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# Solving singular integral equations with orthogonal rational functions

Bernhard Beckermann\* and Ana C. Matos\*

## Abstract

We suggest a new numerical method of solving the signed equilibrium with external field in logarithmic potential theory on a union of distinct real intervals. A reformulation of our problem leads us to a system of integral equations with a weakly singular Cauchy kernel. We then recall a polynomial spectral method and its error analysis, and suggest a new spectral method using orthogonal rational functions in order to solve our problem. Choosing appropriate and explicitly given poles allows to speed up computation. Such techniques have been exploited also in the famous adaptative multipole algorithm for particle simulations [8] or more recently for solving the Laplace equation [28].

**Keywords:** equilibrium problems, singular integral equations, orthogonal rational functions, spectral methods

## 1 Introduction

We are interested in solving the following signed equilibrium problem in logarithmic potential theory, which can be seen also as a particular singular integral equation with logarithmic kernel.

**Problem 1.1.** Given a union  $I$  of  $m$  real compact and disjoint intervals  $I_j = [a_j, b_j]$  and  $Q \in \mathcal{C}(\mathbb{R})$ , find a (signed) measure  $\mu$  of mass  $\int d\mu = 1$  with  $\text{supp}(\mu) \subset I$  and  $F \in \mathbb{R}$  s.t.

$$\forall x \in I : \quad V^\mu(x) + Q(x) = F, \quad (1.1)$$

with logarithmic potential  $V^\mu(x) = \int \log\left(\frac{1}{|x-y|}\right) d\mu(y)$ . □

In the literature [4, §1.3], a measure  $\mu$  as in Problem 1.1 often is referred to as a signed equilibrium measure, namely the one with within all signed measures  $\nu$  of mass 1 which

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minimizes the weighted logarithmic energy  $\nu \mapsto \int V^\nu d\nu + 2 \int Q d\nu$ . Defining the measures supported on  $I_j$  by

$$d\omega_j(x) = \frac{dx}{\pi \sqrt{(b_j - x)(x - a_j)}}, \quad d\eta_j(x) = \frac{\sqrt{(b_j - x)(x - a_j)} dx}{\pi}, \quad (1.2)$$

then we show in Theorem 2.2 below that, provided that  $Q' \in L^2(\eta_j)$  for all  $j$ , we have one and only one solution with density (with respect to  $\omega_j$ )

$$\rho_j(x) := \frac{d\mu|_{I_j}}{d\omega_j}(x) = \pi \sqrt{(x - a_j)(b_j - x)} \frac{d\mu|_{I_j}}{dx}(x) \in L^2(\omega_j) \quad (1.3)$$

for  $j = 1, \dots, m$ . Notice that the choice of our functional spaces is essential, for existence and uniqueness in different  $L^p$  spaces we refer the reader to the classical reference [20, §10, 11, 18 and 19], or [1, 2, 18] and the recent paper [15] for the case of several intervals.

**Example 1.2.** In the case  $m = 1$  of a single interval  $I = I_1 = [a_1, b_1]$  and external field  $Q = 0$ , it is known that the solution of Problem 1.1 is given by  $\mu = \omega_1$  and hence  $\rho_1 = 1$ . Even for a general external field  $Q$  we may explicitly write down the solution in terms of  $T_n$ , the Chebyshev polynomials of the first kind, being orthogonal (but not orthonormal) with respect to  $\omega_1$ : the Ansatz

$$\rho_1(x) = \frac{d\mu}{d\omega_1}(x) = T_0\left(\frac{2x - a_1 - b_1}{b_1 - a_1}\right) + \sum_{n=1}^{\infty} c_n T_n\left(\frac{2x - a_1 - b_1}{b_1 - a_1}\right) \in L^2(\omega_1)$$

is known to give for  $x \in I_1$  the logarithmic potential

$$V^\mu(x) = \log\left(\frac{4}{b_1 - a_1}\right) + \sum_{n=1}^{\infty} \frac{c_n}{n} T_n\left(\frac{2x - a_1 - b_1}{b_1 - a_1}\right),$$

and we find the coefficients  $c_n$  by comparing with the Chebyshev expansion of  $Q$  on  $I_1$ .  $\square$

In the case of a single interval  $m = 1$ , the previous example shows that the choice of the basis of Chebyshev polynomials allowed us to diagonalize the map  $\mu \mapsto V^\mu$ , being compact, with unbounded inverse. It turns out that, by taking derivatives in (1.1), we are left with a Cauchy kernel instead of a logarithmic kernel, which allows to apply Fredholm theory. It is well known that derivatives of

$$x \mapsto V^{\mu|_{I_k}}(x) = \int_{I_k} \log\left(\frac{1}{|x - y|}\right) \rho_k(y) d\omega_k(y)$$

can be expressed in terms of Cauchy and Hilbert transforms for  $x \in \mathbb{R}$ , but the scaling factor in the definition of such transforms is unfortunately not unique, compare with, e.g., [10, §6.7 p. 169 ff] and [17]. We will choose a scaling such that the derivative is given by  $\mathcal{C}_k(\rho_k)(x)$  for  $x \in \mathbb{R} \setminus I_k$ , and by  $\mathcal{H}_k(\rho_k)(x)$  for  $x \in I_k$ .

**Definition 1.3.** We define the weighted Cauchy transform, and weighted Hilbert transform, respectively, by

$$\mathcal{C}_k(\rho_k)(x) = \int_{I_k} \frac{\rho_k(y)}{y - x} d\omega_k(y) \text{ if } x \notin I_k,$$

$$\mathcal{H}_k(\rho_k)(x) := \lim_{\epsilon \rightarrow 0^+} \operatorname{Re} \mathcal{C}_k(\rho_k)(x + i\epsilon) = \int_{I_k} \frac{\rho_k(y)}{y - x} d\omega_k(y) \text{ if } x \in I_k,$$

with the principal value integral  $f$ .  $\square$

Then our initial Problem 1.1 can be reformulated as follows.

**Problem 1.4.** Find  $\rho_j = \frac{d\mu|_{I_j}}{d\omega_{I_j}} \in \mathcal{L}^2(\omega_j)$  for  $j = 1, \dots, m$  such that

(M) The total mass of the measure is 1:

$$\int d\mu = \sum_{k=1}^m \int_{I_k} \rho_k d\omega_k = 1.$$

(D) The derivative  $(V^\mu + Q)'$  vanishes on each subinterval:

$$\forall j = 1, \dots, m \quad \forall x \in I_j : \quad Q'(x) + \mathcal{H}_j(\rho_j)(x) + \sum_{k \neq j} \mathcal{C}_k(\rho_k)(x) = 0.$$

(C)  $V^\mu + Q$  is equal to the same constant on each subinterval:

$$\forall j = 2, \dots, m : \quad \int_{b_{j-1}}^{a_j} \left( Q'(x) + \sum_{k=1}^m \mathcal{C}_k(\rho_k)(x) \right) dx = 0.$$

□

In (D) we therefore have derived a new system of integral equations, containing one singular integral operator  $\mathcal{H}_j$  with a Cauchy kernel, and other integral operators with a smooth kernel, which describe the interaction between intervals. As it is typically done in spectral methods, we will choose a basis in which the action of  $\mathcal{H}_j$  is easy to describe, compare with Example 1.2, and show that the interaction can be well approximated by simpler finite rank approximations, leading to an effective way of computing the unknown densities  $\rho_j$ , where we provide a complete error analysis. Spectral methods for (systems of) integral equations have been considered before by many authors. In the setting of Problem 1.4, we refer the readers for instance to [23] based on bases of Chebyshev orthogonal polynomials, and to [9] where piecewise linear splines are used, each time without error analysis. In contrast, we will show that a basis of orthogonal rational functions (ORF) allows for high precision with a small number of degrees of freedom, even in the critical case where two of the intervals  $I_j$  are close to each other. Using rational functions is also implicit in the so-called multi-pole method [8] for single/double layer potential problems, where, roughly speaking, the precise form of  $\mu$  in the far field, that is, for  $x$  sufficiently far from  $I_j$ , is not so important: for obtaining the equilibrium (1.1) for  $x \in I_j$ , we may replace  $\mu|_{I_k}$  for  $k \neq j$  by a suitable weighted sum of Dirac measures, leading for the potential to a weighted sum of logarithmic terms, with a derivative being a rational function with poles at the supports of the Dirac mass points. Hence the interaction terms in (D) should be well approximable by rational functions with fixed poles, as long as the poles are carefully chosen. Our choice of poles will be inspired by the third Zolotarev extremal problem, which recently gained a lot of attention in the context of row rank approximations of matrices with small displacement rank [3].

Before going further, let us first briefly discuss the link with Riemann-Hilbert techniques.

**Remark 1.5.** As shown in [10, §6.7 p. 169 ff], the Cauchy transform  $\mathcal{C}_j$  has boundary values by approaching  $x \in \text{Int}(I_j)$  through the complex plane from above or below, and more precisely,

$$x \in \text{Int}(I_j) : \quad \mathcal{C}_j(\rho_j)(x \pm i0) = \mathcal{H}_j(\rho_j)(x) \pm i\pi \frac{d\mu}{dx}(x). \quad (1.4)$$

Thus the function  $f(z) := \sum_{k=1}^m \mathcal{C}_k(\rho_k)(z)$  with  $\rho_k$  as in Problem 1.4 has the following properties: it is obviously analytic in  $\overline{\mathbb{C}} \setminus I$ , behaves like  $f(z) = \frac{1}{z} + \mathcal{O}(\frac{1}{z^2})_{z \rightarrow \infty}$  by (M), satisfies the jump condition  $f(x+i0) + f(x-i0) + 2Q'(x) = 0$  for all  $x \in \text{Int}(I)$  by (D), and finally satisfies the side conditions  $\int_{b_{j-1}}^{a_j} (f(x) + Q'(x)) dx = 0$  for  $j = 2, \dots, m$  by (C). In other words,  $f$  is a solution of a Riemann-Hilbert (RH) problem. Conversely, any numerical method for solving this RH-problem provides the required measure  $\mu$  through (1.4). However, those numerical methods for solving RH-problems we are aware of are based on the a non-trivial task of numerical evaluation of singular integrals, which explains the different approach taken in this paper.

## 2 A polynomial spectral method

We start by recalling orthonormal (and dense) bases of  $L^2(\omega_j)$  and  $L^2(\eta_j)$ , both measures being defined in (1.2), and describe the action of  $\mathcal{H}_j$  and  $\mathcal{C}_k$  on these bases. Let  $\{T_n(z)\}_n$  and  $\{U_n(z)\}_n$  be the sequences of Chebyshev polynomials of first and second kind in the interval  $[-1, 1]$ . We may obtain orthonormal bases of the above spaces by a linear change of variables plus a suitable scaling, which is easiest described in terms of the Riemann conformal map  $\phi_j : \mathbb{C} \setminus I_j \mapsto \mathbb{C} \setminus \overline{\mathbb{D}}$ , with  $\mathbb{D}$  the open unit disk,

$$w = \phi_j(z) \quad \iff \quad \frac{1}{2}(w + \frac{1}{w}) = \frac{2z - a_j - b_j}{b_j - a_j}.$$

Then the orthonormal bases are given by shifted normalized Chebyshev polynomials of first/second kind in  $[a_j, b_j]$ , defined by  $T_{j,0} = 1$  and, for  $n \geq 1$ ,

$$\begin{aligned} T_{j,n}(z) &= \sqrt{2}T_n\left(\frac{2z - a_j - b_j}{b_j - a_j}\right) = \frac{1}{\sqrt{2}}\left(w^n + \frac{1}{w^n}\right), \\ \frac{b_j - a_j}{2}U_{j,n-1}(z) &= \sqrt{2}U_{n-1}\left(\frac{2z - a_j - b_j}{b_j - a_j}\right) = \sqrt{2}\frac{w^n - \frac{1}{w^n}}{w - \frac{1}{w}}. \end{aligned}$$

The action of  $\mathcal{H}_j$  and  $\mathcal{C}_k$  on these polynomials is well known [17, §11.12 p. 567 ff].

