Sediment transport models in Generalized shear shallow water flow equations

Arno Roland Ndengna Ngatcha, Boniface Nkonga

Applications in Engineering Science 15 (2023) 100148

Contents lists available at ScienceDirect

Applications in Engineering Science

journal homepage: www.elsevier.com/locate/apples

A sediment transport theory based on distortion-free-boundary nonhomogeneous fluid flows

Arno Roland Ndengna Ngatcha [a,*], Boniface Nkonga [b]

[a] Laboratory E3M, National Higher Polytechnic school of Douala, University of Douala, P.O. BOX 2107, Douala, Cameroon
[b] Université Côte d'Azur & Inria/CASTOR/AmFoDuc, CNRS, LJAD, Nice Sophia-Antipolis, France

ARTICLE INFO

Keywords:
Sediment transport phenomena
Distortion-free-boundary nonhomogeneous fluid flows
Approximation of long wave theory
Phase-lag
Sediment transport model
Coastal environments

ABSTRACT

Over the last 50 years several sediment transport models in coastal environments based on Shallow Water(SW) type models have been developed in the literature. The water flow over an abrupt moving topography quickly spatially variable becomes accelerated and strongly varied arising the turbulence (distortion). The acceleration and strong variation of the flow facilitate the transport of a large quantity of sediments present at the bottom while modifying it. The mathematical models based on SW type models widely used to describe the sediment transport phenomena do not account the distortion effects. Indeed, it is well-known that the SW models are derived from first order approximation of long wave theory. The acceleration and strong variation of the water flow near the bottom is due to the distortion of the horizontal velocity profile along the vertical direction. One can regard distortion as a combination of strain and rotation. The effect of the rotational component is to weaken the effect of the strain somewhat. In this work, we put in place a king theory of sediment transport derived from the second order approximation of long wave theory that can describe sediment transport processes in distortion-free-boundary nonhomogeneous fluid flows. The derived model accounts the distortion (fluctuation with great correlation lengths) that creates the turbulence. Moreover, the model differentiates the fluid velocity from sediment velocity (phase-lag) near the sediment bed. The proposed theory significantly reduces the modeling errors observed in several sediment transport models based on nonhomogeneous shallow water equations and has a great potential to increase the predictive power of sediment transport models in rivers, lakes, coastal flows, ocean basins and so on. The proposed theory improves several existing sediment transport theories recently developed in the literature and can be apply with some degree confidence.

1. Introduction

Erosion (by split effect) and deposition (by gravitation) of sediments due to free surface water flow is a coastal engineering problem of great interest in many countries. Floods and their consequences, problems silting up of river and maritime ports and hydroelectric dam reservoirs disrupt the economic and social stability of developing countries. It is therefore important to model the Shallow Water (SW) phenomena by integrating the realistic and observable physical and hydrodynamic parameters. Particularly, SW phenomena are well-known in coastal environments and are involved in different coastal engineering problems (Fotsi et al., 2019). These phenomena are often studied with many simplifications such as considering only the mean water velocity at the first order Ngatcha et al. (2022a), Ngatcha et al. (2022b,c), Ngatcha and Njifenjou (2022). In many situations, this simplification makes the desired solution inaccurate and without physical meaning. In the coastal zone, the flow is accelerated and strongly varied and cannot be modeled by a system integrating only the horizontal averaged water velocity at the first order. In coastal flows, assume that fluctuations fluid velocity are negligible is unrealistic. The water free-surface flow in presence of abrupt topography becomes strongly turbulent. The integration of turbulence in SW phenomena requires an approximation of the averaged velocity at least to the second order. Note that a satisfactory computation of sediment transport requires a high order accuracy of surface flow computations. The averaged second-order vertical fluctuation effects $\mathcal{O}(\overline{u'u'}) = \mathcal{O}(\overline{u'v'}) = \mathcal{O}(\overline{v'v'}) = \mathcal{O}(u^{*2})$ must be included in a sediment transport model. The idea of fluid dynamics with shear stress tensor id due to Reynolds who discussed the subject in detail by Reynolds (1895). These fluctuations have been identified in term of stress due to agitation motions by Lorentz (1907). The classical averaged sediment transport models neglect the second-order vertical fluctuation effect. Several averaged sediment transport models have been previously developed on the basis of the assumption the uniform

* Corresponding author.
E-mail address: arnongatcha@gmail.com (A.R.N. Ngatcha).

https://doi.org/10.1016/j.apples.2023.100148
Received 24 August 2023; Accepted 27 August 2023
Available online 7 September 2023

mean current and of small-scale tank experiments (see for instance Cao et al., 2004; Clare et al., 2021; Hu and Cao, 2009; Gonzalez-Aguirre et al., 2020; Einstein, 1942; Engelund and Hansen, 1967; Nielsen, 1992; Soulsby et al., 2012; Paola and Voller, 2005; Rijn, 1984; Yalin, 1977). The models are different from each other by the parametrization of the relevant mechanisms (deposition, erosion, water/sediment entrainment, turbulence, etc.) (Bouchut et al., 2008; Hu and Cao, 2009; Exner, 1925; Lajeunesse et al., 2010; Ngatcha and Njifenjou, 2022). In this work, we improve the averaged sediment transport models based on SW type models recently developed and widely used in the literature (see (Audusse et al., 2021; Bouchut et al., 2008; Cao et al., 2004; Castro Diaz et al., 2009; Clare et al., 2021; Garcia and Parker, 1991; Hudson and Sweby, 2003; Holly and Rahuel, 1990; Lai, 2020; Ngatcha et al., 2022a; Ngatcha et al., 2022b,c; Paola and Voller, 2005; Warner et al., 2008)) by integrating the hydrodynamic and physical concepts as the turbulence (or distortion) and the phase-lag. These well-known mechanisms that appear frequently in coastal flows are relevant in the sediment transport models.

These mechanisms are relevant in the sediment transport models. In fact, a coastal water flow is likely to distortion and give rise to turbulence. The sediment transport models in shallow water equations have several drawbacks. First, do not capture several physical and hydrodynamic processes (as strong fluid/sediment interactions) present in coastal environments. Second, they neglect the large spatial and/or temporal variations of the bottom. Third, the strong sediment-fluid and fluid-fluid interactions (appearing for example during a dam break) are neglected or poorly described. Fourth, do not capture a realistic description of the hydrodynamics of coastal flows. All these classical models are based on homogeneous or nonhomogeneous shallow water equations at the first order under shallowness assumption $\varepsilon = \frac{H}{L} \ll 1$ where H and L denote the vertical and length scales. The nonhomogeneous equations can write in $Q_T = \Omega\times]0,T] = \{(t,x,y), t \in \mathbb{R}_+^*, \text{with}(x,y) \in \mathbb{R}^2\}$ as:

$$\frac{\partial h}{\partial t} + \nabla.(h\overline{\mathbf{u}}) = G^1 + G^2, \tag{1a}$$

$$\frac{\partial h\overline{\mathbf{u}}}{\partial t} + \nabla.\left(h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\right) + \frac{gh^2\delta\rho}{2\rho}\nabla\rho + gh\nabla\eta + \mathcal{O}(\varepsilon) = G^1_{\mathbf{u}} + G^2_{\mathbf{u}} + \frac{\mathcal{F}}{\rho}, \tag{1b}$$

$$\frac{\partial h\rho}{\partial t} + \nabla.(h\overline{\mathbf{u}}\rho) = \nabla.\left(hD_L\nabla\rho\right), \tag{1c}$$

where $\overline{\mathbf{u}}$ is the averaged velocity, h the water depth, $\eta = h + Z_b$ is the free surface (Z_b being the topography), $\mathcal{F}$ is the source term, ρ is the mixture density and D_L is the diffusion coefficient, $G^{1,2}$ are two functions (due to kinematic conditions) describing the exchange and entrainment processes and $G^{1,2}_{\mathbf{u}}$ the momentum transfers of these processes.

To derive SWE given by (1a)–(1c), we have considered the motion equations of incompressible nonhomogeneous fluid with free boundary combined with a hydrostatic assumption without random values of $\mathbf{u}$. Note that the model (1a)–(1c) is developed without account random values of $\mathbf{u}$ (denoted $\mathbf{u}'$) and the horizontal velocity is computed out of the boundary layer and at bed level. If we assume that the fluid is turbulent or that $\mathbf{u}$ has random value, the classical model given by (1a)–(1c) is not applicable. Using the random values of $\mathbf{u}$ we have:

$$\mathbf{u}\otimes\mathbf{u} = \overline{\mathbf{u}}\otimes\overline{\mathbf{u}} + \overline{\mathbf{u}}\otimes\mathbf{u}' + \mathbf{u}'\otimes\overline{\mathbf{u}} + \mathbf{u}'\otimes\mathbf{u}'. \tag{2}$$

We can use also the random values of pressure or the mixture density. Considering random values of $\mathbf{u}$ two equations appear from Navier-Stokes equations: the first is the $\overline{\mathbf{u}}$ equation and the second is the $\mathbf{u}'$ equation with $\nabla.\mathbf{u}' = 0$. Thus we can obtain a $\mathbf{u}'\otimes\mathbf{u}'$ equation by multiplying the $\mathbf{u}'$ equation by $\mathbf{u}'$ and using the symmetric property of $\mathbf{u}'\otimes\mathbf{u}'$. This simple manipulation lead to existence of third order fluctuation term $\nabla.\left(\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'\right)$ that is not considered in this work.

In coastal flows (assumed almost potential), we can use shallowness assumption and vertical integration over the water depth. This leads to account a new variable $\widehat{\mathcal{U}} = \overline{\mathbf{u}'\otimes\mathbf{u}'}$ named distortion velocity tensor (or Reynolds shear stress) that plays the role of acceleration of the water flow near the bottom. The term $h\widehat{\mathcal{U}}$ modifies the momentum equations and the bedload equation while increasing the number of equations of the system. The presence of this term leads to write a kinetic energy conservation equation (or $\overline{K}$ equation, $\overline{K} = \frac{1}{2}\left(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}} + \widehat{\mathcal{U}}\right)$) or $\widehat{\mathcal{U}}$ equation to close the fully coupled system. We note that this tensor is assumed symmetric and definite positive i.e. $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}_{ij} = \widehat{\mathcal{U}}_{ji};\ \ i,j = 1,2$ and $\widehat{\mathcal{U}}_{ji} > 0, \forall i,j$. It is essential to know that $\mathcal{O}(\widehat{\mathcal{U}}_{12}) \ll \mathcal{O}(\widehat{\mathcal{U}}_{11}), \mathcal{O}(\widehat{\mathcal{U}}_{22})$ to maintain the positivity of $\widehat{\mathcal{U}}$.

SW model neglects this contribution since it derive from first approximation long wave theory that considers $|\mathbf{u} - \overline{\mathbf{u}}| = \mathcal{O}(\varepsilon)$. Often, the currents in coastal flows are very strong (several meters per second) and result in very important distortion effects $h\widehat{\mathcal{U}}$ which are essential in a realistic description of the dynamics of sediment and the morphodynamic in these flows. In coastal flows the distortion intensity of a current ($\approx \overline{\mathbf{u}'\otimes\mathbf{u}'}/\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}$) near the bottom can become greater.

The distortion velocities modify considerably the momentum equations of the classical SWE and increase the number of equations of the system. The fluctuating motion is transported by a fluctuating characteristic velocity $\overline{a}$ given such that $\overline{a}^2 = \widehat{\mathcal{U}}$ (since $\widehat{\mathcal{U}}$ is positive). Accounting the distortion of water flow velocity given in Eq. (2), the new momentum equations take form:

$$\frac{\partial h\overline{\mathbf{u}}}{\partial t} + 2\nabla.\left(h\overline{K}\right) + gh\nabla\eta + \frac{gh^2\delta\rho}{2\rho}\nabla\rho + \mathcal{O}(\varepsilon^\delta) = G^1_{\mathbf{u}} + G^2_{\mathbf{u}} + \frac{\mathcal{F}}{\rho}, \tag{3}$$

where $\delta = \min\{2,\alpha\},\ \ \alpha > 1$.

Here, the spatial variations of the density of the water sediment mixture are ignored in the convective part and are only taken into account into the pressure term. Moreover, the shear stress components $\widehat{\mathcal{U}}_{ij}$ is assumed strongly positive one (that is responsible for the essential 3D of turbulent fluctuations). The availability of turbulent tensions $\widehat{\mathcal{U}}$ in the flow can lead to the velocity depending on component of $\widehat{\mathcal{U}}$ and $\overline{\mathbf{u}}$. The same observation is done by Troshkin (1990) for homogeneous fluid flow. The proposed derivation will contributes greatly to improve his results. Under some considerations, we can find easily a two-velocity shallow water model. We recall that for SWE we have approximated the velocity as:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathcal{O}(\varepsilon), \quad \widehat{\mathcal{U}} = \mathcal{O}(\varepsilon), \ \text{ and } \ \partial_z\mathbf{u} = \mathcal{O}(\varepsilon). \tag{4}$$

For a distortion SW model(DSWM), one has:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathcal{O}(\varepsilon^{2\beta}), \quad \widehat{\mathcal{U}} = \mathcal{O}(\varepsilon^{2\beta}), \ \ \beta < 1. \tag{5}$$

If we consider a vorticity $\omega = \partial_z\mathbf{u} - \varepsilon^2\nabla w$ conserved along the trajectories $\partial_t\omega + u\partial_x\omega + v\partial_y\omega + w\partial_z\omega = 0$, we have in DSW context:

$$\omega = \partial_z\mathbf{u} + \mathcal{O}(\varepsilon^2) \tag{6}$$

particularly we write:

$$\partial_z\mathbf{u} = \mathcal{O}(\varepsilon^\delta), \quad \forall t \geq 0, \delta = \min\{2,\alpha\}, \ \ \alpha > 1. \tag{7}$$

In the above vortex consideration, we have neglected the vortex stretching (or vortex contraction) and the vortex diffusion. In this work the vorticity is neglected in all the conservation equations of the derived system. We consider an unsteady flows between two mobile surfaces without vortex (see Eq. (4)).

Such similar SW for homogeneous fluid flows modeling including distortion effects without vortex in long waves approximation has been also introduced by Teshukov (2007) and further accounting the vortex by Richards and Gavrilyuk (2013). The numerical studies of these classical distortion (or shear) shallow water models can be found in Chandrashekar et al. (2020), Nkonga and Chandrashekar (2021).

All these flow structures admit waves of small disturbances that can propagate with a finite velocity depending on component of $\widehat{\mathcal{U}}$ and $\overline{\mathbf{u}}$. For example One of this wave velocity has the form $\overline{\mathbf{u}}.\mathbf{n} \pm \sqrt{gh + 3\widehat{\mathcal{U}} : \mathbf{n}\otimes\mathbf{n}}$. We can prove using an analogy between waves of small disturbances of inviscous and incompressible turbulent mixing

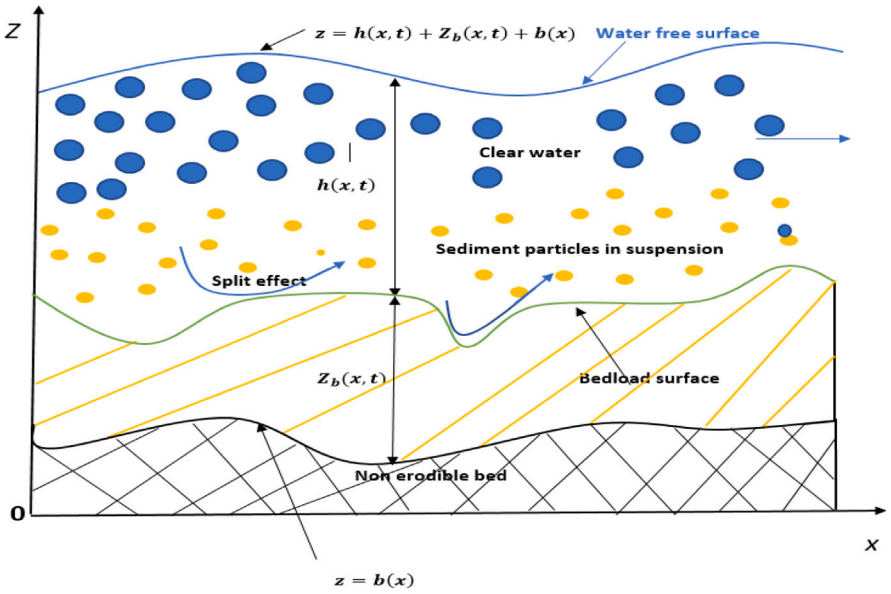

Fig. 1. Multi-physics; multi-components flow models: clear water, water/sediment mixture, bedload surface, and non-erodible bottom.

flow and electromagnetic waves. The classical DSW models neglect the density variation of the flow that modifies its behavior. It well-known that the density variation influences the turbulent shear flows (see Brown and Roshko (1971)). In presence of sediments, the water fluid becomes dense and the isotherm fluid density depends on sediment concentration. Additionally, water near the bottom is more dense than on the free surface. The mathematical model derived in this work is a multi-fluid and multi-physics based system (see Fig. 1). Our main goal consists of developing a new sediment transport model that integrates all the processes involved in the three zones (clear water, water/sediment, sediment zones) and that accounts the density differences. The water/sediment zone (or suspended particles layer) is located at the interface between pure water layer and sediment bed layer (at $\eta \le z \le Z_b^*$ where the free surface is $\eta(\mathbf{x},t)$ and the bed level is $Z_b^*(\mathbf{x},t)$). This multifluid aspect, has been neglected in some flow structures that include distortion effect available in the literature.

