ell_1(k))/(1-k)} d\tau \quad (6.9)$$

which holds for all  $t \in [0, T_1)$ . From (6.9), we infer that  $\boldsymbol{\eta}_k$  satisfies the following differential inequality

$$\frac{d\boldsymbol{\eta}_k}{dt}(t) \geq \frac{(1-k)\ell_2(k)}{k+\nu+1} \boldsymbol{\eta}_k(t)^{(1-k-\ell_1(k))/(1-k)}, \quad t \in [0, T_1).$$

By integrating the above differential inequality over the interval  $[0, t]$ , we derive that, for  $t \in [0, T_1)$ ,

$$\boldsymbol{\eta}_k(t)^{\ell_1(k)/(1-k)} \leq \boldsymbol{\eta}_k(0)^{\ell_1(k)/(1-k)} + \frac{\ell_1(k)\ell_2(k)}{k+\nu+1} t,$$

recalling that  $\ell_1(k) < 0$ . Using the above inequality and (6.9), we obtain

$$\boldsymbol{\mu}_k(u(t))^{\ell_1(k)/(1-k)} \leq \boldsymbol{\eta}_k(t)^{\ell_1(k)/(1-k)} \leq \boldsymbol{\mu}_k(u^{\text{in}})^{\ell_1(k)/(1-k)} + \frac{\ell_1(k)\ell_2(k)}{k+\nu+1} t,$$

for  $t \in [0, T_1)$ . Since  $\boldsymbol{\mu}_k(u(t))^{\ell_1(k)/(1-k)} \geq 0$ , we have

$$t \leq \frac{k+\nu+1}{|\ell_1(k)|\ell_2(k)} \boldsymbol{\mu}_k(u^{\text{in}})^{\ell_1(k)/(1-k)},$$

for  $t \in [0, T_1)$ . If we let  $t \rightarrow T_1$  in the above inequality, then we conclude that

$$T_1 \leq \frac{k+\nu+1}{|\ell_1(k)|\ell_2(k)} \boldsymbol{\mu}_k(u^{\text{in}})^{\ell_1(k)/(1-k)}, \quad (6.10)$$

and this inequality is valid for any  $k \in (|\nu| - 1, 1)$ . Now, we note that

$$\boldsymbol{\mu}_k(u^{\text{in}})^{k_0/(1+k_0-k)} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{(1-k)/(1+k_0-k)} \geq \boldsymbol{\mu}_1(u^{\text{in}}) = \rho,$$

so that

$$\boldsymbol{\mu}_k(u^{\text{in}})^{\ell_1(k)/(1-k)} \leq \rho^{\ell_1(k)(1+k_0-k)/k_0(1-k)} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{|\ell_1(k)|/k_0}.$$

Consequently, since

$$\lim_{k \rightarrow -\nu-1} \ell_1(k) = \lambda_1 + \nu + 1 + (\lambda_2 - \nu - 2)_+ < 0,$$

$$\lim_{k \rightarrow -\nu-1} \ell_2(k) = \rho^{(\lambda_1+\nu+1)/(\nu+2)} \min\{\rho^{\lambda_2/(\nu+2)}, \rho^{(\nu+2+k_0-\lambda_2)/k_0} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{(\lambda_2-\nu-2)/k_0}\} > 0,$$

we may let  $k \rightarrow -\nu - 1$  in (6.10) and obtain

$$T_1 \leq \liminf_{k \rightarrow -\nu-1} \left\{ \frac{k+\nu+1}{|\ell_1(k)|\ell_2(k)} \rho^{\ell_1(k)(1+k_0-k)/k_0(1-k)} \boldsymbol{\mu}_{1+k_0}(u^{\text{in}})^{|\ell_1(k)|/k_0} \right\} = 0,$$