**Lemma 2.1.** For  $n \geq 1$ ,

$$\begin{aligned} \mathcal{C}_k(T_{k,n})(x) &= -\sqrt{2}\frac{\phi'_k(x)}{\phi_k(x)}\frac{1}{\phi_k(x)^n}, \quad \mathcal{C}_k(T_{k,0})(x) = -\frac{\phi'_k(x)}{\phi_k(x)}, \\ \mathcal{H}_j(T_{j,n})(x) &= U_{j,n-1}(x), \quad \mathcal{H}_j(T_{j,0})(x) = 0, \\ \int_{b_{j-1}}^{a_j} \mathcal{C}_k(T_{k,n})dx &= \sqrt{2}\frac{\phi_k(a_j)^{-n} - \phi_k(b_{j-1})^{-n}}{n}, \\ \int_{b_{j-1}}^{a_j} \mathcal{C}_k(T_{k,0})dx &= \log\left(\frac{\phi_k(b_{j-1})}{\phi_k(a_j)}\right). \end{aligned}$$

*Proof.* The first 4 identities are a particular case of Lemma 3.3 below, the last two are obtained by integration.  $\square$

Setting

$$\mathcal{C}_{j,k} : L^2(\omega_k) \mapsto L^2(\eta_j), \quad \mathcal{C}_{j,k}(g) = \mathcal{C}_k(g)|_{I_j},$$

we notice that condition (D) holds iff  $\mathcal{H}_j(\rho_j) + Q' + \sum_{k \neq j} \mathcal{C}_{j,k} \rho_k$  equals  $0 \in L^2(\eta_j)$ , or, in other words, is  $\eta_j$ -orthogonal to  $U_{j,\ell-1}$  for all  $\ell \geq 1$ . This allows us to formulate Problem 1.4 as a system of linear equations in  $(\ell^2)^m$  of the form

$$(I + K)x = c, \tag{2.1}$$

where  $K$  (and accordingly  $x$  and  $b$ ) is partitioned into blocks  $K_{j,k} : \ell^2 \mapsto \ell^2$  for  $j, k = 1, 2, \dots, m$ , the  $k$ th block of the vector  $x$  of unknowns being

$$(x)_k = (\rho_{k,0}, \rho_{k,1}, \dots)^T \quad \text{with} \quad \rho_k(x) = \sum_{n=0}^{\infty} \rho_{k,n} T_{k,n}(x) \tag{2.2}$$

allowing to describe in each equation the contribution of the density  $\rho_k$ , and the  $j$ th block of the equations describing in the 0th row an exceptional equation (namely condition (M) for  $j = 1$  and (C) for  $j = 2, \dots, m$ ), and in the  $\ell$ th row for  $\ell \geq 1$  the orthogonality with  $U_{j,\ell-1}$  coming from condition (C). This leads to the matrix

$$((I + K)_{j,k})_{\ell,n} = \begin{cases} \delta_{n,0} & \text{if } j = 1, \ell = 0, \\ \int_{b_{j-1}}^{a_j} \mathcal{C}_k(T_{k,n})(x) dx & \text{if } j > 1, \ell = 0, \\ \langle \mathcal{C}_{j,k} T_{k,n}, U_{j,\ell-1} \rangle_{L^2(\eta_j)} & \text{if } j \neq k, \ell > 0, \\ \delta_{\ell,n} & \text{if } j = k, \ell > 0, \end{cases} \tag{2.3}$$

and the right-hand side

$$((c)_j)_\ell = \begin{cases} 1 & \text{if } j = 1, \ell = 0, \\ -\int_{b_{j-1}}^{a_j} Q'(x) dx = Q(b_{j-1}) - Q(a_j) & \text{if } j > 1, \ell = 0, \\ -\langle Q', U_{j,\ell-1} \rangle_{L^2(\eta_j)} & \text{if } \ell > 0. \end{cases} \tag{2.4}$$

If we suppose that  $Q$  is given on each interval  $I_j$  as a Chebyshev series of the first kind,

$$\forall j = 1, 2, \dots, m, \forall x \in I_j : \quad Q|_{I_j}(x) = \sum_{\ell=0}^{\infty} Q_{j,\ell} T_{j,\ell}(x), \tag{2.5}$$

then, by taking derivatives,  $-\langle Q', U_{j,\ell-1} \rangle_{L^2(\eta_j)} = -\ell Q_{j,\ell}$  for  $\ell \geq 1$ , since  $\ell U_{j,\ell-1}$  is the derivative of  $T_{j,\ell}$ . Thus all entries of  $c$  occurring in (2.4) are computable. Using Lemma 2.1, also all entries of  $K$  occurring in (2.3) including  $\mathcal{C}_{j,k} T_{k,n}$  are computable, up to the scalar product with  $U_{j,\ell-1}$ , which we will approach later on by some quadrature formula. The last case in (2.3) shows that  $K_{j,j}$  is the zero matrix except the entries in the 0th row coming from conditions (M) or (C), moreover, for  $j > 1$  the decay of these entries is rather slow. Notice that  $(x)_k \in \ell^2$  iff  $\rho_k \in L^2(\omega_k)$ , and  $(c)_j \in \ell^2$  iff  $Q' \in L^2(\eta_j)$ . Moreover, it is not too difficult to check using Lemma 2.1 that each row of  $K_{j,k}$  is in  $\ell^2$  so that  $(I + K)x$  makes sense (though it is not yet clear that  $K_{j,k}$  is a bounded operator acting on  $\ell^2$ ). Existence and uniqueness of a solution of (2.1) is shown in the following statement.

**Theorem 2.2.** *The operator  $K : (\ell^2)^m \mapsto (\ell^2)^m$  defined in (2.3) is a compact operator of Hilbert-Schmidt class, and  $(I + K)$  is bijective and boundedly invertible.*

*In particular, if  $Q^j \in L^2(\eta_j)$  for  $j = 1, 2, \dots, m$ , then Problems 1.1 or 1.4 have one and only one solution with  $\rho_k \in L^2(\omega_k)$  for  $k = 1, 2, \dots, m$ .*

*Proof.* We start by showing that  $K$  has finite Hilbert-Schmidt norm. Since the exceptional rows of index 0 are elements of  $\ell^2$  by the last two formulas of Lemma 2.1 and there are a finite number of intervals, it is sufficient to show that  $\mathcal{C}_{j,k}$  is Hilbert-Schmidt for all  $j, k \in \{1, \dots, m\}$ ,  $j \neq k$ . This is known to be true for any integral operator with smooth kernel, see for instance [16, Example V.2.19], but also follows from Theorem 3.6 below.

In order to conclude, by the Fredholm alternative, we only have to show that  $I + K$  is injective (or 0 is not an eigenvalue of  $I + K$ ). Suppose on the contrary that there is  $y \in (\ell^2)^m$ ,  $y \neq 0$  and  $(I + K)y = 0$ . Then as in (2.2) and Remark 1.5 we construct  $\rho_k \in L^2(\eta_k)$ , and

$$f(z) = \sum_{k=1}^m \mathcal{C}_k(\rho_k)(z) \neq 0$$

being analytic in  $\overline{\mathbb{C}} \setminus I$ , and  $f(z) = O(1/z^2)_{z \rightarrow \infty}$  by the homogeneous counterpart of condition (M). The homogeneous counterpart of (D) gives the jump behavior  $f(x + i0) + f(x - i0) = 0$  for  $x \in I$ . From [20, §84] we know that such a function  $f$  necessarily has the form

$$f(z) = \frac{P(z)}{\sqrt{R(z)}}, \quad R(z) = \prod_{j=1}^m (z - a_j)(z - b_j),$$

where  $P$  is a non-trivial polynomial with real coefficients of degree  $\leq m - 2$ . Thus there is at least one index  $j$  such both  $P$  and  $\sqrt{R}$  do not change sign on  $[b_{j-1}, a_j]$ , implying that  $\int_{b_{j-1}}^{a_j} f(x)dx \neq 0$ , in contradiction with the homogeneous counterpart of condition (C).  $\square$

On a computer, we have to change the system (2.1) into

$$(I + \underline{K})\underline{x} = \underline{c} \tag{2.6}$$

with  $\underline{c} \in (\ell^2)^m$  close to  $c$ , and  $\underline{K}$  of finite rank and close to  $K$ , in order to get a finite number of equations and unknowns. In what follows we will only consider the case where there exists some matrix  $F$  with a finite number of columns which are mutually orthogonal, and which span both  $\underline{c}$  and the image of  $\underline{K}$ . The following lemma shows that in this case the solution  $\underline{x}$  of (2.6) can be obtained by solving a linear system with a finite number of unknowns and equations.