All the three zones described in Fig. 1 are non-miscible and the sediment is assumed constituted a homogeneous medium. We have:

$$\rho|_{z=\eta} \simeq \rho_w, \quad \rho|_{z=Z_b^*} \simeq (1-\phi^*), \quad \rho|_{\eta\le z\le Z_b^*} = \rho_s c + \rho_w(1-c),$$

where ϕ^* is the bed porosity, c is the sediment concentration, ρ_w and ρ_s are the water density and sediment density respectively, $Z_b^* = Z_b + b$. The density variation modifies the behavior of the averaged flow structure distorted that accounts random values of $\mathbf{u}$. Thus the Froude number in presence of sediment becomes $Fr = \dfrac{\overline{\mathbf{u}}.\mathbf{n}}{\sqrt{g'h + 3\widehat{\mathcal{U}} : \mathbf{n}\otimes\mathbf{n}}}$, g' being the reduced gravity accounting the mixture density. The model presented here describes both suspended and bedload transports in a new averaged flow structure admitting a new Froude number given above.

An inherent limitation of shallow flow systems is not to give a good representation of phenomena that happens in the water near the bottom. As the shallow water phenomena consist of average over the water depth Navier-Stockes or Euler equations, they not make any difference between the interface $z = Z_b^*$ and the rest of the water. Near the bottom the vertical velocity is weak. The horizontal gradient velocity is also weak. The pressure gradient near the bottom is similar to that of layer out of the boundary layer. Note that at $z = Z_b^*$ the water discharge can vary quickly increasing the water velocity and modify the profile of velocity (creation of the distortion). Here, according to Fig. 1, we study the sediment transport phenomena in a domain $\mathbf{D}_t \subseteq \mathbb{R}^3 \times \mathbb{R}_+^*$ defined by:

$$\mathbf{D}_t = (x,y,z)^T = \left\{(x,y,z), x,y \in \mathbb{R}, Z_b^*(x,y,t) \le z \le \eta(x,y,t), t \in \mathbb{R}_+^*\right\}. \tag{8}$$

In this work, a nonhomogeneous distortion shallow water flow model is designed and derived to describe the motion of fluid mixing in a domain by dynamical water surface and mobile abrupt topography. In the proposed model, the mass and momentum transfer respectively in the continuity, momentum and kinetic energy equations for the water-sediment mixture are exclusively account with formal mechanical justifications. In such case, for any hydrodynamic functions ψ and φ defined in $\mathbb{R}^2$ to $\mathbb{R}^2$ one has:

$$\frac{\partial}{\partial \mathbf{x}}\left(\int_{Z_b^*}^{\eta} \psi'\varphi' dz\right) = \mathcal{O}(\varepsilon^{2\beta}), \quad \beta < 1. \tag{9}$$

The model developed here can accurately simulate turbidity currents (as in Hu and Cao (2009) and Bouchut et al. (2008)) and generalizes all the turbidity currents models based on SWE and widely used in the literature. The pressure in a such fluid is given by:

$$P = P_a + g\int_{Z_b^*}^{\eta} \rho d\gamma - \int_{Z_b^*}^{\eta} \rho a_3 d\gamma = P_h + P_N, \tag{10}$$

where $P_h = P_a + g\int_{Z_b^*}^{\eta} \rho d\gamma$ and $P_N = \int_{Z_b^*}^{\eta} \rho a_3 d\gamma$ are the hydrostatic pressure and non-hydrostatic pressure. Here, a_3 accounts the vertical acceleration effects near the bottom and the Eddy viscosity effect.

In the nonhomogeneous flow structure model proposed here, the pressure is assumed hydrostatic i.e. $\mathcal{O}(a_3) \ll g$ (the acceleration and eddy viscosity term in $z-$momentum equation is smaller than the gravitational acceleration). Really we have still $|P_N| \ll P_H$ but we do not have always $|\nabla P_N| \gg \nabla P_H$, since ∇P_H does not vary with both x and z – coordinates. In this work, we do not consider the non-hydrostatic pressure. Therefore, the proposed hydrostatic nonhomogeneous distortion shallow water model (or turbidity model) has 7 equations (1 mass equation, 2 momentum equations, 3 kinetic energy equations and 1 density evolution equation). This later extends the classical DSW model cited above that has 6 equations and that cannot include the space–time density variation effects. The Distortion nonhomogeneous SWE are coupled with a bedload evolution equation that describes the morphodynamics of sediment bed.

Write the equation of dynamic of bed interface (morphodynamics) is then necessary to describe its behavior. Here, we propose a new bedload equation that accounts phase-lag and that allows to still ensure the hyperbolicity of the model. To the best knowledge, our model is one of the more general existing in the literature and can be adapted to several other physics (turbidity currents, pollute transport, etc.) well-observed in nature like the turbulence due to distortion of the horizontal velocity. The main objective of this work is to propose a new sediment transport theory taking into account the distortion-free-boundary nonhomogeneous fluid flows (turbulent mixing flows). The main novelties are that: (i) The proposed new averaged sediment transport model takes into account the distortion of horizontal profile velocity in the vertical direction (ii) The model includes the density difference, erosion/deposition exchange, water/sediment entrainment and some transfer processes (iii) In the new model there is a differentiation between the fluid velocity and the sediment velocity via a new bedload equation. In this paper, we propose a coupled model for suspended and bedload sediment transport in the turbulent shallow water framework with all the physical and mechanical justifications is formally derived and presented. Additionally, we propose a brief hyperbolicity study of the derived model. All the relations used through the text are rigorously demonstrated via some physical and mathematical tools. The rest of the paper is organized as follows. The Section 2 presents the assumption, the governing equations based on two-phase flow strategy and the boundary conditions. In Section 3, we present the boundary conditions used to develop the model. Using a vertical averaging technique we derive the model in Section 4. Section 5 is devoted to presenting some properties of the model.

2. Assumptions and governing equations

The governing equations and some assumptions are presented here to describe the physics and mechanics associated with the model. These equations are the basis of many solutions of fluid mechanics problems. Particularly problems of sediment transport. We expose also all the boundary and kinematic conditions use in the averaged equations.

2.1. Assumptions

The following assumptions have been used: (i) Long waves propagating assumption $\varepsilon = \dfrac{\mathbf{H}}{\mathbf{L}} \ll 1$; (ii) The fluid is incompressible, no heart transfer (the horizontal gradient temperature is zero) and no breaking waves; (iii) The suspension is assumed to be sufficiently dilute to justify the use of the Boussinesq approximation. (iv) The sediment diameters d_{50} are uniform. (v) The water surface varies gradually meaning thereby that the pressure distribution along a vertical is assumed hydrostatic; i.e. $\dfrac{\partial P}{\partial z} = \mathcal{O}(\varepsilon)$; (vi) The interactions between the grain are neglected (noncohesive grains); (vii) $\mathbf{u} = \mathbf{u}' + \overline{\mathbf{u}}$ where $\overline{\mathbf{u}} = \dfrac{1}{h}\int_I \mathbf{u} dz$ and where the turbulent fluctuation $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$ such that $\dfrac{1}{h}\int_I \mathbf{u}' dz = 0$; (viii) $|\mathbf{u} - \overline{\mathbf{u}}| = \mathcal{O}(\varepsilon^2)$; (ix) $\dfrac{1}{h}\int_I \mathbf{u}' \otimes \mathbf{u}' dz = \widehat{\mathcal{U}}$, where $\widehat{\mathcal{U}} = \overline{\mathbf{u}' \otimes \mathbf{u}'}$ is the Reynolds tensor or stress tensor that describes the distortion effect; (x) We have $|\widehat{\mathcal{U}}_{ij}| = \mathcal{O}(\varepsilon^{2\beta}) \gg \mathcal{O}(\varepsilon^2)$, $\beta < 1$; (xi) The terms $|\widetilde{\mathcal{U}}_{ijk}| = \mathcal{O}(\varepsilon^{3\beta})$, with $\varepsilon^{3\beta} \ll \varepsilon^{2\beta}$ for $\beta < 1$ are omitted; (xii) There no exist turbulence in free surface ($\widehat{\mathcal{U}} = 0$ at $z = \eta$).

2.2. Governing equations based on two-phase flows

In our approach, the starting point is to consider that the motions of each phase are governed by two conservation equations of mass and momentum. Thus, by notice k the index designing the fluid phase or solid phase, the conservation equations for the k-phase are given by the following equations:

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \nabla.(\alpha_k \rho_k \mathbf{u}_k) + \frac{\partial \alpha_k \rho_k w}{\partial z} = 0, \tag{11}$$

$$\frac{\partial \alpha_k \rho_k \mathbf{u}_k}{\partial t} + \nabla.(\alpha_k \rho_k \mathbf{u}_k \otimes \mathbf{u}_k) + \frac{\partial \alpha_k \rho_k \mathbf{u}_k w_k}{\partial z} + \nabla P_k = \mathcal{F}_{k,x,y},$$
$$\frac{\partial \alpha_k \rho_k w_k}{\partial t} + \nabla.\left(\alpha_k \rho_k \mathbf{u}_k w_k\right) + \frac{\partial \alpha_k \rho_k w_k w_k}{\partial z} + \frac{\partial P_k}{\partial z} = \mathcal{F}_{k,z},$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where α_k, $\mathbf{u}_k = (u_k, v_k)$, w_k ρ_k represent respectively the volume fraction, the horizontal velocity vector, the vertical velocity and volume mass of the phase k. $\mathcal{F}_k$ is the source term for each phase of components $\mathcal{F}_{k,x,y}$ and $\mathcal{F}_{k,z}$ in each direction. given by:

$$\mathcal{F}_{k,x,y} = \nabla.\left(2\mu_k(\mathbf{D}_\mathbf{x}\mathbf{u})\right) + \frac{\partial}{\partial z}\left(\mu_k\left(\frac{\partial \mathbf{u}_k}{\partial z} + \frac{\partial w_k}{\partial x} + \frac{\partial w_k}{\partial y}\right)\right)$$

and

$$\mathcal{F}_{k,z} = \frac{\partial}{\partial z}\left(2\mu_k\left(\frac{\partial w_k}{\partial z}\right)\right) + \nabla.\left(\mu_k\left(\nabla w_k + \partial_z u_k\right)\right)$$

where $\mathbf{D}_\mathbf{x}$ is the symmetric gradient $\mathbf{D}_\mathbf{x} = \dfrac{\nabla_\mathbf{x} + (\nabla_\mathbf{x})^T}{2}$. $div(.)$ or $\nabla.(.)$ is two-dimensional divergence calculated with respect to vector variable x_i, $i = 1, 2$ or x_α, $\alpha = 1, 2$.

For $\xi = b, \eta, Z_b$, we define the normal and a base of tangent by respectively:

$$\mathbf{n}_\xi = \frac{1}{\sqrt{1 + |\nabla\xi|^2}}\begin{pmatrix}\nabla\xi \\ 1\end{pmatrix} \quad \text{and} \quad t_\xi^{x_1,x_2} = \frac{1}{\sqrt{1 + \left|\partial_{x_1,x_2}\xi\right|^2}}\begin{pmatrix}n_{x_1,x_2} \\ \partial_{x_1,x_2}\xi\end{pmatrix} \tag{12}$$

The viscosity tensor is defined for each phase by:

$$\tau_k = 2\mu_k\begin{bmatrix}\dfrac{\nabla \mathbf{u}_k}{2} + \dfrac{(\nabla \mathbf{u}_k)^T}{2} & \dfrac{\partial_\mathbf{x} w_k + \partial_z \mathbf{u}_k}{2} \\ \dfrac{\partial_\mathbf{x} w_k + \partial_z \mathbf{u}_k}{2} & \partial_z w_k\end{bmatrix}. \tag{13}$$

The continuity of this tensor at the free surface and the water-sediment interface is:

$$(-P_f.\mathbb{I} + \tau_f)\mathbf{n}_\eta = 0 \quad \text{at} \quad z = \eta, \tag{14}$$

$$(-P_s.\mathbb{I} + \tau_s)\mathbf{n}_{Z_b} = (-P_f.\mathbb{I} + \tau_f)\mathbf{n}_{Z_b} \quad \text{at} \quad z = Z_b^*, \tag{15}$$

The 3D equations are completed with initial data for each phase k:

$$\mathbf{u}_k(\mathbf{x}, z, 0) = \mathbf{u}_k^0(\mathbf{x}, z), \quad w_k(\mathbf{x}, z, 0) = w_k^0(\mathbf{x}, z), \quad \rho_k(\mathbf{x}, z, 0) = \rho_k^0(\mathbf{x}, z). \tag{16}$$

These initial values must satisfy the compatibility condition following:

$$\nabla \mathbf{u}_k^0 + \partial_z w_k^0 = 0, \quad \mathbf{u}_{fs}\nabla Z_b^0 - w_{fs} = 0, \quad u_f^0.\nabla b = 0. \tag{17}$$

where Z_b^0 is the initial bottom.

All the above equations can be used to solve a fluid–structure problem where the structure is the sediment. The above stress formulation given by (11)-(17) generalizes several fluid-sediment stress formulation recently published and available in the literature. Such stress formulation for sediment transport can be resolved by a splitting technique as in Minev and Usubov (2022) but this scheme can be computational cost. The approach here does not consist in solving the above stress formulation problem but to write an equation of conservation of momentum for the mixture by summing Eq. (11) on the two phases. It is possible to sum this stress formulation because the distance between each phase is small compared to their size. By summing on two phases the Eqs. (11), we get three-dimensional mixing flows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho u_i\right)}{\partial x_i} = 0, \tag{18}$$
$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_j\right)}{\partial x_j} + \frac{\partial P}{\partial x_i} = \mathcal{F}_i,$$
$$\frac{\partial u_i}{\partial x_i} = 0,$$

where $x_i, i = 1,2,3$ are the space components, $u_i, i = 1,2,3$ are component of velocity, $\mathcal{F}_i$ is the friction term and P is the pressure term.

In the context of the application of a nonhomogeneous Distortion Shallow Water model to sediment transport problems in coastal and estuarine environments, the use of an Eulerian method for the fluid and the sediments is well suited. Due to the large number of particles, the Lagrangian method is not suitable. The Eulerian approach allows us to average the local instantaneous equations of mass and momentum conservation. The averaging procedure gives a macroscopic description of the flow properties and facilitates the finding of solutions of the problem. In practice, the averaging procedure excludes the high frequencies of the local instantaneous equations (fluctuations) which are necessary in an averaged formulation. Indeed, the statistic properties of the latter influence the macroscopic processes.

2.3. The hydrostatic and Boussinesq assumptions

Hydrostatic assumption

The hydrostatic assumption consists in neglecting the vertical acceleration of the fluid:

$$\rho\left(\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial w^2}{\partial z}\right)\approx 0. \tag{19}$$

Thus, the z-direction of 3D momentum equation in (18) using hydrostatic assumption (19) reduces to

$$\frac{\partial P}{\partial z}=\rho g. \tag{20}$$

This implies that the pressure distribution over the vertical direction is hydrostatic.

The Boussinesq assumption

In the suspension zone, the density variations is assumed small and this allows justifying the Boussinesq assumption which means that density variation is considered only in the gravitational forces. This assumption is valid in various regimes but is not adapted for long time phenomena (waves propagation, shoaling) and does not a conservation of kinetic energy. In the suspension layer the situation where the density is influenced by the sediment concentration c is considered which mean the effects of concentration is taken into account:

$$\rho=\rho_w(1-c)+\rho_s c. \tag{21}$$

With Eq. (21), we can determine the water-sediment mixture velocity as $\mathbf{u}_m = (1-c)\mathbf{u}_f + c\mathbf{u}_s$ where $\mathbf{u}_k, k = f,s$ are the solid and fluid velocities respectively. Assuming that $\mathbf{u}^f = \mathbf{u}^s$ we find $\mathbf{u}^m = \mathbf{u}$ (the property is continuous between the sediment and the fluid). These equations assume that the velocity of sediment and fluid is almost equal in the horizontal direction. The density variation is assumed small(in flow) and this allows justifying the Boussinesq assumption which means that the density variation is considered only in the gravitational forces. Thus second equation of (18) is the momentum-balance for the fluid-sediment. Note that when the volume concentrations of sediment is small enough i.e. $\rho_w(1-c) \gg \rho_s c$, the mixture density is almost constant.

We assume that ρ_s, ρ_w are constant and $\rho_s \neq \rho_w$. Now we assume that the volumetric sediment concentration c satisfies the equation:

$$\frac{\partial c}{\partial t}+\frac{\partial cu}{\partial x}+\frac{\partial cv}{\partial y}+\frac{\partial cw}{\partial z}=0, \tag{22}$$

2.4. Bed evolution equation

To finish, we consider the bed evolution equation, based on the mass balanced equation on a arbitrary control volume $\Omega(t)$. The Grass-based approximation (Grass, 1981) has been used and next reformulated by including distortion effect. Here we assume that there no vertical discontinuity of the bed and we denote by V_s the mass of sediment within the control volume Ω. The equation for the conservation of mass V_s present on the bed is given by:

$$\frac{dV_s}{dt}=0 \longrightarrow \int_{z=b}^{z=Z_b^*}\int_{\partial\Omega}(\rho_b(1-p))dS \tag{23}$$

ρ_b is the bed density.

From Eqs. (18), (22) and (23), we add the followings boundary equations.

3. Boundary conditions

The model developed here is thus a 2D sediment transport model. Here, the region $\mathbf{D}_t \subset \mathbb{R}^3 \times \mathbb{R}_+^*$ occupied by the flow is given by:

$$\mathbf{D}_t=\left\{(\mathbf{x},z),\ (\mathbf{x})\in\mathbb{R}^2,\ Z_b^*(\mathbf{x},t)\le z\le Z_b^*(\mathbf{x},t)+h(\mathbf{x},t), t\in\mathbb{R}_+^*\right\}, \tag{24}$$

where $Z_b^*(\mathbf{x},t)$ and $h(\mathbf{x},t)$ are respectively the water depth and the bed level. The free surface is $\eta(\mathbf{x},t)=Z_b^*(\mathbf{x},t)+h(\mathbf{x},t)$ and its perturbations due to the turbulence are neglected. We assume that the perturbations of the free surface are due to the movement of sediment bed form. Using these assumptions some conditions should be imposed on the bed and the free surface.