thereby completing the proof of Theorem 2.4.  $\square$

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## REFERENCES

- [1] J. Banasiak. On a non-uniqueness in fragmentation models. *Mathematical Methods in the Applied Sciences*, 25(7):541–556, 2002.
- [2] J. Banasiak. Conservative and shattering solutions for some classes of fragmentation models. *Mathematical Models and Methods in Applied Sciences*, 14(04):483–501, 2004.
- [3] J. Banasiak. Shattering and non-uniqueness in fragmentation models—an analytic approach. *Physica D: Nonlinear Phenomena*, 222(1-2):63–72, 2006.
- [4] J. Banasiak, W. Lamb, and Ph. Laurençot. *Analytic methods for coagulation-fragmentation models*. CRC Press, 2019.
- [5] J. Bertoin. Self-similar fragmentations. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 38(3):319–340, 2002.
- [6] J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2006.
- [7] W. Biedrzycka and M. Tyran-Kaminska. Self-similar solutions of fragmentation equations revisited. *Discrete & Continuous Dynamical Systems-Series B*, 23(1):13–27, 2018.
- [8] J. Carr and F. P. da Costa. Instantaneous gelation in coagulation dynamics. *Z. Angew. Math. Phys.*, 43(6):974–983, 1992.
- [9] Z. Cheng and S. Redner. Scaling theory of fragmentation. *Physical Review Letters*, 60(24):2450–2453, 1988.
- [10] Z. Cheng and S. Redner. Kinetics of fragmentation. *Journal of Physics A: Mathematical and General*, 23(7):1233–1258, 1990.
- [11] C. de la Vallée Poussin. Sur l’intégrale de Lebesgue. *Transactions of the American Mathematical Society*, 16(4):435–501, 1915.
- [12] M. H. Ernst and I. Pagonabarraga. The nonlinear fragmentation equation. *Journal of Physics A: Mathematical and Theoretical*, 40(17):F331–F337, 2007.
- [13] M. Escobedo, S. Mischler, and M. Rodriguez Ricard. On self-similarity and stationary problem for fragmentation and coagulation models. *Annales de l’Institut Henri Poincaré Analyse non Linéaire*, 22(1):99–125, 2005.
- [14] A. Filippov. On the distribution of the sizes of particles which undergo splitting. *Theory of Probability & Its Applications*, 6(3):275–294, 1961.
- [15] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations:  $L^p$  spaces*. Springer Science & Business Media, 2007.
- [16] A. K. Giri. On the uniqueness for coagulation and multiple fragmentation equation. *Kinetic and Related Models*, 6(3):589–599, 2013.
- [17] A. K. Giri and Ph. Laurençot. Existence and nonexistence for the collision-induced breakage equation. *SIAM Journal on Mathematical Analysis*, 53(4):4605–4636, 2021.
- [18] A. K. Giri and Ph. Laurençot. Weak solutions to the collision-induced breakage equation with dominating coagulation. *Journal of Differential Equations*, 280:690–729, 2021.
- [19] B. Haas. Loss of mass in deterministic and random fragmentations. *Stochastic Processes Appl.*, 106(2):245–277, 2003.

- [20] P. Kapur. Self-preserving size spectra of comminuted particles. *Chemical Engineering Science*, 27(2):425–431, 1972.
- [21] M. Kostoglou and A. Karabelas. A study of the nonlinear breakage equation: analytical and asymptotic solutions. *Journal of Physics A: Mathematical and General*, 33(6):1221–1232, 2000.
- [22] M. Kostoglou and A. Karabelas. A study of the collisional fragmentation problem using the Gamma distribution approximation. *Journal of Colloid and Interface Science*, 303(2):419–429, 2006.
- [23] P. Krapivsky and E. Ben-Naim. Shattering transitions in collision-induced fragmentation. *Physical Review E*, 68(2):021102, 2003.
- [24] Ph. Laurençot. Mass-conserving solutions to coagulation-fragmentation equations with non-integrable fragment distribution function. *Quarterly of Applied Mathematics*, 76(4):767–785, 2018.
- [25] E. McGrady and R. M. Ziff. “Shattering” transition in fragmentation. *Physical Review Letters*, 58(9):892–895, 1987.
- [26] S. Redner. Fragmentation. In H. Herrmann and S. Roux, editors, *Statistical models for the fracture of disordered media*, chapter 10, pages 321–348. Elsevier, 1990.
- [27] I. Stewart. A global existence theorem for the general coagulation–fragmentation equation with unbounded kernels. *Mathematical Methods in the Applied Sciences*, 11(5):627–648, 1989.
- [28] I. Stewart. A uniqueness theorem for the coagulation-fragmentation equation. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 107, pages 573–578. Cambridge University Press, 1990.
- [29] P. G. J. van Dongen. On the possible occurrence of instantaneous gelation in Smoluchowski’s coagulation equation. *J. Phys. A*, 20(5-6):1889–1904, 1987.
- [30] R. M. Ziff. New solutions to the fragmentation equation. *Journal of Physics A: Mathematical and General*, 24(12):2821–2828, 1991.