**Lemma 2.3.** *Let  $FF^*\underline{K} = \underline{K}$ ,  $FF^*\underline{c} = \underline{c}$ ,  $F^*F = I$ , and suppose that  $\|K - \underline{K}\| \|(I + K)^{-1}\| < 1$ . Then system (2.6) has a unique solution given by*

$$\underline{x} = Fy, \quad \text{with } y \text{ unique solution of } (I + F^*\underline{K}F)y = F^*\underline{c}.$$

*We furthermore have the error estimates, both for the residual and the error,*

$$\begin{aligned} \|c - (I + K)\underline{x}\| &\leq \|K - \underline{K}\| \|\underline{x}\| + \|c - \underline{c}\|, \\ \|\underline{x} - \underline{x}\| &\leq \|(I + K)^{-1}\| \left( \|K - \underline{K}\| \|\underline{x}\| + \|c - \underline{c}\| \right). \end{aligned}$$

*Proof.* Since  $I + \underline{K} = (I + K)(I + (I + K)^{-1}(\underline{K} - K))$ , a simple Neumann series argument shows that  $I + \underline{K}$  is invertible, and hence (2.6) has a unique solution  $\underline{x}$ . By assumption on  $\underline{K}$  and  $\underline{c}$  we have that

$$\underline{x} = (\underline{c} - \underline{K}\underline{x}) = FF^*(\underline{c} - \underline{K}\underline{x}) = FF^*\underline{x},$$

that is, there exists a unique  $y = F^*\underline{x}$  (with a finite number of components) such that  $\underline{x} = Fy$ . Inserting this information into (2.6) gives

$$(I + \underline{K})\underline{x} - \underline{c} = F(y + F^*\underline{K}Fy - F^*\underline{c}).$$

Since the columns of  $F$  are linearly independent, we see that  $y = F^*\underline{x}$  is unique solution of  $(I + F^*\underline{K}F)y = F^*\underline{c}$ , as claimed above. The error estimates follow by observing that

$$c - (I + K)\underline{x} = (I + K)(x - \underline{x}) = (\underline{K} - K)\underline{x} + (c - \underline{c}).$$

□

**Example 2.4.** Since  $K$  is Hilbert-Schmidt, a first attempt could be to follow the idea of a finite section method,<sup>1</sup> that is, to keep in the block  $(\underline{K})_{j,k}$  only the first  $(N_j + 1)$  rows from  $K_{j,k}$ , and to complete by zeros: here

$$\underline{K} = FF^*K, \quad F^* = \text{diag} \left[ [ I_{N_j+1} \ 0 ]_{j=1, \dots, m} \right], \quad \text{and} \quad \underline{c} = FF^*c. \quad (2.7)$$

In this case, with  $y$  as in Lemma 2.3, computed in terms of  $F^*\underline{K}F$  and  $F^*c$ , composed of the the principal submatrices of  $K_{j,k}$  of size  $(N_j + 1) \times (N_k + 1)$ , and of the first  $N_j + 1$  components of each block  $c_j$ , we find that the  $k$ th block  $\underline{x}_k$  of  $\underline{x}$  has the components  $(\underline{x}_k)_n = (y_k)_n$  for  $n = 0, \dots, N_k$ , and  $(\underline{x}_k)_n = 0$  for  $n > N_k$ , which corresponds to the ansatz of the form

$$\underline{\rho}_k(x) = \sum_{n=0}^{N_j} (\underline{x}_k)_n T_{k,n}(x) \quad (2.8)$$

in order to get an approximate density  $\underline{\rho}_k$  approaching  $\rho_k$ . Notice that our estimates of Lemma 2.3 for both the residual and the error will be only interesting if the entries in  $c - \underline{c} = (I - FF^*)c$  decay sufficiently fast. According to (2.4), this vector contains coefficients of order  $n > N_j$  of the Chebyshev expansion of  $Q'$  on  $I_j$ , see (2.9), with their decay depending on the smoothness of the external field  $Q$ . □

The drawback of the finite section approach suggested in Example 2.4 is that, according to (2.3), it requires to compute the scalar products  $\langle \mathcal{C}_{j,k}T_{k,n}, U_{j,\ell-1} \rangle_{L^2(\eta_j)}$ , where we have explicit formulas for the two arguments, but no explicit formula for their scalar product. Here the quantity  $(\mathcal{C}_{j,k}T_{k,n})$  occurring in condition (D) is smooth in  $I_j$ . We will show in Corollary 2.7 that it can be successfully approximated by a polynomial interpolant, where the abscissa come from a Gauss quadrature rule related to  $\eta_j$ .

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<sup>1</sup>Citations finite section method



**Definition 2.5.** For  $j = 1, \dots, m$ , and integers  $N_j > 0$  to be fixed later, denote by  $d_{j,h}^2$  and  $x_{j,h} \in I_j$ ,  $h = 1, \dots, N_j$ , the weights/abscissa of the  $(N_j)$ th Gauss quadrature rule for  $\eta_j$ , that is,  $d_{j,h} = \frac{b_j - a_j}{2\sqrt{N_j+1}} \sin(\frac{\pi h}{N_j+1})$ , and  $x_{j,h} = \frac{a_j + b_j}{2} + \frac{b_j - a_j}{2} \cos(\frac{\pi h}{N_j+1})$  are the roots of  $U_{j,N_j}$  such that<sup>2</sup>

$$M_j^{-1} = M_j^*, \quad M_j := [ d_{j,h} U_{j,\ell-1}(x_{j,h}) ]_{h,\ell=1,\dots,N_j}.$$

Finally, denote by  $\Pi_j$  the corresponding polynomial interpolation operator: for a function  $g$  defined on  $I_j$ , we denote

$$\Pi_j g(x) = (U_{j,0}(x), \dots, U_{j,N_j-1}(x)) M_j^* [ d_{j,h} g(x_{j,h}) ]_{h=1,\dots,N_j}.$$

□

It is easily checked that computing  $\langle \Pi_j g, U_{j,\ell-1} \rangle_{L^2(\eta_j)}$  in case  $\ell \leq N_j$  is equivalent of approaching the scalar product in  $\langle g, U_{j,\ell-1} \rangle_{L^2(\eta_j)}$  by our Gauss quadrature rule, and  $\langle \Pi_j g, U_{j,\ell-1} \rangle_{L^2(\eta_j)} = 0$  in case  $\ell > N_j$ . We thus define  $\underline{K}$  as  $K$  in (2.3), but replace the entry  $\langle \mathcal{C}_{j,k} T_{k,n}, U_{j,\ell-1} \rangle_{L^2(\eta_j)}$  for  $\ell \geq 1$  and  $j \neq k$  by

$$\langle \Pi_j(\mathcal{C}_{j,k} T_{k,n}), U_{j,\ell-1} \rangle_{L^2(\eta_j)} = \sum_{h=1}^{N_j} d_{j,h}^2 U_{j,\ell-1}(x_{j,h}) (\mathcal{C}_{j,k} T_{k,n})(x_{j,h})$$

for  $\ell \leq N_j$ , and by 0 if  $\ell > N_j$ . As a consequence, we enter in the framework of Lemma 2.3 with  $\underline{K} = FF^* \underline{K}$  and  $F$  and  $\underline{c} = FF^* c$  as in (2.7), but  $\underline{K}$  differing from the one in (2.7). Here, our approximate density  $\underline{\rho}_k$  is obtained as in (2.8) by solving the (finite dimensional) system of Lemma 2.3, where each entry of the matrix  $F^* \underline{K} F$  is computed using quadrature, and the right-hand side  $F^* c$  using (2.4) and (2.5).

**Remark 2.6.** By multiplying the  $j$ th block of our finite-dimensional system by  $\text{diag}(1, M_j)$ , it is not too difficult to check that, at least for polynomial  $Q_j$ , we may restate our polynomial spectral method as a collocation method, where condition (D) is replaced by

$$\forall j = 1, \dots, m \quad \forall h \in 1, \dots, N_j : \quad \mathcal{H}_j(\underline{\rho}_j)(x_{j,h}) + \sum_{k \neq j} \mathcal{C}_{j,k}(\underline{\rho}_k)(x_{j,h}) + Q'(x_{j,h}) = 0,$$

and conditions (M) and (C) are translated into additional  $m$  equations. □

Turning to the error estimates of Lemma 2.3, we first observe that

$$\|c - \underline{c}\|^2 = \sum_{j=1}^m \sum_{\ell=N_j+1}^{\infty} |\ell Q_{j,\ell}|^2 \tag{2.9}$$

since we have replaced  $Q'_j$  by its partial orthogonal sum. Using a special case of Theorem 3.6 with  $w_{j,n} = 0$  for all  $j, n$  we also may conclude on the Hilbert-Schmidt norm of the error in the matrix.

---

<sup>2</sup>A direct proof makes use of the fast Fourier transform DST1.

**Corollary 2.7.** *Suppose without loss of generality that  $a_1 < b_1 < \dots < a_m < b_m$ . Provided that the integers  $N_1, \dots, N_m$  are chosen such that*

$$\begin{aligned} \forall j = 1, \dots, m-1 : & \quad \frac{b_j - a_j}{2\sqrt{2}(a_{j+1} - b_j)} |\phi_j(a_{j+1})|^{-N_j} \leq \epsilon, \\ \forall j = 2, \dots, m : & \quad \frac{b_j - a_j}{2\sqrt{2}(a_j - b_{j-1})} |\phi_j(b_{j-1})|^{-N_j} \leq \epsilon, \end{aligned}$$

we find that  $\|K_{j,k} - \underline{K}_{j,k}\|_{HS} = \|(I - \Pi_j)\mathcal{C}_{j,k}\|_{HS} \leq \epsilon$ .

In particular, for our polynomial spectral method we have geometric convergence in case of smooth right-hand sides  $Q_1, \dots, Q_m$ .

We notice that the above error estimate is particularly interesting if both  $|\phi_j(a_{j+1})|$  and  $|\phi_j(b_{j-1})|$  are  $\gg 1$ , in other words,  $I_j$  is well separated both from  $I_{j-1}$  and  $I_{j+1}$ . In the considerations to follow we will improve the rate of convergence also in the cases where consecutive intervals may be close.

**Remark 2.8.** We claim that, for  $j \neq k$ , the block  $K_{j,k}$  has a much smaller numerical rank as the one predicted by Corollary 2.7. For this purpose, denote by  $X$  the (infinite) matrix obtained from  $K_{j,k}$  by omitting the exceptional row of index 0. Let  $B_k$  denote the Jacobi matrix associated to the polynomials  $T_{k,n}$  for  $n = 0, 1, \dots$ , and  $A_j$  the Jacobi matrix associated to the polynomials  $U_{j,\ell-1}$  for  $\ell = 1, 2, \dots$ . Then, formally,

$$(xI - A_j)(U_{j,0}(x), U_{j,1}(x), \dots)^T = 0, \quad (T_{k,0}(y), T_{k,1}(y), \dots)(yI - B_k) = 0,$$

with disjoint spectra  $\sigma(A_j) = I_j$  and  $\sigma(B_k) = I_k$ . As a consequence,

$$(A_j X - X B_k)_{\ell,n} = \int \int U_{j,\ell-1}(x) T_{k,n}(y) d\eta_j(x) d\omega_k(y) = \frac{\delta_{\ell,1} \delta_{n,0}}{U_{j,0} T_{k,0}} = \delta_{\ell,1} \delta_{n,0} \frac{b_j - a_j}{2\sqrt{2}},$$

in other words,  $X$  has a displacement rank given by  $\text{rang}(A_j X - X B_k) = 1$ . We may therefore apply [3, Theorem 2.1] which expresses the decay of the singular values  $\sigma_0(X) \geq \sigma_1(X) \dots$  of  $X$  in terms of the so-called Zolotarev numbers

$$Z_{\ell,n}(E, F) = \min_{R \in \mathcal{R}_{\ell,n}} \max_{x \in E} |r(x)| \max_{y \in F} \frac{1}{|r(y)|},$$

where  $\mathcal{R}_{\ell,n} = \{P/Q : P, Q \text{ polynomials of degree } \leq \ell, \text{ and } \leq n, \text{ respectively}\}$ . Namely, for  $0 \leq n \leq \min\{N_j - 1, N_k - 1\}$  there holds that

$$\sigma_n(X)/\sigma_0(X) \leq Z_{n,n}(I_j, I_k) \in [q^n, 4q^n], \quad \text{with } q = \exp\left(\frac{-1}{\text{cap}(I_j, I_k)}\right), \quad (2.10)$$