3.1. Free boundary equations

One approach to determining boundary conditions at the free surface is to use the flux conditions.

Sediment flux at the free surface

At the free surface, the total sediment flux is assumed to be zero. In other words, there is no sediment flux across the free boundary surface

$$\mathbf{W}_s c+\sigma_z\frac{\partial c}{\partial z}=0, \quad \text{at} \quad z=\eta \tag{25}$$

where $\mathbf{W}_s$ is the effective settling velocity obtained by averaged Lagrange equation of the motion over the set of particles and depending of local concentration:

$$\frac{\mathbf{W}_s}{W_s}=(1-c)^{2.29}, \tag{26}$$

where the settling velocity W_s (see Appendix).

A material point on a free surface $M(\mathbf{x},z,t) = -z+\eta(t,\mathbf{x})$. The free boundary is subjected to the kinematic and dynamic boundary conditions:

$$\begin{aligned}\frac{dM}{dt} &= -\frac{dz}{dt}+\frac{d\eta}{dt}\\ &= -\frac{\partial z}{\partial t}+\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\frac{dx}{dt}++\frac{\partial\eta}{\partial y}\frac{dy}{dt}\\ &= \frac{\partial\eta}{\partial t}+\mathbf{u}(t,\mathbf{x},\eta)\nabla\eta-w(t,\mathbf{x},\eta).\end{aligned}$$

Thus $\frac{dM}{dt}=0$ leads to:

$$\frac{\partial\eta}{\partial t}+\mathbf{u}(t,\mathbf{x},\eta)\nabla\eta-w(t,\mathbf{x},\eta)=0. \tag{27}$$

The relation given by Eq. (27) is obtained under the assumption that, any fluid particle which is on the free surface of the fluid at the initial time will remain on the free surface for any further time. We also have assumed that the functions present in Eq. (27) is sufficiently smooth so that the Jacobian of the transformation between material and spatial coordinates is defined, allowing us to apply the transport theorem of Reynolds:

$$\frac{d}{dt}\int_V M dV=\int_V\frac{\partial M}{\partial t}+\oint_{\partial V} M\otimes\mathbf{u}.\mathbf{n}ds, \tag{28}$$

where $\mathbf{n}$ is the unity normal vector. Indeed, a fluid particle is defined by the position $\mathcal{X}(\mathbf{x}(t),t) = (\mathbf{x}(t),z(t))^T$ and is on the surface when. Taking into account the net water volume rate exchange per unit of

time denoted $F_u(t) = z - \eta(\mathbf{x}, t)$, Eq. (27) can always write more generally as:

$$\frac{\partial \eta}{\partial t} + \mathbf{u}(t, \mathbf{x}, \eta)\nabla\eta - w(t, \mathbf{x}, \eta) = \frac{dF_u}{dt}, \tag{29}$$

or using the normal vector

$$\frac{\partial \eta}{\partial t} + \mathbf{u}(\eta).\mathbf{n}_\eta - w(\eta) = \frac{dF_u}{dt}, \tag{30}$$

where $\mathbf{n}_\eta = \left(\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}\right)^T$. Eq. (30) is more required in the context of rain fall and surface evaporation. Precisely $\frac{dF_u}{dt}$ is the source term which represents the entrainment of water and/or sediment/water mixed by the turbulence in the channel. When $\frac{dF_u}{dt} = 0$ the freshwater should be preserved. This term appears clearly in the momentum conservation and kinetic energy equations (see below).

3.2. Dynamic of the bed interface

The bed boundary is viewed as a phase interface across which the liquid-granular mixture undergoes a transition from solid-to fluid-like behavior. Even mobile bottom $z = Z_b(\mathbf{x}, t)$, we have imposed to continuity of the normal velocity (which vanishes since the bottom is assumed to be the rest) and we can define a quantity $\frac{dF_b}{dt}$ (where $F_b(t) = z(t) - Z_b^*(t, \mathbf{x}(t))$) that describes the erosion/deposition exchange. $\frac{dF_b}{dt}$ can be also seen as the balance between the amount of sediment left behind by the current and the amount of sediment carried away from the bed-load interface. Using a similar manner as above we write:

$$\frac{\partial Z_b^*(\mathbf{x}, t)}{\partial t} + \mathbf{u}(t, \mathbf{x}, Z_b^*)\nabla Z_b^*(\mathbf{x}, t) - w(t, \mathbf{x}, Z_b^*) = -\frac{dF_b}{dt}. \tag{31}$$

Using the normal vector, we write:

$$\frac{\partial Z_b^*}{\partial t} + \mathbf{u}\left(t, \mathbf{x}, Z_b^*\right).\mathbf{n}_{Z_b} - w(Z_b^*) = -\frac{dF_b}{dt}, \quad \mathbf{n}_{Z_b} = \left(\frac{\partial Z_b}{\partial x}, \frac{\partial Z_b^*}{\partial y}\right). \tag{32}$$

One has again used Eq. (28). When the layer considered is the entire water column, the top of the layer is typically taken to be the mean water level, which is assumed to be constant with time and across which there is no sediment flux. The derivative $\frac{dF_b}{dt}$ is seen as function of the balance between the particles that are eroded or deposited on the surface and of the local bed porosity.

Sediment flux at the bottom

One approach to determine boundary condition at the free surface is to use flux conditions. At the bottom boundary we can use the vertical sediment flux (Neumann type) or concentration condition. For the first case we have:

$$W_s c + \sigma_z \frac{\partial c}{\partial z} = D - \mathcal{E}, \quad \text{at} \quad z = Z_b^* \tag{33}$$

According to Garcia and Parker (1991), which assumed that the dis-equilibrium introduced by the unsteadiness remains mild, the erosion flux (E) can be considered to be equal to the entrainment rate ($\mathcal{E}$) under equilibrium condition and related to the reference concentration value through the settling velocity. From Eq. (33), we can have:

$$-\sigma_z \frac{\partial c}{\partial z} = \mathcal{E}, \quad \text{at} \quad z = Z_b^* \tag{34}$$

The sediment flux at the bed is determined by the rates of deposition (D) and erosion (E) as in Cao et al. (2004).

In the above equations, $\mathbf{n}_\eta$ and $\mathbf{n}_{Z_b}$ are the unit outward normals at the bottom and at the free surface respectively. The no-penetration condition is assumed at the water-sediment interface and at the sub-stratum

$$\frac{\partial Z_b^*}{\partial t} + \mathbf{u}_{Z_b}\nabla Z_b^* - w(Z_b) = 0 \tag{35}$$

$$\mathbf{u}_{Z_b}.\nabla B - w(B) = 0 \tag{36}$$

In bed evolution $\mathbf{u}_{Z_b}$ represents the characteristic advection velocity that measures the velocity with which the bottom moves. This allows to expect that the bed evolution equation has a hyperbolic nature and this will facilitate the finding of eigenvalues of the proposed model.

Bedload transport approximation

The estimation of bedload transport is difficult to predict and remains an open problem. In fact, near the bed some hydrodynamic and physical processes are difficult to assess. The mass conservation of moving particles is then applied to formulate the transport at the bedload interface in term of a transport discharge flux Q_b:

$$\mathbf{u}(t, \mathbf{x}, Z_b^*)\nabla Z_b^*(\mathbf{x}, t) - u_3(\mathbf{x}, t, Z_b) = \nabla.Q_b. \tag{37}$$

Here, the motion at the bedload interface is balanced by the gradient of the horizontal mass sediment flux Q_b. Indeed, with an interface normal $\mathbf{n}_{Z_b^*}$, the LHS of (37) can be view as an asymptotic limit of a divergence formulation. The evolution of the bedload interface elevation can be then given by the Exner-type of the form:

$$\frac{\partial Z_b^*}{\partial t} + \frac{A_g}{(1-\phi^*)}\left(\frac{\partial \overline{(u^2+v^2)u}}{\partial x_1} + \frac{\partial \overline{(u^2+v^2)v}}{\partial x_2}\right) = \frac{D-E}{1-\phi^*}. \tag{38}$$

In above equation, A_g is a coefficient usually obtained experimentally by taking into account the grain diameter and the kinematic viscosity of the sediment mixture. The function $\overline{(u^2+v^2)\mathbf{u}}$ is the depth-average of the horizontal speed approximated in Liu et al. (2015) as:

$$\overline{(u^2+v^2)\mathbf{u}} \approx (\overline{u}^2 + \overline{v}^2)\overline{\mathbf{u}}.$$

Next section, we will propose to extend several averaging sediment transport models, by accounting the horizontal velocity distortion along the vertical direction. Nevertheless, given the numerical difficulties (loss of hyperbolicity) encountered with this flux-formulation of the bedload transport, another alternative that is numerically suitable is to formulate a new bedload equation.

4. Derivation of the model by a vertical averaging technique

We introduce here the averaged of a function ψ by

$$\overline{\psi} = \frac{1}{h}\int_{Z_b}^{\eta} \psi(\mathbf{x}, t)dz \quad \text{where} \quad h(\mathbf{x}, t) = \eta(\mathbf{x}, t) - Z_b^*(\mathbf{x}, t) \tag{39}$$

ψ' is the fluctuation with respect to the average. It value is $\psi' = \psi - \overline{\psi}$ and clearly the average of fluctuation ψ' is zero. We also write:

$$h\overline{\psi\varphi} + \int_{Z_b}^{\eta} \psi'(\mathbf{x}, t)\varphi'(\mathbf{x}, t)dz = \int_{Z_b}^{\eta} \psi(\mathbf{x}, t)\varphi(\mathbf{x}, t)dz. \tag{40}$$

We introduce also the Leibniz relations:

$$\frac{\partial h\overline{\psi}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\int_{Z_b}^{\eta} \psi dz = \int_{Z_b}^{\eta} \frac{\partial \psi}{\partial \mathbf{x}}dz - \psi(\eta)\frac{\partial \eta}{\partial \mathbf{x}} + \psi(Z_b)\frac{\partial Z_b}{\partial \mathbf{x}}, \tag{41}$$

and

$$\frac{\partial h\overline{\psi}}{\partial t} = \frac{\partial}{\partial t}\int_{Z_b}^{\eta} \psi dz = \int_{Z_b}^{\eta} \frac{\partial \psi}{\partial t}dz - \psi(\eta)\frac{\partial \eta}{\partial t} + \psi(Z_b)\frac{\partial Z_b}{\partial t}. \tag{42}$$

Above Leibniz's formula are applied to invert the differential operators and integration.

From the last equation of the (18) and according to the above assumptions we have:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0, \quad \text{and} \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0.$$

This means that the horizontal and vertical velocity fluctuations are at zero divergence. The velocity can be decomposed in three different parts: a stretching part, distorting part and a vorticity part. For most coastal engineering applications the stretching part and vorticity part are often neglected than the stress that prevails in coastal flows. Moreover, when the fluid flow becomes strongly viscous the vorticity

part does not have any contribution (Lee, 2021). By defining a basis plane vector (i, j), a fluid velocity can be divided in in-plane $\overline{\mathbf{u}}$ and out-plane $\mathbf{u}'$ contributions. The resulting stress tensor from out plane basis will be injected as distortion velocities tensor (in in-plane basis) which contributes to acceleration of the water flow.

4.1. Derivation continuity equation

Integrating the divergence free equation over the depth of water and using the previous relations for $\psi = \mathbf{u}$ and the boundaries condition given by Eqs. (30) and (32) yields:

$$\frac{\partial h}{\partial t} - \mathbf{u}\left(Z_b^*\right).\mathbf{n}_F - \mathbf{u}\left(\eta\right).\mathbf{n}_s + w(Z_b^*) - w(\eta) = \frac{dF_b}{dt} + \frac{dF_u}{dt} \tag{43}$$

The Leibniz relation gives:

$$\overline{div(\mathbf{u})} = div(h\overline{\mathbf{u}}) + \mathbf{u}(Z_b^*).\mathbf{n}_{Z_b^*} + \mathbf{u}(\eta).\mathbf{n}_s, \tag{44}$$

And as $\overline{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}} = \overline{div(\mathbf{u})} + \overline{\frac{\partial w}{\partial z}} = 0$, one writes:

$$div(h\overline{\mathbf{u}}) + \mathbf{u}(Z_b^*).\mathbf{n}_{Z_b^*} - \mathbf{u}(\eta).\mathbf{n}_s - w(\eta) + w(Z_b) = 0. \tag{45}$$

We obtain finally the averaged conservation mass equation:

$$\frac{\partial h}{\partial t} + div(h\overline{\mathbf{u}}) = \frac{dF_u}{dt} + \frac{dF_b}{dt}. \tag{46}$$

The mass conservation equation of sediment/water mixture given by (46) means that the erosion/deposition exchange modifies the fresh-water mass. The two terms on the RHS quantify the rate of water entrainment and bed deformation, respectively.

4.2. Depth averaged momentum equation

In the present analysis it is assumed that the fluid and sediment velocities are almost equal in the horizontal directions. Then we have:

$$\rho\overline{\mathbf{u}} = \rho_w(1-c)\mathbf{u}^f + \rho_s c\mathbf{u}^s, \tag{47}$$

with $\mathbf{u}^f = \mathbf{u}^s$. Let H, L denote the vertical and horizontal length scales; The pressure is assumed to be hydrostatic. Let V denote the horizontal velocity scale, then the continuity equation implies that $w = \mathcal{O}(\varepsilon V)$. Using the time and pressure scale as L/V and ρV^2, the x_3-momentum equation can be no-dimensionalized as:

$$\varepsilon^2\frac{Dw'}{Dt'} + \frac{\partial P'}{\partial z'} = \frac{1}{Fr^2}\rho', \tag{48}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\nabla$, the prime quantities are non-dimensional and $Fr = \frac{V}{\sqrt{gH}}$, Neglecting the terms in $\mathcal{O}(\varepsilon^2)$, we obtain the hydrostatic approximation:

$$\frac{\partial P}{\partial z} = -\rho g \Rightarrow P - P_a = -\rho g(z-\eta), \tag{49}$$

$$\Rightarrow P = P_a - \rho g(z-\eta), \tag{50}$$

where P_a is the atmospheric pressure at the free surface which may be taken to be constant. We have the following derivation of pressure term:

$$\begin{aligned}\frac{\partial P}{\partial \mathbf{x}} &= \frac{\partial \rho g(\eta - z)}{\partial \mathbf{x}} = \rho g\frac{\partial \eta}{\partial \mathbf{x}} - g(\eta - z)\frac{\partial \rho}{\partial \mathbf{x}},\\ \frac{1}{\rho}\frac{\partial P}{\partial \mathbf{x}} &= g\frac{\partial \eta}{\partial \mathbf{x}} - g(\eta - z)\frac{\partial \rho}{\partial \mathbf{x}}.\end{aligned} \tag{51}$$

Now the horizontal momentum equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla.(\mathbf{u}\otimes\mathbf{u}) + \frac{\partial(\mathbf{u}w)}{\partial z} + \frac{1}{\rho}\nabla P = \frac{\mathcal{F}}{\rho}. \tag{52}$$

The averaged equation of (52)

$$\overline{\frac{\partial \mathbf{u}}{\partial t}} + \overline{div(\mathbf{u}\otimes\mathbf{u})} + \overline{\frac{\partial(\mathbf{u}w)}{\partial z}} + \overline{\frac{\nabla P}{\rho}} = \overline{\frac{\mathcal{F}}{\rho}}. \tag{53}$$

We have respectively using the above Leibniz relations:

$$h\overline{\frac{\partial \mathbf{u}}{\partial t}} = \frac{\partial}{\partial t}\int_{Z_b^*}^{\eta}\mathbf{u}dz - \mathbf{u}(\eta)\frac{\partial \eta}{\partial t} + \mathbf{u}(Z_b^*)\frac{\partial Z_b^*}{\partial t} \tag{54}$$

$$= \frac{\partial h\mathbf{u}}{\partial t} - \mathbf{u}(\eta)\frac{\partial \eta}{\partial t} + \mathbf{u}(Z_b^*)\frac{\partial Z_b^*}{\partial t}, \tag{55}$$

and thus

$$\begin{aligned}h\overline{div(\mathbf{u}\otimes\mathbf{u})} &= div\int_{Z_b^*}^{\eta}\mathbf{u}\otimes\mathbf{u}dz - (\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta + (\mathbf{u}\otimes\mathbf{u})(Z_b^*)\nabla Z_b^*\\ &\quad + (w\otimes\mathbf{u})(\eta) - (w\otimes\mathbf{u})(Z_b^*)\\ &= div(h\overline{\mathbf{u}\otimes\mathbf{u}}) - (\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta + (\mathbf{u}\otimes\mathbf{u})(Z_b^*)\nabla Z_b^* + (w\mathbf{u})(\eta)\\ &\quad - (w\mathbf{u})(Z_b^*)\\ &= 2div(h\overline{K}) - (\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta + (\mathbf{u}\otimes\mathbf{u})(Z_b^*)\nabla Z_b^* + (w\mathbf{u})(\eta)\\ &\quad - (w\mathbf{u})(Z_b^*),\end{aligned} \tag{56}$$

where

$$\overline{K} = \frac{1}{2}\overline{\mathbf{u}\otimes\mathbf{u}} = \frac{1}{2}\overline{(\overline{\mathbf{u}} + \mathbf{u}')\otimes(\overline{\mathbf{u}} + \mathbf{u}')} = \frac{\overline{\mathbf{u}}\overline{\mathbf{u}}}{2} + \frac{\overline{\mathbf{u}'\mathbf{u}'}}{2} = \frac{\overline{\mathbf{u}}\overline{\mathbf{u}}}{2} + \widehat{\mathcal{U}}.$$

The pressure term is given:

$$\frac{1}{\rho}\frac{\partial P}{\partial \mathbf{x}} = g\frac{\partial \eta}{\partial \mathbf{x}} - g(\eta - z)\frac{\partial \rho}{\partial \mathbf{x}}$$

The averaged equation of (57)

$$\overline{\frac{1}{\rho}\frac{\partial P}{\partial \mathbf{x}}} = g\overline{\frac{\partial \eta}{\partial \mathbf{x}}} - \overline{g(\eta - z)\frac{\partial \rho}{\partial \mathbf{x}}} \Rightarrow \int_{Z_b^*}^{\eta}\frac{1}{\rho}\frac{\partial P}{\partial \mathbf{x}} = gh\frac{\partial \eta}{\partial \mathbf{x}} + \frac{1}{2\rho}gh^2\frac{\partial \rho}{\partial \mathbf{x}}. \tag{57}$$

Thus the averaged momentum equations are given by:

$$\begin{aligned}\frac{\partial h\mathbf{u}}{\partial t} &+ 2\nabla.(h\overline{K}) + gh\nabla\eta + \frac{1}{2\rho}gh^2\nabla\rho - \mathbf{u}(\eta)\left(\frac{\partial \eta}{\partial t} + \mathbf{u}(\eta)\nabla\eta - w(\eta)\right)\\ &+ \mathbf{u}(Z_b^*)\left(\frac{\partial Z_b^*}{\partial t} + \mathbf{u}(Z_b^*)\nabla Z_b^* - w(Z_b^*)\right) = \frac{1}{\rho}h\overline{\mathcal{F}}.\end{aligned} \tag{58}$$

Using the boundary conditions we obtain momentum conservation equations following:

$$\frac{\partial h\mathbf{u}}{\partial t} + 2\nabla.(h\overline{K}) + gh\nabla\eta + \frac{1}{2\rho}gh^2\nabla\rho = \mathbf{u}(\eta)\frac{dF_u}{dt} - \mathbf{u}(Z_b^*)\frac{dF_b}{dt} + \frac{1}{\rho}h\overline{\mathcal{F}}. \tag{59}$$

We see that the deposition/erosion exchange, and water/sediment entrainment have an effect in the momentum equation. Therefore, The above Eq. (59) describes momentum conservation of water/sediment distortion mixture flow. The first two terms on the LHS quantify the production of accelerated motion due to turbulent kinetic energy, likewise the third term represents the density variability effect that prove a turbulent mixing flow character. The first term on the RHS of Eq. (59) features the momentum transfer arising from water entrainment from the clear water. The second term on the RHS describes the sediment transfer due to erosion/deposition exchange between the bed interface at $z = Z_b^*$ and suspension zone. The last term describes the dissipation due to bed friction.

Remark 4.1. The term $\overline{K}$ is not completely known to us and hence the set of Eqs. (46) and (52) does not form a closed system are the interaction between the fluctuation velocity requires modeling. When $\frac{\overline{\mathbf{u}'\otimes\mathbf{u}'}}{2} = 0$, we close Eqs. (46) and (59) and the system obtained is the well-known shallow water model. This simplification can be justified when we assume that $\frac{\partial \mathbf{u}}{\partial z} \ll 1$ (vertical shear).

4.3. Derivation of kinetic energy equation

. When the vertical shear is not small, the second-order velocity fluctuation terms no longer be ignored and in this case from the momentum equations, we have:

$$\mathbf{u}\otimes\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} + \frac{1}{\rho}\nabla P - \frac{\mathcal{F}}{\rho}\right)$$

$$+\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\,\mathbf{u}+w\frac{\partial \mathbf{u}}{\partial z}+\frac{1}{\rho}\nabla P-\frac{\mathcal{F}}{\rho}\right)\otimes\mathbf{u}=0. \tag{60}$$

We recall that using the decomposition $\mathbf{u}=\overline{\mathbf{u}}+\mathbf{u}'$, we can write:

$$\mathbf{u}\otimes\mathbf{u}=\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}+\mathbf{u}\otimes\mathbf{u}'+\mathbf{u}'\otimes\overline{\mathbf{u}}+\mathbf{u}'\otimes\mathbf{u}',$$

where the term $\overline{\mathbf{u}\otimes\mathbf{u}}$ is not zero and where $\mathbf{u}'\otimes\mathbf{u}'$ measure the distortion of instantaneous horizontal velocity profile along the vertical direction.

$$\mathbf{u}\otimes\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\otimes[(\mathbf{u}.\nabla)\,\mathbf{u}]+\mathbf{u}\otimes w\frac{\partial \mathbf{u}}{\partial z}+\mathbf{u}\otimes\left(gh\nabla\eta+\frac{(\eta-z)}{\rho}g\nabla\rho\right)+ \tag{61}$$

$$\frac{\partial \mathbf{u}}{\partial t}\otimes\mathbf{u}+[(\mathbf{u}.\nabla)\,\mathbf{u}]\otimes\mathbf{u}+w\frac{\partial \mathbf{u}}{\partial z}\otimes\mathbf{u}+\left(gh\nabla\eta+\frac{(\eta-z)}{\rho}g\nabla\rho\right)\otimes\mathbf{u}$$

$$=\mathbf{u}\otimes\frac{\mathcal{F}}{\rho}+\frac{\mathcal{F}}{\rho}\otimes\mathbf{u}$$

From Eq. (61), we have:

$$\frac{\partial \mathbf{u}\otimes\mathbf{u}}{\partial t}+\nabla(\mathbf{u}\otimes\mathbf{u}).\mathbf{u}+w\frac{\partial \mathbf{u}\otimes\mathbf{u}}{\partial z}+(g\nabla\eta+(\eta-z)g\nabla\rho)\otimes\mathbf{u}$$

$$+\mathbf{u}\otimes(g\nabla\eta+(\eta-z)g\nabla\rho)=\mathbf{u}\otimes\frac{\mathcal{F}}{\rho}+\frac{\mathcal{F}}{\rho}\otimes\mathbf{u}. \tag{62}$$

We let $\mathbf{u}\otimes[(\mathbf{u}\nabla)\mathbf{u}]+[(\mathbf{u}\nabla)\mathbf{u}]\otimes\mathbf{u}=\nabla(\mathbf{u}\otimes\mathbf{u}).\mathbf{u}$. Notice that

$$\nabla(\mathbf{u}\otimes\mathbf{u}).\mathbf{u}=div(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})-(\mathbf{u}\otimes\mathbf{u})\nabla.\mathbf{u}, \tag{63}$$

and

$$\frac{\partial w(\mathbf{u}\otimes\mathbf{u})}{\partial z}=w\frac{\partial \mathbf{u}\otimes\mathbf{u}}{\partial z}+\mathbf{u}\otimes\mathbf{u}\frac{\partial w}{\partial z}, \tag{64}$$

we can rewrite (62) as:

$$\frac{\partial \mathbf{u}\otimes\mathbf{u}}{\partial t}+\nabla.(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})+\frac{\partial w(\mathbf{u}\otimes\mathbf{u})}{\partial z}+\left(gh\nabla\eta+\frac{(\eta-z)}{\rho}g\nabla\rho\right)\otimes\mathbf{u}$$

$$+\mathbf{u}\otimes\left(gh\nabla\eta+\frac{(\eta-z)}{\rho}g\nabla\rho\right)-\mathbf{u}\otimes\frac{\mathcal{F}}{\rho}-\frac{\mathcal{F}}{\rho}\otimes\mathbf{u}=0. \tag{65}$$

After integration of (62) over the water depth, we get:

$$\underbrace{\int_{Z^*_b}^{\eta}\frac{\partial(\mathbf{u}\otimes\mathbf{u})}{\partial t}dz}_{(I)}+\underbrace{\int_{Z^*_b}^{\eta}\nabla.(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\,dz}_{(II)}+\underbrace{\int_{Z^*_b}^{\eta}\frac{\partial w(\mathbf{u}\otimes\mathbf{u})}{\partial z}dz}_{(III)} \tag{66}$$

$$+\underbrace{\int_{Z^*_b}^{\eta}\mathbf{u}\otimes\left(g\nabla\eta+(\eta-x_3)g\nabla\rho\right)dz}_{(IV)}+\underbrace{\int_{Z^*_b}^{\eta}\left(g\nabla\eta+(\eta-x_3)g\nabla\rho\right)\otimes\mathbf{u}dz}_{(V)}$$

$$=\int_{Z^*_b}^{\eta}\mathbf{u}\otimes\frac{\mathcal{F}}{\rho}dz+\int_{Z^*_b}^{\eta}\frac{\mathcal{F}}{\rho}\otimes\mathbf{u}dz.$$

Now we explain each term (I), (II), (III), (IV) and (V).

$$(I)=\frac{\partial}{\partial t}\int_{Z^*_b}^{\eta}\mathbf{u}\otimes\mathbf{u}dz+(\mathbf{u}\otimes\mathbf{u})\left(Z^*{}_b\right)\frac{\partial Z^*_b}{\partial t}-(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial\eta}{\partial t} \tag{67}$$

$$=\frac{\partial}{\partial t}\left(h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}+h\overline{\mathbf{u}'\otimes\mathbf{u}'}\right)-(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial\eta}{\partial t}+(\mathbf{u}\otimes\mathbf{u})\left(Z^*_b\right)\frac{\partial Z^*_b}{\partial t}, \tag{68}$$

where we have noted that:

$$\overline{\mathbf{u}\otimes\mathbf{u}}=\overline{(\overline{\mathbf{u}}+\mathbf{u}')\otimes(\overline{\mathbf{u}}+\mathbf{u}')}=h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}+2h\widehat{\mathcal{U}}.$$

$$(II)=div\int_{Z^*_b}^{\eta}(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\,dz-(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\left(Z^*{}_b\right)\nabla Z^*_b$$

$$+(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta \tag{69}$$

$$\int_{Z^*_b}^{\eta}(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\,dz=\underbrace{\int_{Z^*_b}^{\eta}\left(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\right)dz}_{(a)}+\underbrace{\int_{Z^*_b}^{\eta}\left(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\otimes\mathbf{u}'\right)dz}_{(b)}$$

$$+\underbrace{\int_{Z^*_b}^{\eta}\left(\mathbf{u}'\otimes\overline{\mathbf{u}}\otimes\mathbf{u}'\right)dz}_{(c)} \tag{70}$$

$$+\underbrace{\int_{Z^*_b}^{\eta}\left(\mathbf{u}'\otimes\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\right)dz}_{(d)}+\underbrace{\int_{Z^*_b}^{\eta}\left(\mathbf{u}'\otimes\mathbf{u}'\otimes\overline{\mathbf{u}}\right)dz}_{(e)}+\underbrace{\int_{Z^*{}_b}^{\eta}\left(\overline{\mathbf{u}}\otimes\mathbf{u}'\otimes\overline{\mathbf{u}}\right)dz}_{(f)} \tag{71}$$

$$+\underbrace{\int_{Z^*_b}^{\eta}\left(\overline{\mathbf{u}}\otimes\mathbf{u}'\otimes\mathbf{u}'\right)dz}_{(g)}+\underbrace{\int_{Z^*_b}^{\eta}\left(\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'\right)dz}_{(h)}. \tag{72}$$

We have naturally $(b)=0$, $(f)=0$, $(d)=0$ and by simple calculations we obtain $(a)=h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}$, $(g)=h\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}$, $(e)=h\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}}$. Now we will evaluate only the term (c). We use a simple divergence decomposition and we have:

$$div\left(\int_{Z^*_b}^{\eta}\left(\mathbf{u}'\otimes\overline{\mathbf{u}}\otimes\mathbf{u}'\right)dz\right)=\frac{\partial}{\partial x_\beta}\rho\int_{Z^*_b}^{\eta}\left(u'{}_\alpha\overline{u}_\lambda u'{}_\beta\right)dz, \tag{73}$$

$$=\frac{\partial}{\partial x_\beta}\left[\overline{u}_\lambda\int_{Z^*_b}^{\eta}u'{}_\alpha u'{}_\beta dz\right]$$

$$=\frac{\partial}{\partial x_\beta}\left[\overline{u}_\lambda\widehat{\mathcal{U}}_{\alpha\beta}\right],$$

$$=\overline{u}_\lambda\frac{\partial\widehat{\mathcal{U}}_{\alpha\beta}}{\partial x_\beta}+\widehat{\mathcal{U}}_{\alpha\beta}\frac{\partial\overline{u}_\lambda}{\partial x_\beta}.$$

Then we obtain:

$$\int_{Z^*_b}^{\eta}div(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\,dz=div\left(h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}\right)+div\left(\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}}\right)$$

$$+div\left(\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}\right) \tag{74}$$

$$+\,div(\widehat{\mathcal{U}})\otimes\overline{\mathbf{u}}+\widehat{\mathcal{U}}\left(\nabla\overline{\mathbf{u}}\right)^T+div\left(\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'\right)-(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta$$

$$+(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\left(Z^*_b\right)\nabla Z^*_b,$$

where the $'T'$ means transposed. The third term gives:

$$(III)=w(\mathbf{u}\otimes\mathbf{u})(\eta)-w(\mathbf{u}\otimes\mathbf{u})\left(Z^*{}_b\right) \tag{75}$$

The fourth and fifth terms are calculated as follows:

$$(IV)=\overline{\mathbf{u}}\otimes\left(gh\nabla\eta+gh^2\frac{1}{2\rho}\nabla\rho\right)+g\overline{\mathbf{u}'\otimes(\eta-x_3)\nabla\rho}, \tag{76}$$

$$(V)=\left(gh\nabla\eta+gh^2\frac{1}{2\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}}+g\overline{(\eta-z)\nabla\rho\otimes\mathbf{u}'}. \tag{77}$$

since ρ is constant in z direction we have:

$$\overline{g(\eta-z)\nabla\rho\otimes\mathbf{u}'}=\frac{\partial\rho}{\partial x_\alpha}\overline{g(\eta-z)\otimes\mathbf{u}'}\quad\text{and}$$

$$g\overline{\mathbf{u}'\otimes(\eta-z)\nabla\rho}=g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho. \tag{78}$$

The term $\overline{\mathbf{u}'\otimes(\eta-z)}$ and $\overline{(\eta-z)\otimes\mathbf{u}'}$ can represent the fluctuation moments on the free surface. These terms are neglected in this work. We will can used in the study of the sediment transport in longshore or crosshore zone. Thus, Eq. (65) implies that:

$$\frac{\partial}{\partial t}\left(h\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}'\otimes\mathbf{u}'}\right)+2div\left(h\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}}\right)+2div(\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}})+2div(\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}) \tag{79}$$

$$+\,div\left(\overline{\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'}\right)+\left(h\nabla\eta+gh^2\frac{1}{2\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}}\otimes\left(gh\nabla\eta\right.$$

$$\left.+gh^2\frac{1}{2\rho}\nabla\rho\right)$$

$$=\overline{\mathbf{u}}\otimes\frac{\overline{\mathcal{F}}}{\rho}+\frac{\overline{\mathcal{F}}}{\rho}\otimes\overline{\mathbf{u}}-g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'}-\overline{\mathbf{u}'\otimes(\eta-z)}g\nabla\rho$$

$$+(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta-(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\left(Z^*_b\right)\nabla Z^*_b$$

$$-\,w(\mathbf{u}\otimes\mathbf{u})(\eta)-w(\mathbf{u}\otimes\mathbf{u})\left(Z^*{}_b\right)+(\mathbf{u}\otimes\mathbf{u})\left(Z^*{}_b\right)\frac{\partial Z^*_b}{\partial t}+(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial\eta}{\partial t}.$$

recalling that the kinetic energy reads

$$\overline{K}=\frac{1}{2}\left(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}'\otimes\mathbf{u}'}\right),$$

we obtain:

$$
\begin{aligned}
2\frac{\partial\left(h\overline{K}\right)}{\partial t} &+ 2div\left(h\overline{K}\otimes\overline{\mathbf{u}}\right)+2div(\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}})+2div(\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}) \\
&+div\left(\overline{\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'}\right) \\
&+\left(h\nabla\eta+gh^2\frac{1}{2\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}}\otimes\left(gh\nabla\eta+gh^2\frac{1}{2\rho}\nabla\rho\right) \\
&=-g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'}-g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho+(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta \\
&-(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\left(Z_b\right)\nabla Z_b^*+w(\mathbf{u}\otimes\mathbf{u})(\eta)-w(\mathbf{u}\otimes\mathbf{u})\left(Z_b^*\right) \\
&+(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial\eta}{\partial t}-(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial Z_b^*}{\partial t}.
\end{aligned}
\tag{80}
$$

Since $(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})(\eta)\nabla\eta=(\mathbf{u}\otimes\mathbf{u})(\eta)(\mathbf{u}(\eta).\nabla\eta)$ and according to the boundary condition we have the following relations:

$$
(\mathbf{u}\otimes\mathbf{u})\frac{dF_u}{dt}=-w(\mathbf{u}\otimes\mathbf{u})(\eta)(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{\partial\eta}{\partial t}+(\mathbf{u}\otimes\mathbf{u})(\eta)\mathbf{u}(\eta)\nabla\eta
$$

and

$$
\begin{aligned}
(\mathbf{u}\otimes\mathbf{u})\frac{dF_b}{dt}=&-w(\mathbf{u}\otimes\mathbf{u})\left(Z_b^*\right)+(\mathbf{u}\otimes\mathbf{u})\left(Z_b^*\right)\frac{\partial Z_b^*}{\partial t} \\
&+(\mathbf{u}\otimes\mathbf{u})\left(Z_b^*\right)\mathbf{u}\left(Z_b^*\right)\nabla Z_b^*
\end{aligned}
$$