$\text{cap}(E, F)$  being the logarithmic capacity of the condenser with the two disjoint and compact plates  $E$  and  $F$ , see, e.g., [3]. In our particular case of two real intervals, according to [3, Eqns. (3.7) and (3.8)] this capacity can be computed explicitly in terms of the Gröetsch modulus  $\mu(\cdot)$  and the cross-ratio  $\gamma$  of the four endpoints

$$\exp\left(\frac{-1}{\text{cap}(I_j, I_k)}\right) = \exp\left(\frac{-\pi^2}{2\mu(1/\sqrt{\gamma})}\right) \leq \exp\left(\frac{-\pi^2}{\log(16\gamma)}\right), \quad \gamma = \left|\frac{a_k - a_j}{a_k - b_j} \frac{b_k - b_j}{b_k - a_j}\right|. \quad (2.11)$$

A similar argument can be applied for each column block  $(K_{j,k})_{j \neq k}$  for fixed  $k$ , showing that here the singular values decay like

$$Z_{n,n}(I_1 \cup I_2 \cup \dots \cup I_{k-1} \cup I_{k+1} \cup \dots \cup I_m, I_k). \quad (2.12)$$

This has to be compared with the in general much larger quantity <sup>3</sup>

$$Z_{n,0}(I_j, I_k) \in [q^n, 2q^n], \quad \text{with } q = \max\left\{\frac{1}{|\phi_j(a_k)|}, \frac{1}{|\phi_j(b_k)|}\right\}$$

occurring implicitly in Corollary 2.7. One may show that these rates are especially of quite different magnitude if  $I_k$  is close to one of the other intervals.  $\square$

It turns out that the proof of [3, Theorem 2.1] is constructive, a small rank matrix can be explicitly constructed by orthogonal projection of each column block of  $K$  on rational Krylov spaces, with poles related to optimal rational functions in (2.12). Another possibility would be to use the formalism of  $\mathcal{H}$ -matrices which are known to be powerful in case of (systems of) integral equations. Here we will proceed differently, and define different rational orthogonal bases of  $L^2(\omega_k)$ , and of  $L^2(\eta_k)$ , respectively.

### 3 A rational spectral method

In our polynomial spectral method in §2 we decided to project  $L^2(\omega_k)$  onto the finite dimensional subset  $\text{span}\{T_{k,0}, \dots, T_{k,N_k}\}$ , and  $L^2(\eta_j)$  onto  $\text{span}\{U_{j,0}, \dots, U_{j,N_j-1}\}$ , respectively. We now want to replace Chebyshev polynomials by Chebyshev orthogonal rational functions (ORF), with the aim of getting higher precision with less degree of freedoms, by carefully choosing the poles of these rational functions. Chebyshev ORF together with rational Gauss quadrature will be discussed in §3.1. We then introduce in §3.2 our new rational spectral method, and provide a complete error analysis in §3.3, showing how a suitable choice of poles reduces the complexity.

#### 3.1 Chebyshev orthogonal rational functions

We start by recalling orthonormal rational functions in  $L^2(\omega_j)$  and  $L^2(\eta_j)$ , both measures being defined in (1.2), and then describe the action of  $\mathcal{H}_j$  and  $\mathcal{C}_k$  on these bases. Given  $w_{j,0} = 0, w_{j,1}, w_{j,2} \dots \in (-1, 1)$  to be fixed later with  $\sum_{n=0}^{\infty} (1 - w_{j,n}^2) = \infty$ , the Takenaka-Malmquist rational functions [26] are given by

$$\varphi_{j,n}(w) = \frac{\sqrt{1 - w_{j,n}^2}}{1 - ww_{j,n}} B_{j,n}(w), \quad \text{with } B_{j,n}(w) = \prod_{\ell=0}^{n-1} \frac{w - w_{j,\ell}}{1 - ww_{j,\ell}},$$

they are known to be orthogonal in the Hardy space  $H^2$  [12]. Let as before  $w = \phi_j(z)$  be the Riemann conformal map from  $\overline{\mathbb{C}} \setminus I_j$  onto  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  normalized at infinity. Then the Faber-Dzhrbashyan rational functions for the interval  $I_j$  [11, 25] are obtained by taking Faber transforms of the  $\varphi_{j,n}$ .

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<sup>3</sup>Peut-etre un dessin pour les différents taux?

**Definition 3.1.** Define  $t_{j,0}(z) = 1$ , and for  $n \geq 1$  with  $w = \phi_j(z)$ ,

$$t_{j,n}(z) = \frac{1}{\sqrt{2}}(\varphi_{j,n}(w) + \varphi_{j,n}(\frac{1}{w})), \quad u_{j,n-1}(z) = \frac{2}{b_j - a_j} \sqrt{2} \frac{\varphi_{j,n}(w) - \varphi_{j,n}(\frac{1}{w})}{w - \frac{1}{w}}.$$

In the particular case  $w_{j,0} = w_{j,1} = \dots = 0$ , it is not difficult to check that  $\varphi_{j,n}(y) = y^n$ , and thus  $t_{j,n} = T_{j,n}$  and  $u_{j,n} = U_{j,n}$  of the previous chapter. Else, setting  $z_{j,n} \notin I_j$  with  $\phi_j(z_{j,n}) = 1/w_{j,n}$ , one may check that  $t_{j,n} \in \mathcal{R}_{n,n}$  with finite poles  $z_{j,1}, \dots, z_{j,n}$ , whereas  $u_{j,n-1} \in \mathcal{R}_{n-1,n}$  with finite poles  $z_{j,1}, \dots, z_{j,n}$  with obvious generalizations if some of the poles are  $\infty$ .

The following result shows why Bultheel et al in [6] refer to  $t_{j,n}$  and  $u_{j,n}$  as Chebyshev orthogonal rational functions (Chebyshev ORF) of the first kind and of the second kind. For  $u_{j,n}$ , these authors only consider the particular case  $w_{j,1} = 0$ .<sup>4</sup>

**Lemma 3.2.** *For  $j = 1, \dots, m$ , the functions  $\{t_{j,n} : n = 0, 1, \dots\}$  form an orthonormal and complete system of  $L^2(\omega_j)$ , and the functions  $\{u_{j,n} : n = 0, 1, \dots\}$  form an orthonormal and complete system of  $L^2(\eta_j)$ .*

The proof of this and the next result is presented in Appendix A.1. It seems to be an original observation that Lemma 2.1 on the Cauchy and the Hilbert transform for Chebyshev polynomials remains true after replacing  $(T_{j,n}, U_{j,n}, w^{-n})$  by the triplet  $(t_{j,n}, u_{j,n}, \varphi_{j,n}(1/w))$ , that is, for Chebyshev ORF.

**Lemma 3.3.** *For all  $n \geq 1$ ,*

$$\begin{aligned} \mathcal{C}_j(t_{j,n})(x) &= -\sqrt{2} \frac{\phi_j'(x)}{\phi_j(x)} \varphi_{j,n}(\frac{1}{\phi_j(x)}), & \mathcal{C}_j(t_{j,0})(x) &= -\frac{\phi_j'(x)}{\phi_j(x)}. \\ \mathcal{H}_j(t_{j,n})(x) &= u_{j,n-1}(x), & \mathcal{H}_j(t_{j,0})(x) &= 0. \end{aligned}$$

Since we have to introduce rational counterparts, let us write more explicitly  $\Pi_j = \Pi_j^{pol}$  and  $N_j = N_j^{pol}$ . An essential ingredient in our polynomial spectral method of §2 was to approach the  $L^2(\eta_j)$  scalar product by some Gauss quadrature rule, integrating correctly polynomials of degree  $< N_j^{pol}$ . Rational Gauss quadrature for general ORF with respect to a measure on the real line have been discussed by Van Deun, Bultheel and González-Vera in [6, Theorem 2.4], here we will need rational Gauss quadrature both for  $\eta_j$  and  $\omega_k$ .

**Proposition 3.4.** *Let  $N_j^{rat} > 0$  be some integer such that  $w_{j,N_j^{rat}+1} = 0$*

*(a) The Chebyshev ORF  $u_{j,N_j^{rat}}$  has distinct roots  $x_{j,1}^{rat}, \dots, x_{j,N_j^{rat}}^{rat} \in I_j$ . Furthermore, there exist positive real numbers  $d_{j,1}^{rat}, \dots, d_{j,N_j^{rat}}^{rat}$  such that, for all  $f, g \in \text{span}\{u_{j,0}, \dots, u_{j,N_j^{rat}-1}\}$ ,*

$$\int f(x)g(x) d\eta_j(x) = \sum_{n=1}^{N_j^{rat}} (d_{j,n}^{rat})^2 f(x_{j,n}^{rat})g(x_{j,n}^{rat}).$$

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<sup>4</sup>Ici et avant il manquent des references precises, avec numero de theoreme etc... dans [6, 12, 11]. Est-ce que [12] a aussi seulement le cas  $w_{j,1} = 0$ ?

(b) The Chebyshev ORF  $t_{j,N_j^{rat}+1}$  has distinct roots  $\tilde{x}_{j,0}^{rat}, \dots, \tilde{x}_{j,N_j^{rat}}^{rat} \in I_j$ . Furthermore, there exist positive real numbers  $\tilde{d}_{j,0}^{rat}, \dots, \tilde{d}_{j,N_j^{rat}}^{rat}$  such that, for all  $f, g \in \text{span}\{t_{j,0}, \dots, t_{j,N_j^{rat}}\}$ ,

$$\int f(x)g(x) d\omega_j(x) = \sum_{n=0}^{N_j^{rat}} (\tilde{d}_{j,n}^{rat})^2 f(\tilde{x}_{j,n}^{rat})g(\tilde{x}_{j,n}^{rat}).$$

Taking Chebyshev ORF  $f, g$ , we may conclude from Proposition 3.4 that, again, the matrix

$$M_j^{rat} = \left[ d_{j,h}^{rat} u_{j,\ell-1}(x_{j,h}^{rat}) \right]_{h,\ell=1,\dots,N_j^{rat}} \quad (3.1)$$

is orthogonal, which by the way allows to compute the weights  $d_{j,h}^{rat}$  in terms of the values  $u_{j,0}(x_{j,h}^{rat})$ , as mentioned in [6, Theorem 2.4]. Following §2, it will be useful to consider the rational interpolation operator  $\Pi_j^{rat}(g) \in \text{span}\{u_{j,0}, \dots, u_{j,N_j^{rat}-1}\}$  with  $\Pi_j^{rat}(g)(x_{j,h}^{rat}) = g(x_{j,h}^{rat})$  for  $h = 1, \dots, N_j^{rat}$ , and thus

$$\Pi_j^{rat}(g) = (u_{j,0}, \dots, u_{j,N_j^{rat}-1}) \left( M_j^{rat} \right)^* \left[ d_{j,h}^{rat} g(x_{j,h}^{rat}) \right]_{h=1,\dots,N_j^{rat}}.$$

In particular, as before,  $\langle u_{j,\ell-1}, \Pi_j^{rat} g \rangle_{L^2(\eta_j)}$  is zero for  $\ell > N_j^{rat}$ , and else coincides with the rational quadrature approximation of  $\langle u_{j,\ell-1}, g \rangle_{L^2(\eta_j)}$ .

Notice that, since  $w_{j,N_j^{rat}+1} = 0$ , Definition 3.1 tells us that

$$u_{j,N_j^{rat}}(z) = \frac{2}{b_j - a_j} \sqrt{2} \frac{B_{j,N_j^{rat}+1}(w) - B_{j,N_j^{rat}+1}(\frac{1}{w})}{w - \frac{1}{w}}, \quad w = \phi_j(z).$$

Recalling that  $B_{j,N_j^{rat}+1}(1/w) = 1/B_{j,N_j^{rat}+1}(w)$ , we conclude that  $\phi_j(x_{j,h}^{rat})$  are exactly the points on the unit circle different from  $\pm 1$  where the Blaschke product  $B_{j,N_j^{rat}+1}$  takes the value  $\pm 1$ . The authors in [6, Theorem 4.4] give a Weierstrass type method of complexity  $\mathcal{O}((N_j^{rat})^2)$  for computing these zeros, the weights being computable by the same complexity. Alternately, we can use linear algebra techniques, by finding the nodes as eigenvalues of some tridiagonal matrix pencil, and the weights as the first components of the corresponding normalized eigenvector. This is explained in more details in Appendix A.2.

Most what is said for rational Gaussian quadrature in  $L^2(\eta_j)$  is also true in  $L^2(\omega_j)$ , in particular, we can introduce the orthogonal matrix

$$\tilde{M}_j^{rat} = \left[ \tilde{d}_{j,h}^{rat} t_{j,\ell}(\tilde{x}_{j,h}^{rat}) \right]_{h,\ell=0,\dots,N_j^{rat}}$$

together with the rational interpolation operator  $\tilde{\Pi}_j^{rat}(g) \in \text{span}\{t_{j,0}, \dots, t_{j,N_j^{rat}}\}$  with  $\tilde{\Pi}_j^{rat}(g)(\tilde{x}_{j,h}^{rat}) = g(\tilde{x}_{j,h}^{rat})$  for  $h = 0, \dots, N_j^{rat}$ , and thus

$$\tilde{\Pi}_j^{rat}(g) = (t_{j,0}, \dots, t_{j,N_j^{rat}}) \left( \tilde{M}_j^{rat} \right)^* \left[ \tilde{d}_{j,h}^{rat} g(\tilde{x}_{j,h}^{rat}) \right]_{h=0,\dots,N_j^{rat}}.$$

In particular,  $\langle t_{j,\ell}, \tilde{\Pi}_j^{rat} g \rangle_{L^2(\omega_j)}$  is zero for  $\ell > N_j^{rat}$ , and else coincides with the rational quadrature approximation of  $\langle t_{j,\ell}, g \rangle_{L^2(\omega_j)}$ .

indices	$((I + K)_{j,k})_{\ell,n}$	$((I + \underline{K})_{j,k})_{\ell,n}$
$j = 1, \ell = 0$	$\delta_{n,0}$	$\delta_{n,0}$
$j > 1, \ell = 0, k = j$	$\chi_j(b_j)\delta_{n,0}$	$\chi_j(b_j)\delta_{n,0}$
$j > 1, \ell = 0, k = j - 1$	$\chi_j(b_{j-1})\delta_{n,0}$	$\chi_j(b_{j-1})\delta_{n,0}$
$j > 1, \ell = 0, k \neq j - 1, j$	$\langle \chi_j, t_{k,n} \rangle_{L^2(\omega_k)}$	$\langle \tilde{\Pi}_k^{rat}(\chi_j), t_{k,n} \rangle_{L^2(\omega_k)}$
$j \neq k, \ell > 0$	$\langle \mathcal{C}_{j,k} t_{k,n}, u_{j,\ell-1} \rangle_{L^2(\eta_j)}$	$\langle \tilde{\Pi}_j^{rat}(\mathcal{C}_{j,k} t_{k,n}), u_{j,\ell-1} \rangle_{L^2(\eta_j)}$
$j = k, \ell > 0$	$\delta_{\ell,n}$	$\delta_{\ell,n}$

Table 1: Elements  $((I + K)_{j,k})_{\ell,n}$  versus the discretized counterpart  $((I + \underline{K})_{j,k})_{\ell,n}$ . The indices  $\ell, n = 0, 1, \dots$  give the position in the block  $j, k = 1, \dots, m$ .

indices	$((c)_j)_\ell$	$((\underline{c})_j)_\ell$
$j = 1, \ell = 0$	1	1
$j > 1, \ell = 0$	$-\int Q d\omega_{j-1,j} + \int Q d\omega_{j,j-1}$	expansions in $T_{j-1,n}$ or $T_{j,n}$
$\ell > 0$	$-\langle Q', u_{j,\ell-1} \rangle_{L^2(\eta_j)}$	$-\langle \tilde{\Pi}_j^{rat}(Q'), u_{j,\ell-1} \rangle_{L^2(\eta_j)}$

Table 2: Elements  $((c)_j)_\ell$  versus the discretized counterpart  $((\underline{c})_j)_\ell$ . The index  $\ell = 0, 1, \dots$  give the position in the block  $j = 1, \dots, m$ .

### 3.2 Formulation of the new spectral method using ORF

In our polynomial spectral method in §2 we decided to project  $L^2(\omega_k)$  onto the finite dimensional subset  $\text{span}\{T_{k,0}, \dots, T_{k,N_k^{pol}}\}$ , and  $L^2(\eta_j)$  onto  $\text{span}\{U_{j,0}, \dots, U_{j,N_j^{pol}-1}\}$ , respectively. We now present a new spectral method where we project onto  $\text{span}\{t_{k,0}, \dots, t_{k,N_k^{rat}}\}$ , and  $\text{span}\{u_{j,0}, \dots, u_{j,N_j^{rat}-1}\}$ , respectively. Here we will follow very closely the polynomial spectral method in §2: we may reformulate our continuous problem  $(I + K)x = c$  of (2.1) roughly by replacing in (2.2), (2.3), and (2.4) all  $T_{k,n}$  by  $t_{k,n}$  and all  $U_{j,\ell}$  by  $u_{j,\ell}$ , respectively. This is done in Table 1 and Table 2. We then obtain our discretized problem  $(I + \underline{K})\underline{x} = \underline{c}$  of (2.6) with a matrix  $\underline{K}$  and  $\underline{c}$  where all non-explicit scalar products are replaced by a quadrature rule. The link with the finite-dimensional system  $(I + F^* \underline{K} F)y = F^* \underline{c}$  with  $\underline{x} = Fy$  described in Lemma 2.3 is then obtained by using the matrix

$$F^* = \text{diag} \left[ \left[ \begin{array}{cc} I_{N_j^{rat}+1} & 0 \end{array} \right]_{j=1, \dots, m} \right]$$

for some integers  $N_j^{rat}$  to be fixed later.

There are however two essential modifications in our approach. The first one concerns the computation of the entries of the right-hand side  $\underline{c}$ . In the polynomial case we took  $\underline{c} = FF^*c$ , where we supposed that we know explicitly the Chebyshev orthogonal series (2.5) of the external field  $Q$  on each interval  $I_j$ . This is no longer true for ORF since the coefficients in this orthogonal series depend on the choice of the poles. In addition, we have used the fact that  $U_{j,\ell-1}$  is proportional to the derivative of  $T_{j,\ell}$ , a property which is completely lost if we work with ORF. We will therefore approximate a typical entry  $-\langle Q', u_{j,\ell-1} \rangle_{L^2(\eta_j)}$  in  $c$  by  $-\langle \tilde{\Pi}_j^{rat}(Q'), u_{j,\ell-1} \rangle_{L^2(\eta_j)}$  in  $\underline{c}$ , in other words, we use rational Gauss quadrature.

The second more important modification concerns the discretization of condition (C)

in Problem 1.4, that is, the equation of index 0 in the row blocks  $j = 2, \dots, m$ . Remember that condition (D) in Problem 1.4 insures that  $U^\mu + Q$  is equal to some constant  $C_j$  on the interval  $I_j$  for  $j = 1, \dots, m$ , and condition (C) insures that  $C_j = C_{j-1}$  for  $j = 2, \dots, m$ . The formulation given in Problem 1.4 is suitable for a polynomial spectral method since, as shown in Lemma 2.1, here we have explicit formulas for the primitive of the Cauchy transform  $\mathcal{C}_{j,k}(T_{k,n})$ . Of course, using the partial fraction decomposition of  $\varphi_{j,n}(1/y)/y$ , we could derive an explicit formula for the primitive of  $\mathcal{C}_{j,k}(t_{k,n})$  in Lemma 3.3, and evaluate the integral over the interval  $[b_{j-1}, a_j]$ . However, it turns out that, in finite precision, computing the partial fraction decomposition of  $\varphi_{j,n}(1/y)/y$  is highly unstable, and we quickly loose precision even for modest values of  $n$ . Another attempt could consist of evaluating the integral in condition (C) by some quadrature rule, again a delicate procedure due to the end-point singularities of  $\mathcal{C}_{j,k}(t_{k,n})$  at  $a_k$  and  $b_k$ .

We finally decided to work with equilibrium measures of the condenser formed by the two plates  $I_{j-1}$  and  $I_j$ : according to, e.g., [24, Theorem VIII.2.6 and the computations after Theorem VIII.5.1] there exist probability measures  $\omega_{j-1,j}$  on  $I_{j-1}$  and  $\omega_{j,j-1}$  on  $I_j$  such that the potential

$$\chi_j := U^{\omega_{j-1,j} - \omega_{j,j-1}}$$

is equal to the constant  $\chi_j(b_{j-1})$  on  $I_{j-1}$ , equal to the constant  $\chi_j(b_j)$  on  $I_j$ , and else strictly between these two constants. There are explicit formulas for the density

$$\begin{aligned} \frac{d\omega_{j-1,j}}{dx} &= \frac{\gamma_j}{\sqrt{|(x - a_{j-1})(x - b_{j-1})(x - a_j)(x - b_j)|}} \Big|_{I_{j-1}}, \\ \frac{d\omega_{j,j-1}}{dx} &= \frac{\gamma_j}{\sqrt{|(x - a_{j-1})(x - b_{j-1})(x - a_j)(x - b_j)|}} \Big|_{I_j}, \end{aligned}$$

with a suitable normalization constant  $\gamma_j$ . It is known that the corresponding complex potential is analytic in  $\mathbb{C} \setminus (I_{j-1} \cup I_j)$  but multi-valued, and its composition with the exponential function is known to be related to the conformal map sending the doubly connected domain  $\mathcal{C} \setminus (I_{j-1} \cup I_j)$  onto some ring domain [24, Chapter VIII.6]. Notice that there are explicit formulas in terms of elliptic functions in order to evaluate  $\chi_j$  on the real axis. Our implementation for evaluating  $\chi_j$  on the real axis is to evaluate separately the potential of  $\omega_{j-1,j}$  (and similarly for  $\omega_{j,j-1}$ ) by computing a polynomial Chebyshev expansion in  $L^2(\omega_{j-1})$  of

$$\frac{1}{\sqrt{(x - a_j)(x - b_j)}} = \tilde{\gamma}_j \frac{d\omega_{j-1,j}}{d\omega_{j-1}} \quad (3.2)$$

(either by some exact formula or simply by the matlab package `chebfun`), normalize in order to get a probability measure, and then use Example 1.2 in order to express  $U^{\omega_{j-1,j}}$  in arbitrary precision as a polynomial Chebyshev series on  $I_{j-1}$ , or a power series in  $1/\phi_{j-1}(x)$  outside of  $I_{j-1}$ .