These relation will be introduce easily in the system of equations given by:

$$
\begin{aligned}
\frac{\partial\left(h\overline{K}\right)}{\partial t} &+ div\left(h\overline{K}\otimes\overline{\mathbf{u}}+\frac{\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}}{2}\right)+div\left(\overline{\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'}\right) \\
&+\left(\frac{gh}{2}\nabla\eta+gh^2\frac{1}{4\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}} \\
&+\overline{\mathbf{u}}\otimes\left(\frac{gh}{2}\nabla\eta+gh^2\frac{1}{4\rho}\nabla\rho\right)+\frac{1}{2}g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'} \\
&+\frac{1}{2}g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho \\
&=\underbrace{\frac{1}{2}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}})(\eta)\left[\frac{\partial\eta}{\partial t}+\overline{\mathbf{u}}(\eta)\nabla\eta-w(\eta)\right]}_{(VII)} \\
&+\underbrace{\frac{1}{2}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}})\left(Z^*{}_b\right)\left[\frac{\partial Z^*{}_b}{\partial t}+\overline{\mathbf{u}}\left(Z^*{}_b\right)\nabla Z_b^*-w(Z_b^*)\right]}_{(VIII)}+\frac{1}{2}\overline{\mathbf{u}}\otimes\overline{\frac{\mathcal{F}}{\rho}} \\
&+\frac{1}{2}\overline{\frac{\mathcal{F}}{\rho}}\otimes\overline{\mathbf{u}}.
\end{aligned}
\tag{81}
$$

The term (VIII) is the material derivative and does not vanish here. Derivative material is really a function which represents the balance between the particles that are eroded or deposited on the surface and the local bed porosity. The third order fluctuations $\overline{\mathbf{u}'\otimes\mathbf{u}'\otimes\mathbf{u}'}$ is assumed to be smaller than the second order fluctuations thus is neglected in this work. The account of this term

Assuming that $\overline{\mathbf{u}\otimes\frac{\mathcal{F}}{\rho}}\simeq\overline{\mathbf{u}}\otimes\overline{\frac{\mathcal{F}}{\rho}}$ and $\overline{\frac{\mathcal{F}}{\rho}\otimes\mathbf{u}}\simeq\overline{\frac{\mathcal{F}}{\rho}}\otimes\overline{\mathbf{u}}$, then Eq. (81) becomes:

$$
\begin{aligned}
\frac{\partial\left(h\overline{K}\right)}{\partial t} &+ div\left(h\overline{K}\otimes\overline{\mathbf{u}}+\frac{\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}}{2}\right) \\
&+\left(\frac{gh}{2}\nabla\eta+gh^2\frac{1}{4\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}}+\overline{\mathbf{u}}\otimes\left(\frac{gh}{2}\nabla\eta+gh^2\frac{1}{4\rho}\nabla\rho\right) \\
&=\frac{\overline{\mathbf{u}}\otimes\overline{\mathcal{F}}+\overline{\mathcal{F}}\otimes\overline{\mathbf{u}}}{2\rho}-\frac{1}{2}g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'}-\frac{1}{2}g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho \\
&+\frac{1}{2}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}(\eta))\frac{dF_u}{dt}-\frac{1}{2}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}(Z_b^*))\frac{dF_b}{dt},
\end{aligned}
\tag{82}
$$

where $\frac{\mathcal{F}}{\rho}$ integrates only the friction effect in shallow water context:

$$
\overline{\frac{\mathcal{F}}{\rho}}=C_f\|\overline{\mathbf{u}}\|\overline{\mathbf{u}}.
\tag{83}
$$

It is also possible to formulated the bed friction in distortion shallow water context and this is exposed in a recent work of Ngatcha and his collaborators. The term $C_f=\frac{gn^2}{h^{1/3}}$ being a friction factor and n is the Manning coefficient. The terms $(\mathbf{u}\otimes\mathbf{u})(\eta)$ and $(\mathbf{u}\otimes\mathbf{u})(Z_b^*)$ approximated using the averaged velocity value. The term $(\mathbf{u}\otimes\mathbf{u})(Z_b^*)\frac{dF_b}{dt}$ represents the kinetic transfer due to entrainment/deposition exchange between the mean current flow and the erodible bed. The term $(\mathbf{u}\otimes\mathbf{u})(\eta)\frac{dF_u}{dt}$ represents the kinetic transfer arising from water entrainment from water entrainment from clear water layer. The difficulties is to find the proportional coefficients adapted for the context. The exchanges between the bottom and the fluid medium are integrated in the momentum equations of the model. We can use the idea of Rajagopal (2021) to prove that the kinetic energy equation is bounded. And this purpose is in investigation. We also can prove that the derived model is energetically stable and consistent and such result is also in investigation.

Generalized shear shallow water equations: incorporating the non-uniform density

Neglecting the sediment exchange effects in Eqs. (46), (52) and (82), a generalized shear shallow water model (G-SSW for short) can be derived. The model take into account the variability of density of water and is more general than the classical model developed by Teshukov (2007). The equations of Generalized-SSW model in cartesian form are given by:

$$
\frac{\partial h}{\partial t}+\frac{\partial h\overline{u}_j}{\partial x_j}=\frac{dF_u}{dt},
\tag{84}
$$

$$
\frac{\partial h\overline{u}_i}{\partial t}+\frac{\partial 2h\overline{K}_{ij}}{\partial x_j}+gh\frac{\partial h}{\partial x_i}+\frac{gh^2\delta\rho}{2\rho}\frac{\partial\rho}{\partial x_i}=h\mathcal{F}_i+u_i(\eta)\frac{dF_u}{dt},
\tag{85}
$$

$$
\begin{aligned}
&\frac{\partial h\overline{K}_{ij}}{\partial t}+\frac{\partial}{\partial x_k}\left(h\overline{K}_{ij}+h\frac{\widehat{\mathcal{U}}_{jk}\overline{u}_i+\widehat{\mathcal{U}}_{ik}\overline{u}_j}{2}\right)+\frac{gh\overline{u}_j}{2}\frac{\partial h}{\partial x_i}+\frac{gh^2\overline{u}_j}{4\rho}\frac{\partial\rho}{\partial x_i} \\
&+\frac{gh\overline{u}_i}{2}\frac{\partial h}{\partial x_j}+\frac{gh^2\overline{u}_i}{4\rho}\frac{\partial\rho}{\partial x_j}= \\
&\frac{h\overline{u}_i\mathcal{F}_j+h\overline{u}_j\mathcal{F}_i}{2\rho}-\frac{1}{2}g\nabla\rho\overline{(\eta-x_3)u_i'}-\frac{1}{2}g\overline{u_j'(\eta-x_3)}\nabla\rho+\frac{1}{2}(u_iu_j)(\eta)\frac{dF_u}{dt},
\end{aligned}
\tag{86}
$$

$$
\frac{\partial\rho}{\partial t}+\frac{\partial\rho\overline{u}_j}{\partial x_j}=\rho(\eta)\frac{dF_u}{dt}.
\tag{87}
$$

4.4. Derivation of sediment concentration equation

We take an Eulerian approach for the sediment transport equations, rather than the more computationally expensive Lagrangian approach, and use a macroscopic assumption. Thus, we represent the sediment dynamics in suspension zone via an advection-diffusion equation. We consider here only non-cohesive sediment (there no exists physico-chemistry interaction between them). Furthermore, the low concentration and fine sediment size means the settling velocity of the sediment particles can be approximated by that of a single sediment particle in clear water. The equation modeling the sediment concentration is given by:

$$
\frac{\partial c}{\partial t}+\nabla.(\mathbf{u}c)=\nabla.(\mathbf{W}_sc\delta_{3\alpha})-\frac{\partial(wc)}{\partial z},
\tag{88}
$$

where $\delta_{3\alpha}$ is the Kronecker delta applied to the vertical component. Therefore depth-averaged Eq. (88) is:

$$
\underbrace{\int_{Z_b^*}^{\eta}\frac{\partial c}{\partial t}dz}_{A1}+\underbrace{\int_{Z_b^*}^{\eta}\nabla.(\mathbf{u}c)}_{A2}=\underbrace{\int_{Z_b^*}^{\eta}\nabla.(\mathbf{W}_sc\delta_{3\alpha})}_{A3}-\underbrace{\int_{Z_b^*}^{\eta}\frac{\partial(wc)}{\partial z}dz}_{A4}
\tag{89}
$$

where

$$
\begin{aligned}
A1&=\frac{1}{h}\frac{\partial}{\partial t}\int_{Z_b^*}^{\eta}hcdz+\frac{\partial Z_b^*}{\partial t}c\left(Z_f\right)-\frac{\partial\eta}{\partial t}c(\eta), \\
A2&=\nabla.\left(h\overline{\mathbf{u}c}\right)+\frac{\partial Z_b^*}{\partial t}\mathbf{u}\left(Z_b^*\right)c\left(Z_b^*\right)-\frac{\partial\eta}{\partial t}\mathbf{u}(\eta)c(\eta)
\end{aligned}
$$

Due to the coupled nature of our model. The calculation of $\overline{\mathbf{u}c}$ would not be clear. It is possible to calculate the product $\overline{\mathbf{u}}$(from fluid component) and $\overline{c}$ (from sediment transport). It is possible to calculate the product $\overline{\mathbf{u}}$ and $\overline{c}$. But really is not equal to $\overline{\mathbf{u}}\overline{c}$. In fact $\int_I \overline{\mathbf{u}}\overline{c}dx \neq \int_I \overline{\mathbf{u}c}dx,\ I \in \mathbb{R}^N$. We have therefore:

$$\begin{aligned}
&\mathbf{u}c = \overline{\mathbf{u}c} + \delta(\mathbf{u}c) \qquad (90)\\
&\Rightarrow \frac{\mathbf{u}c}{\overline{c}} = \frac{\overline{\mathbf{u}c}}{\overline{c}} + \frac{\delta(\mathbf{u}c)}{\overline{c}}\\
&\Rightarrow \mathbf{u}c = \mathbf{u}_{adv}\overline{c} + \overline{\delta(\mathbf{u})}\overline{c}\\
&\Rightarrow div\left(\frac{1}{h}\int_{Z_b^*}^{\eta} h\mathbf{u}c\,dz\right) = \nabla.\left(h\overline{\mathbf{u}c}\right) = \nabla.\left(h\mathbf{u}_{adv}\overline{c}\right) + \nabla.\left(h\overline{\delta(\mathbf{u})}\overline{c}\right).
\end{aligned}$$

The particles flux is defined as the product of a coefficient and the concentration gradient. Using the Fick's law we have: $\overline{\delta(\mathbf{u})}\overline{c} = -D_L\nabla\overline{c}$. We have also $\overline{\delta(\mathbf{u})}4(1-\overline{c}) = -D_L\nabla(1-\overline{c})$. In this formulation, $D_L = (D_L^x, D_L^y)$ is the diffusion coefficient in both x, y direction that can be assumed to be equal to the clear fluid viscosity ν. The particles are transported by the average and fluctuating motions and dispersed by the turbulence. Then we obtain:

$$\begin{aligned}
A2 &= \nabla.\left(h\mathbf{u}_{adv}\overline{c}\right) + \nabla.\left(-hD_L\nabla\overline{c}\right) + \frac{\partial Z_b}{\partial t}\mathbf{u}\left(Z_b\right)c\left(Z_b\right)\\
&= \nabla.\left(hF_{corr}\overline{\mathbf{u}}\overline{c}\right) + \nabla.\left(-hD_L\nabla\overline{c}\right) + \frac{\partial Z_b}{\partial t}\mathbf{u}\left(Z_b\right)c\left(Z_b\right)\\
A3 &= \nabla.\left(W_s h\overline{c}\delta_{3\alpha}\right),\\
A4 &= -w(\eta)c(\eta) + w\left(Z_b\right)c\left(Z_b\right).
\end{aligned}$$

The sediment concentration equation written:

$$\frac{\partial hC}{\partial t} + \nabla.\left(hF_{corr}\overline{\mathbf{u}}C\right) = \nabla.\left(hD_L\nabla C\right) + (E-D), \qquad (91)$$

where we have denoted $C = \overline{c}$. Eq. (91) represents mass conservation of sediment, which involves the flux of sediment exchange with the bed. In the dilute zone governed by advection/diffusion Eq. (91), the interaction between the particles are assumed weak and the mechanisms involved are the turbulent dispersive, sedimentation and the flocculation. The grains do not act on the fluid and vice-versa. The particles therefore behave as a passive tracer for the flow except for the fall velocity. For the calculation of eigenvalue we c consider simply that $F_{corr} = 1$.

Turbidity current model

Turbidity currents are driven by the buoyancy force arising from the bulk density excess due to the presence of suspended sediment, and occur in numerous man-made and natural situations (Hu and Cao, 2009). Over the recent decades, there have been a number of numerical and theoretical investigations, and mathematical modeling of turbidity currents because of their profound importance for some ocean engineering problems. Mathematical models based on shallow water equations have received great attention in the last several decades, including full three dimensional (3D), vertical two dimensional (2D) and depth-averaged models. These models cannot provide detailed flow structure information along the current depth. Moreover, these models are only approximately applicable to cases with steady (or slightly unsteady) flow. Near the bottom, it is well-known that the flow is turbulence and turbidity models based on Shallow water equations are not applicable. It therefore necessary to include the acceleration effect due to mobile bed in the flow structure. A model of turbidity with active sediment transport can be derived using the above Eqs. (84)–(87) and replacing the last equation instead of (91). However, the bed deformation can influence the evolution of turbidity currents.

4.5. New bed evolution modeling accounting distortion

Herein, the attention is focussed in the development of a general bedload transport equation which integrates the effect of turbulence that can used for both turbulent and steady flows. The bed evolution equation developed in this part is seen as an extension of Exner's equation Exner (1925) coupled with Grass approximation. It is obtained using the Reynolds transport theorem and the mass balance equation on a arbitrary material volume $\Omega(t)$ and integrating the turbulence effect. The time variation of Z_b is balanced by the horizontal flux $\boldsymbol{\Phi}_h$ and the vertical flux $\boldsymbol{\Phi}_v$. We can write the bed evolution equation as:

$$\frac{\partial Z_b}{\partial t} + \boldsymbol{\Phi}_h + \boldsymbol{\Phi}_v = 0. \qquad (92)$$

The horizontal flux accounts the shear effects which creates the turbulence by introducing the terms related to fluctuations. The shear effect influences the shear stress of the bed thus the bedload transport.

The horizontal flux is proportional to shear force gradient and is seen as the energy dissipation near the bed and according to Yalin (1977) and Cheng (2002) this term is closely related to sediment transport. The shear gradient force is given by $\nabla.\left(A_g\overline{\|u^2+v^2\|\mathbf{u}}\right)$, contents the energy due to grain friction and the energy due to turbulence. Note that the grain friction is the shear stress acting on the sediment grains directly both the water sediment move in same direction. Note However that the turbulence maintain the sediment in suspension and intensity of turbulence is proportional to the total boundary shear stress τ_0 which is typically dominated by forcing drag (McLean and Nelson, 1999). It thus necessary to integrate the turbulence in the bedload equation. According to Eq. (31), the vertical flux is seen as the erosion/deposition exchange. This flux can be approximated by RHS of Eq. (31):

$$\boldsymbol{\Phi}_h + \boldsymbol{\Phi}_v = \frac{A_g}{(1-\phi^*)}\left(\frac{\partial\overline{(u^2+v^2)u}}{\partial x} + \frac{\partial\overline{(u^2+v^2)v}}{\partial y}\right) - \frac{(D-E)}{(1-\phi^*)}. \qquad (93)$$

Integrating the distortion effect on the bedload equation via the shear gradient force we have:

$$\begin{aligned}
\overline{(u^2+v^2)\mathbf{u}} &= \overline{\left(\overline{u}+u'\right)\left((\overline{u}+u')^2+\overline{v}+v'\right)+\left(\overline{v}+v'\right)\left((\overline{u}+u')^2+\overline{v}+v'\right)} \qquad (94)\\
&= \overline{\overline{\mathbf{u}}(\overline{v}^2+\overline{u}^2)} + \overline{\overline{\mathbf{u}}(u'^2+v'^2)} + \overline{\mathbf{u}'(\overline{v}^2+\overline{u}^2)} + \overline{\mathbf{u}'(u'^2+v'^2)}\\
&\quad +2\overline{\mathbf{u}'(\overline{u}u'+\overline{v}v')}\\
&= \overline{\mathbf{u}}(\overline{u}_k\overline{u}_k + \overline{\mathbf{u}'\mathbf{u}}) + 2\overline{u'_k(u_k u'_k)} + \overline{\mathbf{u}'(u'^2+v'^2)}\\
&= \overline{\mathbf{u}}(\overline{u}_k\overline{u}_k + \overline{u'_k u'_k}) + 2\overline{u'_k u'_k}\overline{u}_k + \overline{\mathbf{u}'(u'^2+v'^2)}\\
&\approx 2(\overline{u}_k\overline{K}_{kk} + \widehat{\mathcal{U}}_{kk}\overline{u}_k) + \kappa\nabla\widehat{\mathcal{U}},
\end{aligned}$$

where we have approximated the third order fluctuations in term of second order fluctuations as: $\overline{\mathbf{u}'(u'^2+v'^2)} \approx \kappa\nabla\widehat{\mathcal{U}}$ and where $\overline{K}_{kk} = \frac{(\overline{u}_k\overline{u}_k + \overline{u'_k u'_k})}{2}$, $\widehat{\mathcal{U}}_{kk} = \overline{u'_k u'_k}$. Note that we have removed the dissipation term (according to the basis assumptions). Then we obtain an extended Exner equation without dissipation effect is given by:

$$\frac{\partial Z_b^*}{\partial t} + \frac{2A_g}{1-\phi^*}\frac{\partial}{\partial x_k}\left(\overline{K}_{kk}\overline{u}_k + \widehat{\mathcal{U}}_{kk}\overline{u}_k\right) = -\frac{(E-D)}{1-\phi^*}. \qquad (95)$$

We find a 2D bedload model well-known in the literature when we use (95) with $E-D=0$ and $\overline{u'_k u'_k} = 0$ (Castro Diaz et al., 2009; Paola and Voller, 2005) or when only the turbulence is removed (Hu and Cao, 2009; Cao et al., 2004; Gonzalez-Aguirre et al., 2020). In 1D case we have:

$$\frac{\partial Z_b}{\partial t} + \frac{2A_g}{1-\phi^*}\frac{\partial}{\partial x}(\overline{K}_{11}\overline{u} + \widehat{\mathcal{U}}_{11}\overline{u}) = -\frac{(E-D)}{1-\phi^*} \qquad (96)$$

The Well-known classical Exner's equation with Grass formula is found when $\widehat{\mathcal{U}}_{11} = 0$ and $E-D=0$ in Eq. (96) (Audusse et al., 2012; Deltares, 2014; Neary et al., 2001), etc. Near the bed, the intensity of the fluctuations $\frac{1}{2}\overline{\mathbf{u}'\otimes\mathbf{u}'}$ increases with the number of grains involved in the movement. Eq. (95) shows that bedload transport generates second-order velocity fluctuations in the turbulent event regions, whatever the size of the grains. It is expected that the fluctuations play an important role in the mobilization of near-bottom sediments and increase the suspension time of the sediment in the suspension zone.

This Exner-based equation often is not adapted to describe the morphodynamics (Ngatcha et al., 2022a). In fact, the turbulent flows cannot acts directly in the bottom motion since the hydrodynamic time is more small than the morphodynamic time (phase-lag). The phase-lag is another important mechanism appearing in the sediment transport in turbulent mixture flow. According to the observation in nature, the bedform's characteristic velocity, sediment velocity and fluid velocity must be different (Ngatcha et al., 2022a). In this new model there is a differentiation between these velocities. Using the boundary conditions, we propose a bed-load model able to well describe the motion of the bottom.

There is an alternative formulation, using the relation (37), that writes without dissipation effect as:

$$\frac{\partial Z_b^*}{\partial t} + \mathbf{u}(Z_b^*)\nabla Z_b^* = w(Z_b^*) - \frac{(E-D)}{1-\phi^*}. \tag{97}$$

This bedload equation is given without any consideration of Reynolds dissipation term $\nabla.(\kappa\nabla\widehat{\mathcal{U}})$, $\kappa \in \mathbb{R}_+^*$. This term is also removed in the averaged kinetic energy equation. Given a sedimentary form Z_b^*, two mechanisms influence its evolution. The advection and the erosion/deposition exchanges near the bottom which act strongly on its evolution on the one hand allowing it to move in the flow and on the other hand modifying its geometry. The term on the left is related to advection which highlights that the movement of sedimentary bodies is influenced by a mechanism directly related to the shape of the bottom. All together the derived depth-averaged coupled sediment transport model write as:

$$\frac{\partial h}{\partial t} + div(h\overline{\mathbf{u}}) = \frac{dF_u}{dt} + \frac{dF_b}{dt}, \tag{98}$$

$$\frac{\partial h\overline{\mathbf{u}}}{\partial t} + div(h\overline{K}) + gh\nabla\eta + \frac{1}{2\rho}gh^2\nabla\rho = \mathbf{u}(\eta)\frac{dF_u}{dt} - \mathbf{u}(Z_b^*)\frac{dF_b}{dt} + \frac{1}{\rho}h\overline{\mathcal{F}}, \tag{99}$$

$$\frac{\partial\left(h\overline{K}\right)}{\partial t} + div\left(h\overline{K}\otimes\overline{\mathbf{u}} + \frac{\widehat{\mathcal{U}}\otimes\overline{\mathbf{u}} + \overline{\mathbf{u}}\otimes\widehat{\mathcal{U}}}{2}\right) + \nabla\left(\frac{gh}{2}\nabla\eta + gh^2\frac{1}{4\rho}\nabla\rho\right)\otimes\overline{\mathbf{u}} \tag{100}$$

$$+\overline{\mathbf{u}}\otimes\nabla\left(\frac{gh}{2}\nabla\eta + gh^2\frac{1}{4\rho}\nabla\rho\right) = \frac{\overline{\mathbf{u}}\otimes\overline{\mathcal{F}} + \overline{\mathcal{F}}\otimes\overline{\mathbf{u}}}{2\rho} - g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'} - g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho + \frac{1}{2}\left(\mathbf{u}\otimes\mathbf{u}(\eta)\right)\frac{dF_u}{dt} - \frac{1}{2}\left(\mathbf{u}\otimes\mathbf{u}(Z_b)\right)\frac{dF_b}{dt},$$

$$\frac{\partial hC}{\partial t} + \nabla.\left(hF_{corr}\overline{\mathbf{u}}C\right) = \nabla.\left(hD_L\nabla\overline{c}\right) + (E-D), \tag{101}$$

$$\frac{\partial Z_b}{\partial t} + \frac{2A_g}{1-\phi^*}\nabla.(\overline{K}\overline{\mathbf{u}} + \widehat{\mathcal{U}}\overline{\mathbf{u}}) = -\frac{(E-D)}{1-\phi^*}, \tag{102}$$

where all the parameters associated to the model are exposed through the text and all the closure relations of the model given by (98) are presented in Appendix (see below). The approach presented here incorporates more physical concepts than those used in classical literature; to represent the turbulent processes near the bottom or in the suspension zone. Conventional approach dealing with the water/sediment mixture are not consistent with the physics of the flow. An approach integrating turbulence is needed to simulate these effects in the bed morphodynamics. Other terms that are seen as the moment of fluctuation on the free surface needed approximation or more explain:

$$g\nabla\rho\overline{(\eta-z)\otimes\mathbf{u}'} \quad \text{and} \quad g\overline{\mathbf{u}'\otimes(\eta-z)}\nabla\rho.$$

Note that in the case where we constraint $\widehat{\mathcal{U}} = 0$, when vertical fluctuations of the velocity are neglected, the equation for the energy $\overline{K}$ is useless and we recover the model used in Liu et al. (2015). If in addition we use $A_g = 0$, then the present model degenerates to the one developed by Cao et al. (2004). In cartesian coordinates the derived model we have:

$$\frac{\partial h}{\partial t} + \frac{\partial h\overline{u}_j}{\partial x_j}0 = \frac{dF_u}{dt} + \frac{dF_b}{dt}, \tag{103}$$

$$\frac{\partial h\overline{u}_i}{\partial t} + \frac{\partial 2h\overline{K}_{ij}}{\partial x_j} + gh\frac{\partial h}{\partial x_i} + gh\frac{\partial Z_b^*}{\partial x_i} + \frac{gh^2\delta\rho}{2\rho}\frac{\partial C}{\partial x_i} = h\overline{\mathcal{F}}_i + u_i(\eta)\frac{dF_u}{dt} - u_i(Z_b^*)\frac{dF_b}{dt}, \tag{104}$$

$$\frac{\partial h\overline{K}_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(h\overline{K}_{ij} + h\frac{\widehat{\mathcal{U}}_{jk}\overline{u}_i + \widehat{\mathcal{U}}_{ik}\overline{u}_j}{2}\right) + \frac{gh\overline{u}_j}{2}\frac{\partial h}{\partial x_i} + \frac{gh\overline{u}_j}{2}\frac{\partial Z_b^*}{\partial x_i} + \frac{gh^2\delta\rho\overline{u}_j}{4\rho}\frac{\partial C}{\partial x_i} + \frac{gh\overline{u}_i}{2}\frac{\partial h}{\partial x_j} \tag{105}$$

$$+\frac{gh\overline{u}_i}{2}\frac{\partial Z_b^*}{\partial x_j} + \frac{gh^2\delta\rho\overline{u}_i}{4\rho}\frac{\partial C}{\partial x_j} = \frac{hu_i\mathcal{F}_j + hu_j\mathcal{F}_i}{2\rho} + K_{ij}^{\eta}\frac{dF_u}{dt} - K_{ij}^{Z_b^*}\frac{dF_b}{dt},$$

$$\frac{\partial hC}{\partial t} + \frac{\partial F_{corr}hC\overline{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_{ij}^L\frac{\partial C}{\partial x_j}\right), \tag{106}$$

$$\frac{\partial Z_b^*}{\partial t} + \gamma\frac{\partial}{\partial x_k}\left(\overline{K}_{kk}\overline{u}_k + \widehat{\mathcal{U}}_{kk}\overline{u}_k\right) = -\frac{dF_b}{dt}, \tag{107}$$

where $\gamma = \frac{2A_g}{1-\phi^*}$, $K_{ij}^{\eta} = \frac{1}{2}(u_iu_j)(\eta)$, $K_{ij}^{Z_b^*} = \frac{1}{2}(u_iu_j)(Z_b^*)$.

The rest of closure relations are available in Appendix A. To obtain a well-posed problem we add to (103)–(107), some initial conditions: $h(0,\mathbf{x}) = h^0(\mathbf{x})$, $h\overline{\mathbf{u}}(0,\mathbf{x}) = h\overline{\mathbf{u}}^0(\mathbf{x})$, $C(0,\mathbf{x}) = C^0(\mathbf{x})$, $Z_b^*(0,\mathbf{x}) = Z_b^{*0}(\mathbf{x})$, $(h\overline{K})(0,\mathbf{x}) = (h\overline{K})^0(\mathbf{x})$, $\widehat{\mathcal{U}}(0,\mathbf{x}) = \widehat{\mathcal{U}}^0(\mathbf{x})$, and boundary conditions: $h.\mathbf{n} = 0$, $\mathbf{q}.\mathbf{n} = 0$, $(hC).\mathbf{n} = 0$, $\widehat{\mathcal{U}}.\mathbf{n} = 0$, $(h\overline{K}).\mathbf{n} = 0$, $Z_b^*.\mathbf{n} = 0$.

Similarity assumptions

It is usual to express the velocity at the surface and bottom $\mathbf{u}(\mathbf{x},t,\eta(\mathbf{x},t))$ and $\mathbf{u}(\mathbf{x},t,Z_b^*(\mathbf{x},t))$ in term of mean velocity $\overline{\mathbf{u}}$. It is important to know that $\mathbf{u}(\mathbf{x},t,Z_b^*(\mathbf{x},t))$ in momentum equations and $\mathbf{u}(\mathbf{x},t,Z_b^*(\mathbf{x},t))$ in bedload equation do not have same role. We can assume that:

$$\mathbf{u}(\mathbf{x},t,\eta(\mathbf{x},t)) = \overline{\mathbf{u}}(\mathbf{x},t), \text{ and } \mathbf{u}(\mathbf{x},t,Z_b^*(\mathbf{x},t)) = \overline{\mathbf{u}}_s(\mathbf{x},t) \tag{108}$$

The first equation in (108) is obtained by using the fact that at the free surface there no exists turbulent (mean motion). This assertion means that:

$$\overline{(\mathbf{u}-\overline{\mathbf{u}})(\mathbf{u}-\overline{\mathbf{u}})} = 0 \text{ at } z = \eta \tag{109}$$

where $\mathbf{u} = \mathbf{u}(\eta)$.

In the second equation of the sediment velocity can be given by: $\overline{\mathbf{u}}_s(\mathbf{x},t) = \sigma\overline{\mathbf{u}}(\mathbf{x},t), \sigma > 0$. We can take for coastal flows with sediment transport $\sigma = 0.5$ but this can be modified following the environment context.

Thus, we developed a new mathematical model which accounts for the distortion of horizontal velocity profile along the vertical, the effect of sediment exchange, and the non-uniformity of fluid density. This model generalizes several recent averaged sediment transport models widely used in several papers and books.

Remark 4.2 (*Convergence to the Several Classical Models on Distortion Shallow Water and Shallow Water Contexts*)**.** • When $\widehat{\mathcal{U}} = 0$, we find a version of so-called fully-coupled sediment transport models of Cao et al. (2004), Hu and Cao (2009), Gonzalez-Aguirre et al. (2020), Garcia and Parker (1991), Clare et al. (2021), Ngatcha et al. (2022b), Ngatcha et al. (2022a) and Liu et al. (2015). • The proposed theory extend to the one developed by Ngatcha et al. (2022d). • When we removed the contribution of deposition/erosion exchange and assuming the uniform density, we find easily a version of the homogeneous shear shallow water model with friction source term developed by Richards and Gavrilyuk (2013). • When the source terms and the sediment exchange effects in above equation are removed, we find easily the classical distortion (or shear) shallow water equations developed by Teshukov (2007). • In absence of distortion effect and sediment contribution and

source terms, we find the classical shallow water equations of Barré de Saint-Venant (1871). An analytical study of this model is questionable because the mixture flow with distortion moves between two mobile domains (free surface and bottom). The finding of a exact solution is only possible for a reduced model obtained when the sediment contribution is removed (see the work of Chandrashekar et al. (2020), Nkonga and Chandrashekar (2021)). Such study will complete the class of analytical study of unsteady flow performed in Fetecau and Vieru (2020). Also, the stability in term of energy of the proposed nonhomogeneous fluid flows structure is an open problem. The smooth solutions of the model satisfy also an additional energy conservation equation presented below. Moreover its admits eight eigenvalues that ensures the existence of wave solutions of the model.

5. Reformulation, total energy and hyperbolicity

In this section, we propose to reformulate the derived model in quasi nonconservative form to propose a brief hyperbolicity study. We will show that when Grass approximation is used the hyperbolicity can loss. We will also propose a total energy conservation for smooth solution of the model.

5.1. Reformulation of the model

The proposed model is a nonconservative nonlinear hyperbolic that can put in the following compact form:

$$\begin{aligned}&\frac{\partial W}{\partial t}+\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+gh\mathbf{B}_x^*\frac{\partial Z_b^*}{\partial x}+gh\mathbf{B}_y^*\frac{\partial Z_b^*}{\partial y}+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^*\frac{\partial C}{\partial x}\\&+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_y^*\frac{\partial C}{\partial y}+\\&gh\mathbf{B}_x^K\frac{\partial h}{\partial x}+gh\mathbf{B}_y^K\frac{\partial h}{\partial y}=S(W),\end{aligned}\tag{110}$$

where

$$W=\begin{pmatrix}h\\hu\\hv\\h\overline{K}_{11}\\h\overline{K}_{12}\\h\overline{K}_{22}\\hC\\Z_b^*\end{pmatrix},\quad \mathbf{B}_x^*=\mathbf{B}_x^K=\begin{pmatrix}0\\1\\0\\u\\\frac{1}{2}v\\0\\0\\0\end{pmatrix},\quad \mathbf{B}_y^*=\mathbf{B}_y^K=\begin{pmatrix}0\\0\\1\\0\\\frac{1}{2}u\\v\\0\\0\end{pmatrix},\tag{111}$$

and where the physical flux are:

$$F_1=\begin{pmatrix}hu\\h\widehat{U}_{11}+hu^2\\h\widehat{U}_{12}+huv\\\left(h\overline{K}_{11}+h\widehat{U}_{11}\right)u\\h\overline{K}_{12}u+1/2\left(h\widehat{U}_{11}v+h\widehat{U}_{12}u\right)\\h\overline{K}_{22}u+h\widehat{U}_{12}v\\huC\\\gamma\left(\overline{K}_{11}u+\widehat{U}_{11}u\right)\end{pmatrix},$$

$$F_2=\begin{pmatrix}hv\\h\widehat{U}_{12}+huv\\h\widehat{U}_{22}+hv^2\\\left(hv\overline{K}_{11}+hu\widehat{U}_{12}\right)\\h\overline{K}_{12}v+1/2\left(hv\widehat{U}_{12}+hu\widehat{U}_{22}\right)\\h\overline{K}_{22}v+h\widehat{U}_{22}v\\hvC\\\gamma\left(\overline{K}_{22}v+\widehat{U}_{22}v\right)\end{pmatrix},\tag{112}$$

and where

$$S(W)=\begin{pmatrix}\frac{dF_u}{dt}+\frac{dF_b}{dt}\\h\frac{\mathcal{F}_1}{\rho}+u(\eta)\frac{dF_u}{dt}-u(Z_b^*)\frac{dF_b}{dt}\\h\frac{\mathcal{F}_2}{\rho}+v(\eta)\frac{dF_u}{dt}-v(Z_b^*)\frac{dF_b}{dt}\\hu\frac{\mathcal{F}_1}{\rho}+K_{11}^\eta\frac{dF_u}{dt}-K_{11}^{Z_b^*}\frac{dF_b}{dt}\\\frac{hv\mathcal{F}_1+hu\mathcal{F}_2}{2\rho}+K_{12}^\eta\frac{dF_u}{dt}-K_{12}^{Z_b^*}\frac{dF_b}{dt}\\hv\mathcal{F}_2+K_{22}^\eta\frac{dF_u}{dt}-K_{22}^{Z_b^*}\frac{dF_b}{dt}\\\frac{\partial}{\partial x}\left(D_L^x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_L^y\frac{\partial C}{\partial y}\right)+(1-\phi^*)\frac{dF_b}{dt}\\-\frac{dF_b}{dt}\end{pmatrix}.\tag{113}$$