Using Fubini, we find that

$$0 = C_{j-1} - C_j = \int (U^\mu + Q) d(\omega_{j-1,j} - \omega_{j,j-1}) = \int \chi_j d\mu + \int Q d\omega_{j-1,j} - \int Q d\omega_{j,j-1}.$$

The last two integrals on the right can be easily computed from a Chebyshev expansion (2.5) of the external field  $Q$  on  $I_j$  and  $I_{j-1}$  together with the Chebyshev expansion of



(3.2), giving up to a sign the entry of index 0 in the  $j$ th row block of the right-hand side  $c$ . Similarly, the corresponding row in  $K$  has the entries

$$\langle \chi_j, t_{k,n} \rangle_{L^2(\omega_k)} = \begin{cases} \chi_j(b_{j-1})\delta_{n,0} & \text{if } k = j - 1, \\ \chi_j(b_j)\delta_{n,0} & \text{if } k = j, \end{cases}$$

and for  $k \notin \{j - 1, j\}$ , we can safely use a quadrature formula in order to approach  $\langle \chi_j, t_{k,n} \rangle_{L^2(\omega_k)}$  since  $\chi_j$  is very regular in a neighborhood of  $I_k$  (in fact the corresponding complex potential is analytic in  $\mathbb{C} \setminus [a_{j-1}, b_j]$ ). We choose the rational Gauss quadrature rule for  $L^2(\omega_k)$  given in Proposition 3.4(b).

**Remark 3.5.** If we ignore for the moment the exceptional<sup>5</sup> rows  $\ell = 0$  for  $j = 2, \dots, m$ , we see that for constructing the finite dimensional system  $(I + F^* \underline{K} F)y = F^* \underline{c}$  we have to construct the elements at the position  $(\ell, n)$  of the off-diagonal blocks of index  $j \neq k$  of size  $(N_j^{rat} + 1) \times (N_k^{rat} + 1)$ , each of them asking at most  $\mathcal{O}(N_j^{rat})$  operations (and similarly for the right-hand side). In addition, the construction of the rational Gauss quadrature rules requires  $\mathcal{O}((N_j^{rat})^2)$  operations for each  $j = 1, \dots, m$ .

In the next section we will show that  $N_j^{rat} \ll N_j^{pol}$  for all  $j$ , that is, our rational spectral method is much cheaper than the polynomial spectral method suggested in §2.

### 3.3 Error estimates

According to Lemma 2.3, we have to give upper bounds for  $K - \underline{K}$  coming from two sources, namely  $(I - \Pi_j^{rat})\mathcal{C}_{j,k}$  and, in the exceptional rows,  $(I - \tilde{\Pi}_k^{rat})(\chi_j)$ . Subsequently, we have to estimate the error  $c - \underline{c}$  in the right-hand side where a third term  $(I - \Pi_j^{rat})(Q_j')$  occurs.

**Theorem 3.6.** *For fixed  $j, k \in \{1, \dots, m\}$  with  $j \neq k$ , let  $N = N_j^{rat}$ ,  $w_{j,N+1} = 0$ , and consider the Blaschke product  $\tilde{B}_{j,N}(w) = B_{j,N+1}(w)/w = \prod_{\ell=1}^N \frac{w - w_{j,\ell}}{1 - \overline{w_{j,\ell}}w}$ . Then we have the following upper bound for the Hilbert-Schmidt norm*

$$\|(I - \Pi_j^{rat})\mathcal{C}_{j,k}\|_{HS} \leq \frac{1}{2\sqrt{2}} \frac{b_j - a_j}{\text{dist}(I_j, I_k)} \max_{x \in I_k} |\tilde{B}_{j,N}(\frac{1}{\phi_j(x)})|.$$

*Proof.* We start by applying the Hermite representation in order to obtain a compact representation of the interpolation error. Since

$$y \mapsto \frac{1}{x - y} \left( 1 - \frac{u_{j,N}(y)}{u_{j,N}(x)} \right) \in \text{span}\{u_{j,0}(y), \dots, u_{j,N-1}(y)\}$$

coincides with  $1/(x - y)$  for  $y$  being one of the roots of  $u_{j,N}$ , we find for  $y \in I_j$  that

$$(I - \Pi_j^{rat})\mathcal{C}_{j,k}(t_{k,n})(y) = \int_{I_k} \frac{u_{j,N}(y)}{u_{j,N}(x)} \frac{t_{k,n}(x)}{x - y} d\omega_k(x).$$

---

<sup>5</sup>These  $m - 1$  exceptional rows ask for each entry at most  $\mathcal{O}(N_k^{pol})$  operations. This slight increase in cost is required for the sake of numerical stability.



Applying Parseval, we deduce that

$$\begin{aligned}
\|(I - \Pi_j^{rat})\mathcal{C}_{j,k}\|_{HS}^2 &= \sum_{n=0}^{\infty} \|(I - \Pi_j^{rat})\mathcal{C}_{j,k}(t_{k,n})\|_{L^2(\eta_j)}^2 \\
&= \int_{I_j} \sum_{n=0}^{\infty} \left( \int_{I_k} \frac{u_{j,N}(y) t_{k,n}(x)}{u_{j,N}(x) x - y} d\omega_k(x) \right)^2 d\eta_j(y) \\
&= \int_{I_j} \int_{I_k} \left( \frac{u_{j,N}(y)}{u_{j,N}(x)} \right)^2 \frac{d\omega_k(x)}{(x-y)^2} d\eta_j(y) \\
&\leq \frac{1}{\text{dist}(I_j, I_k)^2} \max_{x \in I_k} \frac{1}{|u_{j,N}(x)|^2}.
\end{aligned}$$

Hence the result follows by observing that, with  $w = \phi_j(x) \in \mathbb{R} \setminus [-1, 1]$  and thus  $\tilde{B}_{j,N}(w)^2 > 1$ ,

$$\begin{aligned}
|u_{j,N}(x)| &= \frac{2\sqrt{2}}{b_j - a_j} \left| \frac{B_{j,N+1}(w) - B_{j,N+1}(\frac{1}{w})}{w - \frac{1}{w}} \right| \\
&= \frac{2\sqrt{2}|\tilde{B}_{j,N}(w)|}{b_j - a_j} \left| \frac{1 - \frac{1}{w^2 \tilde{B}_{j,N}(w)^2}}{1 - \frac{1}{w^2}} \right| \geq \frac{2\sqrt{2}|\tilde{B}_{j,N}(w)|}{b_j - a_j}.
\end{aligned}$$

□

**Theorem 3.7.** For fixed  $k \in \{1, \dots, m\}$ ,  $j \in \{2, \dots, m\} \setminus \{k, k+1\}$ , let  $N = N_k^{rat}$ ,  $w_{k,N+1} = 0$ , and consider the Blaschke product  $\tilde{B}_{k,N}$  as in Theorem 3.6. Then we have the following upper bound

$$\begin{aligned}
\|(I - \tilde{\Pi}_k^{rat})(\chi_j)\|_{L^2(\omega_k)} &\leq \sqrt{2} \max \left\{ \left| \log \left( \frac{a_k - a_{j-1}}{a_k - b_j} \right) \right|, \left| \log \left( \frac{b_k - a_{j-1}}{b_k - b_j} \right) \right| \right\} \\
&\quad \times \max_{y \in [a_{j-1}, b_j]} \frac{1}{|\phi_k(y)|} \max_{y \in [a_{j-1}, b_j]} |\tilde{B}_{k,N}(\frac{1}{\phi_k(y)})|.
\end{aligned}$$

*Proof.* Consider the complex potential

$$\tilde{\chi}_j(z) = \int \log \left( \frac{1}{z - x} \right) d(\omega_{j-1,j} - \omega_{j,j-1})(x),$$

which can be shown to be analytic (and single-valued) in  $\mathbb{C} \setminus [a_{j-1}, b_j]$  including at  $\infty$  where  $\tilde{\chi}_j(z) = \mathcal{O}(1/z)_{z \rightarrow \infty}$ . Denoting by  $\Gamma$  a contour encircling once in mathematically positive direction the interval  $I_k$ , we get from the Cauchy formula

$$\forall x \in I_k : \quad \chi_j(x) = \text{Re}(\tilde{\chi}_j(x)) = \tilde{\chi}_j(x) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{\chi}_j(y) \frac{dy}{y - x},$$

and, by the Hermite formula,

$$\forall x \in I_k : \quad ((I - \tilde{\Pi}_k^{rat})(\chi_j))(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{t_{k,N+1}(x)}{t_{k,N+1}(y)} \tilde{\chi}_j(y) \frac{dy}{y - x}.$$

We now deform  $\Gamma$  to encircling once in the mathematically negative direction the interval  $[a_{j-1}, b_j]$ . Since  $1/t_{j,N+1}$  is single-valued around  $[a_{j-1}, b_j]$ , we obtain with the boundary values  $\tilde{\chi}(y+i0)$  the expression

$$\forall x \in I_k : \quad ((I - \tilde{\Pi}_k^{rat})(\chi_j))(x) = \frac{1}{\pi} \int_{a_{j-1}}^{b_j} \frac{t_{k,N+1}(x)}{t_{k,N+1}(y)} \text{Im}(\tilde{\chi}_j)(y+i0) \frac{dy}{y-x}.$$

Using our explicit formula for  $t_{k,N+1}$  we find for  $w = \phi_k(y) \in \mathbb{R} \setminus [-1, 1]$  that

$$\frac{1}{|t_{k,N+1}(y)|} = \frac{\sqrt{2}}{|B_{k,N+1}(w) + 1/B_{k,N+1}(w)|} \leq \sqrt{2} |B_{k,N+1}(\frac{1}{\phi_k(y)})|,$$

and

$$\begin{aligned} |\text{Im}(\tilde{\chi}_j)(y+i0)| &= \left| \int_{a_{j-1}}^{b_j} \arg(y+i0-x) d(\omega_{j-1,j} - \omega_{j,j-1})(x) \right| \\ &= \begin{cases} 0 & \text{for } y \leq a_{j-1} \text{ and } y \geq b_j, \\ \pi & \text{for } y \in [b_{j-1}, a_j], \\ \pi \left| \int_{a_{j-1}}^y d\omega_{j-1,j}(x) \right| \in [0, \pi] & \text{for } y \in [a_{j-1}, b_{j-1}], \\ \pi \left| \int_y^{b_j} d\omega_{j,j-1}(x) \right| \in [0, \pi] & \text{for } y \in [a_j, b_j]. \end{cases} \end{aligned}$$

Thus for  $x \in I_k$

$$\left| ((I - \tilde{\Pi}_k^{rat})(\chi_j))(x) \right| \leq \sqrt{2} \left| \log\left(\frac{x - a_{j-1}}{x - b_j}\right) \right| |t_{k,N+1}(x)| \max_{y \in [a_{j-1}, b_j]} |B_{k,N+1}(\frac{1}{\phi_k(y)})|,$$

and the result follows by recalling that  $B_{k,N+1}(\frac{1}{\phi_k(y)}) = \tilde{B}_{k,N}(\frac{1}{\phi_k(y)})/\phi_k(y)$ .  $\square$

**Remark 3.8.** Up to some technical constants which will be included in Corollary 3.10 below, we may summarize the preceding two Theorems as follows:  $K - \underline{K}$  will be "small" if for  $j = 1, 2, \dots, m$  we may construct Blaschke products of degree  $N_j^{rat}$  such that  $\tilde{B}_{j,N_j^{rat}}(1/\phi_j)$  is "small" on other intervals  $I_k$  for all  $k \neq j$  in Theorem 3.6, and small on all intervals  $[a_{k-1}, b_k]$  for all  $k \notin \{j, j+1\}$  in Theorem 3.7 (after interchanging the roles of  $j$  and  $k$ ). Notice that the second set includes both  $I_{k-1}$  and  $I_k$ , but also the gap  $(b_{k-1}, a_k)$  between these two intervals. In what follows, we include all the gaps in our set where the above Blaschke products should be small, that is, with the notation of Remark 2.8, we want to make the link with the Zolotarev quantities

$$\forall j = 1, \dots, m : \quad Z_{N_j^{rat}, N_j^{rat}}(I_j, [a_{j+1}, b_{j-1}]), \quad a_{m+1} := a_1, \quad b_0 := b_m, \quad (3.3)$$

an upper bound for the Zolotarev quantities mentioned before in equation (2.12) of Remark 2.8 which has been much less studied in the literature.<sup>6</sup>

The following well-known result tells us how to choose our free parameters  $w_{j,n}$  for  $n = 1, 2, \dots, N_j^{rat}$ .