We can decompose the source term into some others well-known source terms as follows:

$$S(W)=S_{F_b}+S_F+S_w(W)+S_C(W),\tag{114}$$

where S_{F_b}, S_F, S_w, S_C, S_D are respectively sediment exchange source term, friction source term, the water exchange source term, the suspended load sediment concentration variation and the dissipation source term given by:

$$S_{F_b}(W)=\begin{pmatrix}\frac{dF_b}{dt}\\-u(Z_b)\frac{dF_b}{dt}\\-v(Z_b)\frac{dF_b}{dt}\\-K_{11}^{Z_b^*}\frac{dF_b}{dt}\\-K_{12}^{Z_b^*}\frac{dF_b}{dt}\\-K_{22}^{Z_b^*}\frac{dF_b}{dt}\\(1-\phi^*)\frac{dF_b}{dt}\\-\frac{dF_b}{dt}\end{pmatrix},\quad S_w(W)=\begin{pmatrix}\frac{dF_u}{dt}\\u(\eta)\frac{dF_u}{dt}\\v(\eta)\frac{dF_u}{dt}\\K_{11}^\eta\frac{dF_u}{dt}\\K_{12}^\eta\frac{dF_u}{dt}\\K_{22}^\eta\frac{dF_u}{dt}\\0\\0\end{pmatrix},$$

$$S_F(W)=\begin{pmatrix}0\\h\frac{\mathcal{F}_1}{\rho}\\h\frac{\mathcal{F}_2}{\rho}\\hu\frac{\mathcal{F}_1}{\rho}\\\frac{hv\mathcal{F}_1+hu\mathcal{F}_2}{2\rho}\\hv\frac{\mathcal{F}_2}{\rho}\\0\\0\end{pmatrix},\tag{115}$$

$$S_C(W)=\begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{\partial}{\partial x}\left(D_L^x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_L^y\frac{\partial C}{\partial y}\right)\\0\end{pmatrix}$$

The system (110) is genuinely nonconservative and its numerical approximation, in the context of finite volume schemes, needs some specific treatments. The Jacobian matrix associated with the conservative fluxes F_1 and F_2 is diagonalizable (Hyperbolic system). When adding the nonconservative contribution associated with the derivative of Z_b^* the hyperbolicity of the LHS (Left-hand side) is no more guaranteed. From a mathematical point of view, the two-dimensional averaged sediment model admits a major difficulty related to hyperbolicity. This will lead to further numerical complications. In this case, the Lagrange theorem or Gerschgorin disc theorem can be used to find easily eigenstructure of the model (Ngatcha et al., 2022b; Liu et al., 2015). The alternative proposed model does not require the use of these theorems and the eigenvalues and eigenvectors are found easily. The system (110) has been obtained using the formulation (95). When we use the alternative formulation for the bedload Eq. (97), we get the following system:

$$\begin{aligned}&\frac{\partial W}{\partial t}+\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+gh\mathbf{B}_x^*\frac{\partial Z_b^*}{\partial x}+gh\mathbf{B}_y^*\frac{\partial Z_b^*}{\partial y}+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^*\frac{\partial C}{\partial x}\\&+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_y^*\frac{\partial C}{\partial y}\\&+\ gh\mathbf{B}_x^K\frac{\partial h}{\partial x}+gh\mathbf{B}_y^K\frac{\partial h}{\partial y}=S(W)\end{aligned}\tag{116}$$

where

$$W=\begin{pmatrix}h\\hu\\hv\\h\overline{K}_{11}\\h\overline{K}_{12}\\h\overline{K}_{22}\\hC\\Z_b^*\end{pmatrix},\quad \widetilde{\mathbf{B}}_x^*=\begin{pmatrix}0\\gh\\0\\ghu\\1/2ghv\\0\\0\\u_{Z_b^*}\end{pmatrix},\quad \widetilde{\mathbf{B}}_y^*=\begin{pmatrix}0\\0\\gh\\0\\1/2ghu\\ghv\\0\\v_{Z_b^*}\end{pmatrix}\tag{117}$$

$$F_1=\begin{pmatrix}hu\\h\widehat{\mathcal{U}}_{11}+hu^2\\h\widehat{\mathcal{U}}_{12}+huv\\\left(h\overline{K}_{11}+h\widehat{\mathcal{U}}_{11}\right)u\\h\overline{K}_{12}u+1/2\left(h\widehat{\mathcal{U}}_{11}v+h\widehat{\mathcal{U}}_{12}u\right)\\h\overline{K}_{22}u+h\widehat{\mathcal{U}}_{12}v\\huC\\0\end{pmatrix},$$

$$F_2=\begin{pmatrix}hv\\h\widehat{\mathcal{U}}_{12}+huv\\h\widehat{\mathcal{U}}_{22}+hv^2\\\left(hv\overline{K}_{11}+hu\widehat{\mathcal{U}}_{12}\right)\\h\overline{K}_{12}v+1/2\left(hv\widehat{\mathcal{U}}_{12}+hu\widehat{\mathcal{U}}_{22}\right)\\h\overline{K}_{22}v+h\widehat{\mathcal{U}}_{22}v\\hvC\\0\end{pmatrix}\tag{118}$$

Or, in matrix form:

$$\begin{aligned}&\frac{\partial W}{\partial t}\ +\ A_1(W)\frac{\partial W}{\partial x}+A_2(W)\frac{\partial W}{\partial y}+gh\mathbf{B}_x^*\frac{\partial Z_b^*}{\partial x}\\&+\ gh\mathbf{B}_y^*\frac{\partial Z_b^*}{\partial y}\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^*\frac{\partial C}{\partial x}\\&+\ \frac{gh^2\delta\rho}{2\rho}\mathbf{B}_y^*\frac{\partial C}{\partial y}+gh\mathbf{B}_x^K\frac{\partial h}{\partial x}+gh\mathbf{B}_y^K\frac{\partial h}{\partial y}=S(W)\end{aligned}\tag{119}$$

We can express (119) as follows:

$$\frac{\partial W}{\partial t}+\mathcal{A}_1(W)\frac{\partial W}{\partial x}+\mathcal{A}_2(W)\frac{\partial W}{\partial y}=S(W),\tag{120}$$

where

$$\mathcal{A}_1(W)\ =\ A_1(W)+gh\mathbf{B}_x^*\frac{\partial Z_b^*}{\partial x}+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^*\frac{\partial C}{\partial x}+gh\mathbf{B}_x^K\frac{\partial h}{\partial x},$$

$$\mathcal{A}_2(W)\ =\ A_1(W)+gh\mathbf{B}_y^*\frac{\partial Z_b^*}{\partial y}+\frac{gh^2\delta\rho}{2\rho}\mathbf{B}_y^*\frac{\partial C}{\partial y}+gh\mathbf{B}_y^K\frac{\partial h}{\partial y}.$$

Remark 5.1. The above HSTM given by (116) is genuinely two-dimensional nonconservative hyperbolic equations that required the use of some concepts associated with nonconservative hyperbolic equations that are widely used in literature. The modification of the formulation is playing a role only at the contact or slip lines, which are linearly degenerated fields.

The smooth solution must satisfy some positivity constraints which leads to the following solution space for physically admissible solutions

$$\mathcal{W}=\left\{W\in\mathbf{R}^8,\quad h>0,C>0\quad \widehat{\mathcal{U}}>0\right\},$$

where means that the symmetric kinetic energy tensor must be positive definite.

5.2. Conservation of the total energy for smooth solutions

Here, we investigate additional conservation laws satisfied by the HSTM presented above. The total energy equation is given by:

$$\begin{aligned}&\frac{\partial\mathcal{E}}{\partial t}+\frac{\partial}{\partial x}\left[\left(\mathcal{E}+h\widehat{\mathcal{U}}_{11}+\frac{1}{2}gh^2\right)u\right]+\frac{\partial}{\partial y}\left[\left(\mathcal{E}+h\widehat{\mathcal{U}}_{22}+\frac{1}{2}gh^2\right)v\right]=\\&hu\mathcal{F}+\left(K_{11}^{\eta}+K_{22}^{\eta}+gh\right)\frac{dF_b}{dt}+\left(K_{11}^{Z_b^*}+K_{22}^{Z_b^*}+gh\right)\frac{dF_u}{dt}+\\&ghu\frac{\partial Z_b^*}{\partial x}+ghv\frac{\partial Z_b^*}{\partial y}+\frac{gh^2\delta\rho}{2\rho}\frac{\partial C}{\partial x}+\frac{gh^2\delta\rho}{2\rho}\frac{\partial C}{\partial y},\end{aligned}\tag{121}$$

where

$$\begin{aligned}\mathcal{E}\ &=\ h\overline{K}_{11}+h\overline{K}_{22}+1/2gh^2,\\&=\ trace(h\overline{K})+1/2gh^2,\\&=\ \frac{h}{2}trace(\widehat{\mathcal{U}})+\frac{h}{2}|\mathbf{u}|^2+1/2gh^2.\end{aligned}$$

This equation is obtained by multiplying h equation (first equation of (103)) by gh and by adding it to $h\overline{K}_{11}$ and $h\overline{K}_{22}$ equations. The quantity $\mathcal{E}=\mathcal{E}(W)$ is a convex function since $trace(\widehat{\mathcal{U}})=\widehat{\mathcal{U}}_{11}+\widehat{\mathcal{U}}_{22}$ no include $\widehat{\mathcal{U}}_{12}$, $\mathcal{E}$ is not strictly convex.

5.3. Hyperbolicity

The stability of the model is closely related with the hyperbolicity of the governing equations. We prove here the stability of the model in terms of hyperbolicity but more analyzes remain necessary. The one-dimensional sediment transport model in distortion shallow water mixing flow can be written in non-conservative form as follows:

$$\frac{\partial W}{\partial t}+\mathcal{A}_1(W)\frac{\partial W}{\partial x}=S(W)\tag{122}$$

where $\mathcal{A}_1$ is defined above and where W is the vector unknowns defined as: $W=\left(h,hu,hv,h\overline{K}_{11},h\overline{K}_{12},h\overline{K}_{22},hC,Z_b^*\right)^T$.

Définition 5.1. The system (110) is said to be hyperbolic if the matrix $\mathcal{A}(W)$ admits N_λ distinct reals values, then the system is said strictly hyperbolic. With such eigenvalues, we define the eigenvectors associated $E_1(W),E_2(W),\dots,E_{N_\lambda}(W)$. Therefore it is possible to define ones matrix $P=(E_m)_{1\le m\le N_\lambda}$ and $P^{-1}(W)$ with the diagonalization property, the Jacobian can be rewritten:

$$\mathcal{A}(W)=P(W)\Lambda(W)P^{-1}(W)\tag{123}$$

where $\Lambda(W)=diag(\lambda^1(W),\dots,\lambda^{N_\lambda}(W))$ is a diagonal matrix composed by the eigenvalues of the Jacobian matrix. The system is supposed to be strictly hyperbolic and the characteristic fields E_i, $i=1,\dots,N_\lambda$ are supposed to be either genuinely nonlinear: $\nabla\lambda_i(W).E_i(W)\neq 0,\forall W\in\Omega$ or linearly degenerate: $\nabla\lambda_i(W).E_i(W)=0,\forall W\in\Omega$. The characteristic polynomial of the coefficients matrix $\mathcal{A}$ is obtained by setting $\det(\mathcal{A}-\lambda I)=0$ where I is the 8×8 identity matrix.

At this step, one solve eight nonconservative laws. To understand the structure of the eigenfields, consider the governing equation x-direction.

The eigenvalues of the matrix $\mathcal{A}_1$ are given by:

$$\lambda_{1,8} = u \pm \sqrt{gh + 3\widehat{\mathcal{U}}_{11}}, \quad \lambda_3 = u_b, \quad \lambda_{4,5,6} = u, \qquad (124)$$
$$\lambda_{2,7} = u \pm \sqrt{\widehat{\mathcal{U}}_{11}}.$$

In two-dimensional case, we have:

$$\mathcal{A} = \mathcal{A}.\mathbf{n} = \mathcal{A}_1.\mathbf{n}_1 + \mathcal{A}_2.\mathbf{n}_2, \qquad (125)$$

and the eigenvalues become:

$$\lambda_{1,8} = \mathbf{u}.\mathbf{n} \pm \sqrt{gh + 3\widehat{\mathcal{U}} : \mathbf{n}\otimes\mathbf{n}}, \quad \lambda_3 = \mathbf{u}_b.\mathbf{n}, \quad \lambda_{4,5,6} = \mathbf{u}.\mathbf{n}, \qquad (126)$$
$$\lambda_{2,7} = \mathbf{u}.\mathbf{n} \pm \sqrt{\widehat{\mathcal{U}} : \mathbf{n}\otimes\mathbf{n}}.$$

Here, $a = \sqrt{gh + 3\mathcal{U} : \mathbf{n}\otimes\mathbf{n}}$ can be considered as the total shock velocities, the presence of shock does not disturb the positive definiteness of $\widehat{\mathcal{U}}$. For classical distortion shallow water with constant density, the fact that shock velocity, coincide with a leads to the equivalence between the conservative of energy and conservative of entropy. The developed model admits four types of waves: a discontinuity propagating with the velocity $\mathbf{u}.\mathbf{n}$, surface gravity wave propagating (or longitudinal waves) with the velocity $\mathbf{u}.\mathbf{n}\pm a$, distortion waves propagating with the velocity $\mathbf{u}.\mathbf{n} \pm \sqrt{\mathcal{P} : \mathbf{n}\otimes\mathbf{n}}$ and bed wave $\mathbf{u}_b.\mathbf{n}$. The eigenfields corresponding to the contact discontinuities, bed waves and shear waves are linearly degenerate in the sense of Lax and Wendroff (1960), while the eigenfields corresponding to the surface gravity waves are genuinely non-linear.

Remark 5.2. The hyperbolicity of the left hand side is more guaranty by using alternative bedload formulation (which incorporates phase-lag effect). The hyperbolicity is obtained without the use the Gerschgorin or Lagrange theorems as in Liu et al. (2015), and Ngatcha et al. (2022b). We can adapt the flux vector splitting approach as in Ngatcha et al. (2022c) and identify one subsystem of conservative equations (advection system) and one of non-conservative equations (pressure system), both having a very simple eigenstructure compared to the full system. The hyperbolicity proved ensures the existence and the stability of wave solutions of the proposed mathematical model. The convergence of this solution from a physical point of view ensures the robustness of the model. Distortion waves in mechanics and distortion wave in our case present a analogy. Several other properties of the model are currently being studied and will be presented in future publications. The sediment transport theory presented here can be applied to several environment contexts without major modifications.

6. Conclusion

In this paper, a derivation of a mathematical model from multi-fluid equations coupled with a vertical integration technique and accounting fluctuating motion (due to the distortion of horizontal fluid velocity) is proposed. The model described here assumes 3D flows and after averaging gives 2D equations that contain the kinetic energy of fluid particle distortion motion, dissipation contribution plus a turbulent kinetic energy evolution equation. The fluctuating motion is represented by a characteristic velocity that modifies all the structures of 2D averaged nonhomogeneous coastal flows and even the bed evolution equation. Particularly, the derived hyperbolic model describes the sediment transport and sediment bed morphodynamic in nonhomogeneous distortion shallow water flow. First, an extension of the classical distortion shallow water flow is presented and the resulting model is named generalized distortion shallow water flows. The Generalized model that accounts the density differences and many momentum transfer terms, extends practically all the classical averaged turbulent flow structure models available in the literature recently published. Second, when the density of the fluid is influenced by the presence of sediment, a more general sediment transport model with bed morphodynamics in generalized distortion shallow water flows is obtained. The theory presented here is viewed as an extension of recent averaged sediment transport models based on shallow water equations (that neglect the fluctuation motion). It is observed that the distortion characteristic velocity modifies the momentum equations in both direction x and y and adding a new conservation law equation that also satisfy the unknowns of the problem. The variable $\widehat{\mathcal{U}}$ also modifies the morphodynamic equation and leads to a new bedload transport model accounting the phase-lag effect. Therefore, we have seen that an averaged sediment transport model for coastal environment applications must integrate turbulence (due to distortion), fluid density variation, exchange and transfer mechanisms, phase-lag and more other relevant physical and hydrodynamic processes. As a result, the validity regime of the present model is extended, which gives a more appropriate framework for the study of coastal flows in real situation. The proposed models will then be the subject of numerical approximations in the context of Multi-dimensional path-conservative central-upwind methods recently put in place by Ngatcha et al. (2022b). The theory remains open to further developments by other anonymous authors with a view to improving it.

Declaration of competing interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Data availability

No data was used for the research described in the article

Acknowledgments

This work was supported both by the Inria Associate team program (AMFoDUC) and the CNRS program Dispositif de Soutien aux Collaborations avec l'Afrique subsaharienne » (AMoFlowCiGUC). The first author thanks EMS SIMONS GRANTS FOR AFRICA. We would like also Abdou Njifenjou for his comments and suggestions.

Appendix A. Closure models

In this subsection, we expose some relations more relevant for the description of erosion/deposition processes. Some of classical relations well-known in the literature are used or modified for the context nonhomogeneous distortion shallow water. However, these relations can be improved for other contexts.

Water/sediment mixture entrainment

The function dF_b is the increment of volume and $\frac{dF_b}{dt}$ describes the morphological change of the bathymetric (is the volume rate exchange per unit of time). $\frac{dF_b}{dt}$ can be seen as the balance between the amount of sediment left behind by the current (deposited) and the amount of sediment carried away (eroded) from the bed-load interface

$$dF_b = dV_s^E - dV_s^D + \phi^* dF_b, \Rightarrow \frac{dF_b}{dt} = \frac{dV_s^E}{dt} - \frac{dV_s^D}{dt} + \phi^* \frac{dF_b}{dt},$$
$$\Rightarrow \left(1-\phi^*\right)\frac{dF_b}{dt} = E - D \qquad (A.1)$$

where $D = \frac{dV_s^D}{dt}$, $\quad E = \frac{dV_s^E}{dt}$.

Then

$$\frac{dF_b}{dt} = \frac{E-D}{1-\phi^*}, \qquad (A.2)$$

We recall that $\frac{dF_u}{dt} = E_w\|\overline{\mathbf{u}}\|$ where $E_w = f(R_i)$ is the entrainment coefficient of water depending on global Richardson number [?]. $E_w = 0.075(1 + 718R_i^{2.4})^{-0.5}$ obtained by regression with important experimental dispersion. When

$R_i \to 0, E_w = 0.075;$
$R_i \to 1, E_w = 0.003;$

R_i is the Richardson number given by: $R_i = \frac{g'h}{\mathbf{u}^2} = \frac{g'h\mathbf{u}}{\mathbf{u}^3}$. where g' is given above.

Erosion/Deposition processes

E and D represent the entrainment and deposition terms determined by assuming that the sediments are non-cohesive. We use empirical relations for E and D[kg/m^2/s] accounting sediment supply condition and longitudinal bottom slope often neglected in several published works.

$$D = W_s(1 - C_a)^m C_a, \tag{A.3}$$

where the settling velocity W_s is given by:

$$W_s = \sqrt{\left(\frac{13.95\nu}{d_{50}}\right)^2 + 1.095sgd_{50}} - 13.95\frac{\nu}{d_{50}}, \tag{A.4}$$

where ν is the kinematic viscosity of water ($\nu = 1.2 \times 10^{-6}$), d is the average diameter of sediment particles, $s = \frac{\rho_s}{\rho_w} - 1$ is the submerged specific gravity of sediment, where ρ_s is the sediment density and ρ_w the water density. The parameter m is the exponent against experimental data indicating the hindered influence of high sediment concentrations on settling velocity. The function C_a is the local near-bed sediment concentration in volume, which can be determined for uniform grain size by:

$$C_a = \alpha_c C, \tag{A.5}$$

where α_c is given by (Holly and Rahuel, 1990):

$$\frac{1}{\alpha_c} = \begin{cases} \frac{|A(1 - A^r)|}{r}, & |Ro - 1| > 10^{-4} \\ |-A\log(A)|, & |Ro - 1| \le 10^{-4} \end{cases} \tag{A.6}$$

where r is given by:

$$r = \begin{cases} \min(Ro - 1, 3), & |Ro - 1| > 10^{-4} \\ 0, & |Ro - 1| \le 10^{-4} \end{cases} \tag{A.7}$$

where $A = \max\left(\frac{\delta_a}{h}, 1\right)$ and where $Ro = \frac{W_s}{\kappa \mathbf{u}_*}$ is the Rouse number used to define the mode of sediment transport (bed-load or suspension). The function δ_a is the height of the bedload zone:

$$\delta_a = \max(0.007(\tau_b - \tau_{cr})\rho_w, 0) + k_s \tag{A.8}$$

The active layer is the top bed layer participating in the sediment exchange between the water depth and the subsurface while the subsurface layers provide sediments to or receive sediments from the active layer.

The function F_{corr} represents the correction of averaged-velocity $\overline{\mathbf{u}}$ (Huybrechts et al., 2010):

$$F_{corr} = \frac{I_2 - \log(B/30)I_1}{I_1\log(eB/30)}, \tag{A.9}$$

where I_1, I_2 are given by (Clare et al., 2021):

$$I_1 = \begin{cases} \frac{1}{1 - Ro}(1 - B^{1-Ro}), R \neq 1 \\ -\log(B), Ro = 1 \end{cases}, \quad \text{and}$$

$$I_2 = \begin{cases} \frac{I_1 + \log(B)B^{1-Ro}}{Ro - 1}, Ro \neq 1 \\ -0.5(\log(B))^2, Ro = 1, \end{cases}$$

where $B = \frac{h}{k_s}$ and where Ro is the Rouse number defined above.

Appendix B. Sediment transport with sediment condition supply

The function $k_s[m]$ is a roughness coefficient that accounts the sediment supply condition. Note that the amount of sediment that can be moving on a sedimentary bedforms or sediment supply is not negligible on the bed morphodynamic. The sediment supply impacts the sediment transport in coastal context and plays a role on deposition/transportation processes. The impact of the sediment supply on the morphodynamics of a sedimentary bed subject to a current was experimentally studied in coastal context by Vah et al. (2018). We have (Rijn, 1984):

$$k_s = 3d_{90} + 1.1h_{eq}1 - \exp\left(\frac{-25h_{eq}}{\lambda_{eq}}\right). \tag{B.1}$$

Here, $d_{90}[m]$ is the representative grain size in which 90% of all particles in the bed are smaller. The functions $h_{eq}[m]$ and $\lambda_{eq}[m]$ are the equivalent height and the bed form length respectively. Accounting the sediment supply conditions, we have:

$$\frac{\lambda_{eq}}{\lambda_{eq_inf}} = 1 - \beta_T \exp\left(\frac{-\delta}{\gamma_T h_{eq_inf}}\right). \tag{B.2}$$

Best results in our simulation are obtained by taking $\beta_T = 0.44$, $\gamma_T = 0.63$.

$$\frac{h_{eq}}{h_{eq_inf}} = 1 - \beta_T \exp\left(\frac{-\delta}{\alpha_T \theta' h_{eq_inf}}\right), \tag{B.3}$$

where $\alpha_T = 8.24$, θ' is the Shields parameter calculated without bedforms or skin friction Shields parameter and is given by $\theta' = \frac{u'_*}{(s-1)gd_{50}}$ with u'_* bottom velocity without bedforms and based on grain diameters. The functions $h_{eq_{inf}}[m]$ and $\lambda_{eq_{inf}}[m]$ are respectively the height and length at the equilibrium state for unlimited sediment supply conditions. Following Soulsby et al. (2012) (see also Ngatcha et al., 2022a) we have:

$$h_{eq_{inf}} = 202d_{50}D_*^{-0.554}, \quad \lambda_{eq_{inf}} = d_{50}\left(500 + 1881D_*^{-1.5}\right). \tag{B.4}$$

The particle shape influences significantly the bedload transport estimation. Considering the medium sand size class, the grains transported by bedload follow the same trend. In this case, the influence of circularity index $\sqrt{C_{l,50}}$ on the transport in this size class is not visible. In practice the influence of circularity index is visible and the term d_{50} can be replaced by $d_{l,50}/\sqrt{C_{l,50}}$. Following this concept, we integrate this length characteristic in the definition of the Shields number.

Erosion rate without bottom slope

The erosion rate E is calculated by:

$$E := \begin{cases} \frac{\varphi(\theta - \theta_{cr})\sqrt{u^2 + v^2}}{hd^{0.2}}, & \theta > \theta_{cr} \\ 0, \text{otherwise} \end{cases} \tag{B.5}$$

where φ is control the erosion force. The function θ_{cr} is the critical value of Shields parameter θ defined by

$$\theta = \frac{\mathbf{u}_*}{gsd_{50}}, \tag{B.6}$$

where, $\mathbf{u}_*$ is the function velocity defined using the manning coefficient $\mathbf{u}_* = \sqrt{C_f\mathbf{u}^2}$. Another formulation account the sediment supply condition can be given (for dunes):

$$\mathbf{u}_* = \frac{\mathbf{u}}{6 + 2.5\ln(\frac{h}{k_s})}, \tag{B.7}$$

where k_s is a roughness coefficient depending on sediment supply condition (Ngatcha et al., 2022a).

Erosion rate with bottom slope

If we consider the bottom slope $\hat{\alpha}$, the critical Shields accounts the bottom slope and the Shields parameter becomes:

$$\theta^* = \frac{\Psi \mathbf{u}_*}{g\cos(\hat{\alpha})sd_{eff}}, \tag{B.8}$$

where θ^* is the modified Shields parameter, d_{eff} is the effective diameter of sediment taking into account shape of grain. $d_{eff} = \frac{d_{50}}{\sqrt{C_{50}}}$, C_{50} is circularity index of grain, Ψ is the skin friction correction given by:

$$\Psi = \frac{2\kappa^2}{C_h[\log(\frac{11.036h}{k_s})]^2}, \tag{B.9}$$

where C_h is the Nikuradse quadratic drag coefficient given by:

$$C_h = \frac{2\kappa^2}{\log(441.44h)^2}. \tag{B.10}$$

Here, κ is the Von Karman number, and where $k_s[m]$ is a roughness coefficient which accounts the sediment condition supply.

References

Audusse, E., Berthon, C., Chalons, C., Delestre, O., Goutal, N., Jodeau, M., Sainte-Marie, J., Giesselmann, J.Jan., Sadaka, G., 2012. Sediment transport modelling: relaxation schemes for saint-venant-exner and three layer models. ESAIM: Proc. 38, 78–98.

Audusse, E., Boittin, L., Parisot, M., 2021. Asymptotic derivation and simulations of a non-local exner model in large viscosity regime. ESAIM: M2AN 55, 1635–1668.

Barré de Saint-Venant, AJC., 1871. Théorie du mouvement non-permanent des eaux, avec application aux crues des rivière et à l'introduction des marées dans leur lit. C. R. L'Acad. Sci. 73, 147–154.

Bouchut, F., Fernandez-Nieto, E., Mangeney, A., Lagree, P-Y., 2008. On new erosion models of Savage-Hutter type for avalanches. Math. Acta Mech..

Brown, G., Roshko, A., 1971. The effect of density differences on the turbulent mixing layer, Turbulent Shear Flows. AGARD-CP-93 23-1, 23-12.

Cao, Z., Pender, G., Wallis, S., Carling, P., 2004. Computational dam-break hydraulics over 331 erodible sediment bed. J. Hydraul. Eng. 130 (7), 689–703.

Castro Diaz, M.J., Fernandez-Nieto, E.D., Ferreiro, A.M., 2009. Two-dimensional sediment transport models in shallow water equations, a second order finite volume approach on unstructured meshes. Comput. Methods Appl. Mech. Engrg. 198, 2520–2538.

Chandrashekar, P., Nkonga, B., Meena, K., Bhole, A., 2020. A path-conservative finite volume method for shear shallow water model. J. Comput. Phys. 413, 109457.

Cheng, N.S., 2002. Exponential formula for bedload transport. J. Hydraul. Eng. 128 (10), 942–946.

Clare, Mariana C.A., Percival, James R., Angeloudis, Althanasios, Cotter, Colin J., Piggot, M., 2021. Hydro-morphodynamics 2D modelling using a discontinuous Galerkin discretization. Comput. Geosci. 146, 104658.

Deltares, 2014. Delft3D-FLOW simulation of multi-dimensional hydrodynamic flows and transport phenomena including sediments. In: User Manual. 320 Delft, The Netherlands, URL: https://oss.deltares.nl/documents/183920/185723/Delft3D-FLOWUserManual.pdf.

Einstein, H.A., 1942. Formulas for the transportation of bed load. Trans. Am. Soc. Civ. Eng. 107, 561–573.

Engelund, Hansen, E., 1967. A Monograph on Sediment Transport in Alluvial Streams. Teknisk forlag.

Exner, F.M., 1925. Über Die Wechselwirkung Zwischen Wasser Und Geschiebe in Flüssen Akademie Der Wissenschaften Sitzungsberichte. Wien, Austria, p. 134.

Fetecau, C., Vieru, D., 2020. Exact solutions for unsteady motion between parallel plates of some fluids with power-law dependence of viscosity on the pressure. Appl. Eng. Sci. 1, 100003.

Fotsi, Yannick Fossi, Pouvreau, Nicolas, Brenon, Isabelle, Onguene, Raphael, Etame, Jacques, 2019. Temporal (1948–2012) and dynamic evolution of the Wouri Estuary coastline within the Gulf of Guinea. J. Mar. Sci. Eng. 7, 343.

Garcia, M., Parker, G., 1991. Entrainment of bed sediment into suspension. J. Hydraul. Eng. 117 (4), 414–435.

Gonzalez-Aguirre, J.C., Castro, M.J., Morales de Luna, T., 2020. A robust model for rapidly varying flows over movable bottom with suspended and bedload transport: Modelling and numerical approach. Adv. Water Resour. 140, 103575.

Grass, A.J., 1981. Sediment Transport By Waves and Currents Department of Civil Engineering. University college, London.

Holly, F.M., Rahuel, J.M., 1990. New numerical/physical framework for mobile bed modelling, part1: Numerical and physical principles. J. Hydraul. Res. 28, 401–416.

Hu, P., Cao, Z., 2009. Fully coupled mathematical modeling of turbidity currents over erodible bed. Adv. Water Resour. 32 (1), 1–15.

Hudson, J., Sweby, P.K., 2003. Formations for numerically approximating hyperbolic systems governing sediment transport. J. Sci. Comput. 19, 225–252.

Huybrechts, N., Villaret, C., Hervouet, J.M., Comparison between 2D and 3D modelling of sediment transport. In: Application to the Dune Evolution in Proceedings of the 5th International Conference on Fluvial Hydraulics, Braunschweig, Germany.

Lai, Y.G., 2020. A two-dimensional depth averaged sediment transport mobile-bed model with polygonal meshes. Water 12 (1032), http://dx.doi.org/10.3390/w12041032.

Lajeunesse, E., Malverti, L, Charru, F., 2010. Bed load transport in turbulent flow at the grain scale: Experiments and modeling. J. Geophys. Res. 115.

Lax, P., Wendroff, B., 1960. Systems of conservation laws. Comm. Pure Appl. Math. 13, 217–237.

Lee, H-I., 2021. Shears and vortices of rotational couette flow in a cylindrical gap with radial injection and suction. Appl. Eng. Sci. 8, 100071.

Liu, Xin, Mohammadian, Abdolmajid, Kurganov, Alexander, Infante Sedano, Julio Angel, 2015. Well-balanced central-upwind scheme for a fully coupled shallow water system modeling flows over erodible bed. J. Comput. Phys. 300, 202–218.

Lorentz, Hendrik Antoon, 1907. Über die entstehung turbulenter flüssigkeitsbewegungen und über den einfluss dieser bewegungen bei der strömung durch röhren, revised article of "over den weerstand dien een vloeistofstroom in eene cilindrische buis ondervindt, amsterdam, zittingsverslag akad. v. wet. 6, 1897, p. 28. Hendrik Antoon Lorentz: Abhandlungen über theoretische Physik. Leipzig: Teubner 1, 43–71.

McLean, S.R., Nelson, J.M., 1999. Predicting boundary shear and sediment transport over bed form. J. Hydraul. Eng. 125 (10), 725–736.

Minev, P., Usubov, R., 2022. Splitting schemes for the stress formulation of fluid–structure interaction problems. Appl. Eng. Sci. 9, 100082.

Neary, V.S., Wright, S.A., Bereciartua, P., 2001. Case study: Sediment transport in proposed geomorphic channel for Napa river. J. Hydraul. Eng. 127 (11), 901–910.

Ngatcha, Ndengna Arno Roland, Mimbeu, Yves, Onguene, Raphael, Nguiya, Sévérin, Njifenjou, Abdou, 2022a. A novel sediment transport model accounting phase lag effect. A resonance condition. WSEAS Trans. Fluid Mech. 17, 189–211. http://dx.doi.org/10.37394/232013.2022.17.19.

Ngatcha, Ndengna Arno Roland, Njifenjou, Abdou, 2022. A AENO path-conservative central-upwind scheme for a sediment transport model. Ocean Eng. Syst. 12 (3), available in https://doi.org/10.12989/ose.2022.12.3.359.

Ngatcha, Arno Roland, Njifenjou, Abdou, Onguéné, Raphael, 2022b. Fast, robust, efficient and well-balanced path-conservative based methods on unstructured meshes for nonconservative problems. available in https://hal.archives-ouvertes.fr/hal-03668107 submitted for publications.

Ngatcha, N.A.R., Nkonga, B., Njifenjou, A., 2022c. Finite volume AENO methods with time-steps discretization for a averaged sediment transport model. available in https://hal.archives-ouvertes.fr/hal-03668098 submitted for publications.

Ngatcha, Ndengna Arno Roland, Nkonga, Boniface., Njifenjou, Abdou., Onguene, Raphael., 2022d. Sediment transport models in Generalized shear shallow water flow equations. In: CARI 2022. Dschang, Cameroon, https://hal.archives-ouvertes.fr/hal-03735893.

Nielsen, P., 1992. Coastal Bottom Boundary Layers and Sediment Transport. In: Advanced series on ocean engineering, vol. 4, World scientific.

Nkonga, Boniface, Chandrashekar, Praveen, 2021. An exact Riemann solver for a shear shallow water model. In: Arxiv 19 Aout.

Paola, C, Voller, V.R., 2005. A generalized Exner equation for sediment mass balance. J. Geophys. Res. 110.

Rajagopal, K.R., 2021. A note on the stability of flows of fluids whose symmetric part of the velocity gradient is a function of the stress. Appl. Eng. Sci. 8, 100072.

Reynolds, Osborne, 1895. On the dynamic theory of incompressible viscous fluids and the determination of the criterion. Philos. Trans. R. Soc..

Richards, G.L., Gavrilyuk, S.L., 2013. A new model of roll waves comparison layers. Geophysical astrophysical fluid dynamics. J. Fluid Mech. 698, 374–405.

Soulsby, R.L., Whitehouse, R.J.S., Marten, K.V., 2012. Prediction of time-evolving sand ripples in shelf seas. Cont. Shelf Res. (38), 47–62.

Teshukov, V.M., 2007. Gas-Dynamic analogy for vortex free-boundary flows. J. Appl. Mech. Tech. Phys. 48 (3), 303–309.

Troshkin, O.V., 1990. On wave properties of an incompressible turbulent fluid. Physica A 168, 881–899.

Vah, M., Jarno, A., Marin, F., Le Bot, S., 2018. Experimental study on sediment supply-limited bedforms in a coastal context. In: Sixth International Conference on Estuaries and Coasts (ICEC-2018), August 20-23. Caen, France.

Van Rijn, L.C., 1984. Sediment transport, part iii: Bed forms and alluvial roughness. J. Hydraul. Eng. 110, 1733–1754.

Warner, John C., Sherwood, Christopher R., Signell, Richard P., Harris, Courtney K., Arango, Hernan G., 2008. Development of a three-dimensional, regional, coupled wave, current, and sediment-transport model. Comput. Geosci. 34, 1284–1306.

Yalin, M.S., 1977. Mechanics of Sediment Transport. Pergamon Oxford U.K.