---

<sup>6</sup>Notice that Theorem 3.6 only requires to monitor (2.12). We leave for further research to investigate a different discretization of our exceptional rows which treats differently the gaps between our  $m$  intervals and thus only requires to monitor (2.12) instead of (3.3).

**Lemma 3.9.** *Let  $j \in \{1, 2, \dots, m\}$ ,  $N = N_j^{rat}$ , and  $\gamma_j := \left| \frac{a_{j+1}-a_j}{a_{j+1}-b_j} \frac{b_{j-1}-b_j}{b_{j-1}-a_j} \right|$ , then*

$$\min_{w_{j,1}, \dots, w_{j,N} \in \mathbb{C}} \max_{x \in [a_{j+1}, b_{j-1}]} \left| \tilde{B}_{j,N} \left( \frac{1}{\phi_j(x)} \right) \right| \leq 2 \exp \left( \frac{-N}{\text{cap}([a_{j+1}, b_{j-1}], I_j)} \right) \leq 2 \exp \left( \frac{-\pi^2 N}{\log(16\gamma_j)} \right), \quad (3.4)$$

*the minimum being attained for some  $w_{j,1}, \dots, w_{j,N} \in \mathbb{E} := [1/\phi_j(b_{j-1}), 1/\phi_j(a_{j+1})] \subset (-1, 1)$ , and expressible explicitly in terms of elliptic functions as follows: if we denote by  $w = T(u)$  the Blaschke factor with  $\mathbb{E} = T([- \lambda, \lambda])$  for some  $\lambda \in (0, 1)$ , then*

$$\text{for } \ell = 1, \dots, N: \quad w_{j,\ell} = T \left( \lambda \operatorname{sn} \left( K(\lambda^2) \left( -1 + \frac{2\ell - 1}{N} \right); \lambda^2 \right) \right). \quad (3.5)$$

*Proof.* We first observe that a composition of a Blaschke product with a Blaschke factor is again a Blaschke product. Hence the change of variables  $w = T(u)$  implies that

$$\min_{B \text{ Blaschke of order } N} \max_{w \in \mathbb{E}} |B(w)| = \min_{B \text{ Blaschke of order } N} \max_{u \in [-\lambda, \lambda]} |B(u)|.$$

The problem of minimal Blaschke products of order  $N$  on a symmetric interval  $[-\lambda, \lambda]$  with modulus  $k = \lambda^2 \in (0, 1)$  has been reviewed by many authors, see for instance [21]. We cite from [21, Section 3.2] an explicit formula of such a minimal Blaschke product  $B_N$  for a symmetric interval  $[-\sqrt{k}, \sqrt{k}]$ , in terms of Jacobian elliptic functions [22, §22]

$$B_N(u) = \sqrt{k_N} \operatorname{cd}(2NK(k_N)v; k_N), \quad u = \sqrt{k} \operatorname{cd}(2K(k)v; k), \quad \operatorname{cd}(x; k) = \frac{\operatorname{cn}(x; k)}{\operatorname{dn}(x; k)},$$

where  $K(k)$  denotes the complete elliptic integral of modulus  $k$  [22, (19.2.8)],  $K'(k) = K(\sqrt{1-k^2})$ , and  $k_N \in (0, 1)$  is uniquely defined<sup>7</sup> by  $K'(k_N)/K(k_N) = NK'(k)/K(k)$ . In particular [21, Section 3.2], the roots of  $B_N$  are given by  $u_\ell = \sqrt{k} \operatorname{cd}(K(k)^{\frac{2\ell-1}{N}}; k) = -\sqrt{k} \operatorname{sn}(K(k)(-1 + \frac{2\ell-1}{N}); k)$  for  $\ell = 1, 2, \dots, N$ , where in the last equality we have used a quarter-period shift of variables of [22, Table 22.4.3]. Recalling that  $\operatorname{sn}$  is an odd function, formula (3.5) follows after change of variables  $w = T(u)$ . In [21, Proposition 4.1(a)], the authors give the following link with a Zolotarev problem on two real intervals

$$\min_{B \text{ Blaschke of order } N} \max_{u \in [-\lambda, \lambda]} |B(u)| = \sqrt{Z_{N,N}([-\lambda, \lambda], [1/\lambda, -1/\lambda])},$$

and a combination with [3, Eqns. (3.7) and (3.8)] gives

$$\sqrt{Z_{N,N}([-\lambda, \lambda], [1/\lambda, -1/\lambda])} \leq 2 \exp \left( -\frac{N}{2 \operatorname{cap}([-\lambda, \lambda], [1/\lambda, -1/\lambda])} \right).$$

Recall that, for a doubly connected set  $\mathbb{C} \setminus (E \cup F)$ , there exists a conformal map mapping  $\mathbb{C} \setminus (E \cup F)$  onto the ring domain  $\{w \in \mathbb{C} : c < |w| < 1\}$ , with the conformal invariant  $c = \exp(-1/\operatorname{cap}(E, F))$ . Hence, by conformal invariance,

$$2 \operatorname{cap} \left( ([-\lambda, \lambda], [\frac{1}{\lambda}, \frac{-1}{\lambda}]) \right) = 2 \operatorname{cap}(\mathbb{E}, 1/\mathbb{E}) = \operatorname{cap}(1/\mathbb{E}, \overline{\mathbb{D}}) = \operatorname{cap}([a_{j+1}, b_{j-1}], I_j),$$

showing the first inequality in (3.4), the second one following from (2.10) and (2.11).  $\square$

<sup>7</sup>Contrary to [22], the Jacobian elliptic functions  $\tilde{cd}(\cdot; \tau)$  of [21] use a purely imaginary second argument  $\tau \in i(0, +\infty)$ . Comparing the periods  $2\omega_1(\tau)$  and  $\tau\omega_1(\tau)$  of  $\tilde{cd}(\cdot; \tau)$  and  $4K(k)$  and  $2iK'(k)$  of  $cd(\cdot; k)$ , we may conclude that  $\tilde{cd}(\cdot; \tau) = cd(\cdot; k)$  with  $\tau = iK'(k)/K(k)$ , in particular the quantities  $\omega_1(\tau), k(\tau), k(N\tau), \omega_1(N\tau)$  in [21] have to be replaced by  $2K(k), k, k_N, 2K(k_N)$  in the terminology of [22].

Comparing with (2.10) and (2.11), we see that the minimum in Lemma 3.9 is at most twice as large as the Zolotarev quantity of (3.3) (and in fact can be shown to be asymptotically sharper by a factor 2).

**Corollary 3.10.** *Given  $\epsilon > 0$ , provided that the integers  $N_1^{rat}, \dots, N_m^{rat}$  are chosen such that for  $j = 1, \dots, m$  and  $k \in \{1, \dots, m\} \setminus \{j\}$*

$$\frac{1}{\sqrt{2}} \frac{b_j - a_j}{\text{dist}(I_j, I_k)} \exp\left(-\frac{\pi^2 N_j^{rat}}{\log(16\gamma_j)}\right) \leq \epsilon,$$

and for  $k \in \{1, \dots, m\}$ ,  $j \in \{2, \dots, m\} \setminus \{k, k+1\}$

$$2\sqrt{2} \max\left\{\left|\log \frac{a_k - a_{j-1}}{a_k - b_j}\right|, \left|\log \frac{b_k - a_{j-1}}{b_k - b_j}\right|\right\} \max_{y \in [a_{j-1}, b_j]} \frac{1}{|\phi_k(y)|} \exp\left(-\frac{\pi^2 N_k^{rat}}{\log(16\gamma_k)}\right) \leq \epsilon,$$

and the parameters  $w_{j,n}$  for  $j = 1, \dots, m$  and  $n = 1, \dots, N_j^{rat}$  are chosen as in Lemma 3.9, then for all  $j \neq k$  there holds  $\|K_{j,k} - \underline{K}_{j,k}\|_{HS} \leq \sqrt{2}\epsilon$ .

We still have to discuss the error  $\|c - \underline{c}\|$  introduced by using a quadrature formula. In the next two statements we suggest two different approaches.

**Remark 3.11.** If  $Q'|_{I_j}$  is an element of  $\mathcal{R}_{\ell-1,\ell}$  (including a polynomial of degree  $\ell - 1$ ) with real poles<sup>8</sup> out of  $I_j$ , then we may choose the parameters  $w_{j,1}, \dots, w_{j,\ell} \in (-1, 1)$  such that  $Q'|_{I_j} \in \text{span}(u_{j,0}, \dots, u_{j,\ell-1})$ , and thus for  $N_j^{rat} \geq \ell$  and the  $j$ th row block of the right-hand side we find that

$$\|c_j - \underline{c}_j\| = \|(I - \Pi_j^{rat})(Q')\|_{L^2(\eta_j)} = 0.$$

The parameters  $w_{j,\ell+1}, \dots, w_{j,N_j^{rat}}$  will then be chosen to minimize the Blaschke product  $\prod_{n=\ell+1}^{N_j^{rat}} \left|\frac{w - w_{j,n}}{1 - \overline{w}w_{j,n}}\right|$ . Formulated differently, our conclusions of Corollary 3.10 remains valid if we shift  $N_j^{rat}$  and the index  $n$  of the corresponding parameters  $w_{j,n}$  by  $\ell$ .

For introducing our second approach, recall that  $(\underline{c}_j)_\ell = 0$  for  $\ell \geq N_j$ , and hence

$$\|(I - \Pi_j^{rat})(Q')\|_{L^2(\eta_j)} \geq \|(I - R_{j,N_j^{rat}})(Q')\|_{L^2(\eta_j)}$$

$R_{j,N}(g)$  denoting the orthogonal projection of  $g \in L^2(\eta_j)$  onto  $\text{span}(u_{j,0}, \dots, u_{j,N-1})$ . For real analytic  $Q'$ , we can show that both terms have approximately the same magnitude.

**Theorem 3.12.** *Let  $N = N_j^{rat}$ , and suppose that  $g$  is real analytic on  $I_j$  and more precisely has an analytic continuation in the open set  $\Omega \supset I_j$ , and is continuous up to the boundary. Then*

$$\|(I - R_{j,N})(g)\|_{L^2(\eta_j)} \leq Cq, \tag{3.6}$$

$$\|(I - R_{j,N})(g)\|_{L^2(\eta_j)} \leq \|(I - \Pi_j^{rat})(g)\|_{L^2(\eta_j)} \leq \tag{3.7}$$

$$\leq 2 \|(I - R_{j,N})(g)\|_{L^2(\eta_j)} + \frac{Cq^2}{1-q}, \tag{3.8}$$

---

<sup>8</sup>We believe that a similar approach works for non-real poles though this is not studied in the present manuscript.

with the two constants<sup>9</sup>

$$C := \frac{1}{\sqrt{2\pi}} \int_{\partial\Omega} |g(\zeta)| \frac{|d\zeta|}{\sqrt{1 - 1/|\phi_j(\zeta)|^2}}, \quad q := \max_{z \in \partial\Omega} |B_{j,N+1}(\frac{1}{\phi_j(z)})| \in (0, 1).$$

*Proof.* Let

$$g(x) = \sum_{\ell=0}^{\infty} g_{\ell} u_{j,\ell}(x), \quad g_{\ell} = \langle g, u_{j,\ell} \rangle_{L^2(\eta_j)}.$$

We begin by proving (3.6). Using the Cauchy formula for  $g$  and the change of variables  $x = \psi_j(w)$  (where  $\psi_j$  is the inverse of the conformal mapping  $\phi_j$ ) we get

$$\begin{aligned} g_{n-1} &= \frac{1}{2\pi i} \int_{\partial\Omega} g(\zeta) \int_{|w|=1} \frac{\varphi_{j,n}(w)}{\zeta - \psi_j(w)} (w - \frac{1}{w}) \frac{i}{w} dw d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} g(\zeta) \frac{4i}{b_j - a_j} \int_{|w|=1} \frac{\varphi_{j,n}(w)(w^2 - 1)}{\phi_j(\zeta) - w} \frac{1}{w - \frac{1}{\phi_j(\zeta)}} \frac{1}{w} dw d\zeta \end{aligned}$$

Observing that  $\frac{\varphi_{j,n}(w)(w^2-1)}{\phi_j(\zeta)-w}$  is analytic on the unit disk and using the residue theorem we finally get

$$g_{n-1} = \langle g, u_{j,n-1} \rangle_{L^2(\eta_j)} = \frac{\sqrt{2}}{2\pi i} \int_{\partial\Omega} \varphi_{j,n}(\frac{1}{\phi_j(\zeta)}) g(\zeta) d\zeta, \quad (3.9)$$

We easily show that

$$\overline{B_{j,n+1}(u)} B_{j,n+1}(v) - \overline{B_{j,n}(u)} B_{j,n}(v) = \overline{\varphi_{j,n}(u)} \varphi_{j,n}(v) (\bar{u}v - 1)$$

and so

$$\sum_{\ell=0}^n \overline{\varphi_{j,\ell}(u)} \varphi_{j,\ell}(v) = \frac{1}{1 - \bar{u}v} (1 - \overline{B_{j,n+1}(u)} B_{j,n+1}(v)) \quad (3.10)$$

As

$$\frac{1}{2\pi i} \int_{|v|=1} \frac{1}{1 - \bar{u}v} \overline{\varphi_{j,n}(v)} \frac{dv}{v} = \overline{\varphi_{j,n}(u)}$$

we get the expansion

$$\frac{1}{1 - \bar{u}v} = \sum_{\ell=0}^{\infty} \overline{\varphi_{j,n}(u)} \varphi_{j,\ell}(v) \quad (3.11)$$

Combining (3.10) and (3.11) and replacing in the expression of the error of the orthogonal projection we get

$$\|(I - R_{j,N})g\|_{L^2(\omega_j)}^2 = \sum_{n=N}^{\infty} |g_n|^2 = \frac{1}{2\pi^2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{B_{j,N+1}(\frac{1}{\phi_j(z)}) \overline{B_{j,N+1}(\frac{1}{\phi_j(\zeta)})}}{1 - \frac{1}{\phi_j(z)} \frac{1}{\phi_j(\zeta)}} g(z) dz \overline{g(\zeta)} d\zeta.$$

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<sup>9</sup>If for instance  $\Omega$  is an ellipse of parameter  $\rho > 1$  with foci given by the endpoints  $a_j$  and  $b_j$  (or, in other words,  $\mathbb{C} \setminus \Omega = \{z \in \mathbb{C} \setminus I_j : |\phi_j(z)| \geq \rho\}$ ), then  $q$  is the maximum of the Blaschke product  $B_{j,N+1}$  on the circle  $|w| = 1/\rho < 1$ , which is strictly less than 1, and in general (depending on the choice of  $N$  and the roots of  $B_{j,N+1}$ ) much smaller than 1, so that terms of order  $\mathcal{O}(q^2)$  can be ignored. We conclude from (3.6)–(3.8) that we expect  $\|(I - R_{j,N})(g)\|_{L^2(\eta_j)}$  and  $\|(I - \Pi_j^{rat})(g)\|_{L^2(\eta_j)}$  to be of the same magnitude, at most  $\mathcal{O}(q)$ .

Observing that

$$\left| \frac{1}{1 - \frac{1}{\phi_j(z)} \frac{1}{\phi_j(\zeta)}} \right| \leq \frac{1}{\sqrt{1 - \frac{1}{|\phi_j(z)|^2}}} \frac{1}{\sqrt{1 - \frac{1}{|\phi_j(\zeta)|^2}}},$$

we arrive at (3.6).

Let us now prove (3.8). First we choose carefully the parameters  $w_{j,\ell}$  for  $\ell > N + 1$ ,  $N = N_j^{rat}$ , such that via some aliasing we may explicitly compute  $\Pi_j^{rat}(g)$  in (3.12). Let for each integer  $\nu > 0$ , and  $\ell \in \{0, \dots, N + 1\}$

$$w_{\nu(2N+2)+\ell} = w_{\nu(2N+2)-\ell} = w_\ell,$$

then  $B_{j,\nu(2N+2)}(w) = (B_{j,N+1}(w))^{2\nu}$ , and

$$\varphi_{j,\nu(2N+2)+\ell}(w) = B_{j,\nu(2N+2)}(w)\varphi_{j,\ell}(w), \quad \varphi_{j,\nu(2N+2)-\ell}\left(\frac{1}{w}\right) = B_{j,\nu(2N+2)}\left(\frac{1}{w}\right)\varphi_{j,\ell}(w).$$

Since the quadrature nodes  $x_{j,h}^{rat}$  satisfy  $(B_{j,N+1}(\phi_j(x_{j,h}^{rat})))^2 = 1$ , we conclude that, for  $\ell = 1, \dots, N$ ,

$$u_{j,\nu(2N+2)+\ell-1}(x_{j,h}^{rat}) = u_{j,\ell-1}(x_{j,h}^{rat}) = -u_{j,\nu(2N+2)-\ell-1}(x_{j,h}^{rat}), \quad u_{j,\nu(N+1)-1}(x_{j,h}^{rat}) = 0,$$

implying that

$$\Pi_j^{rat}(u_{j,\nu(2N+2)+\ell-1}) = -\Pi_j^{rat}(u_{j,\nu(2N+2)-\ell-1}) = u_{j,\ell-1}, \quad \Pi_j^{rat}(u_{j,\nu(N+1)-1}) = 0,$$

and thus

$$\|(R_{j,N} - \Pi_j^{rat})g\|_{L^2(\eta_j)}^2 = \sum_{\ell=1}^N \left( \sum_{\nu=1}^{\infty} (g_{\nu(2N+2)+\ell-1} - g_{\nu(2N+2)-\ell-1}) \right)^2. \quad (3.12)$$

We may write

$$\begin{aligned} (\Pi_j^{rat} - R_{j,N})(g) &= (\Pi_j^{rat} - R_{j,N})(I - R_{j,N})(g) \\ &= \sum_{\nu=1}^{\infty} (\Pi_j^{rat} - R_{j,N})(R_{j,(\nu+1)(N+1)-1} - R_{j,\nu(N+1)-1})(g) \\ &= \sum_{\nu=1}^{\infty} (\Pi_j^{rat} - R_{j,N}) \left( \sum_{\ell=1}^{N+1} g_{\nu(N+1)-1+\ell} u_{j,\nu(N+1)-1+\ell} \right) \end{aligned}$$

Equation (3.12) allows us to conclude that

$$\begin{aligned} \|(\Pi_j^{rat} - R_{j,N}) \left( \sum_{\ell=1}^{N+1} g_{\nu(N+1)-1+\ell} u_{j,\nu(N+1)-1+\ell} \right)\|_{L^2(\eta_j)}^2 &= \sum_{\ell=1}^N g_{\nu(N+1)+\ell}^2 \leq \\ &\leq \| (R_{j,(\nu+1)(N+1)-1} - R_{j,\nu(N+1)-1})(g) \|_{L^2(\eta_j)}^2 \\ &\leq \| (I - R_{j,\nu(N+1)-1})(g) \|_{L^2(\eta_j)}^2 \end{aligned}$$

. Then

$$\|(\Pi_j^{rat} - R_{j,N})(g)\|_{L^2(\eta_j)} \leq \sum_{\nu=1}^{\infty} \| (I - R_{j,\nu(N+1)-1})(g) \|_{L^2(\eta_j)},$$

and we conclude using (3.6).  $\square$

As a consequence of Theorem 3.12 and Corollary 3.10, we may conclude that our rational spectral method has geometric convergence in case of a real analytic external field. Notice that the rate of convergence for the right-hand side depends on the shape of  $\Omega$ . Choosing level sets for the conformal mapping sending the doubly connected domain  $\mathbb{C} \setminus (I_j \cup [a_{j+1}, b_{j-1}])$  onto a ring domain would allow to compare the rate in these two statements, and it can be shown that we obtain the same rate if the restriction of  $Q'$  on  $I_j$  has an analytic continuation outside  $[a_{j+1}, b_{j-1}]$ .

## 4 Conclusion

We have suggested a new numerical method for solving the signed equilibrium with external field in logarithmic potential theory on a union of distinct real intervals. A reformulation of our problem did lead us to a system of integral equations with a weakly singular Cauchy kernel. We then recalled a polynomial spectral method and its error analysis, and suggest a new spectral method using orthogonal rational functions in order to solve our problem.

Choosing appropriate and explicitly given poles depending only on the intervals allowed us to speed up computation, in particular in the delicate situation where two intervals are close to each other. We still need more numerical evidence that our method is superior to polynomial spectral methods which have been used before. A particular issue is how to implement the evaluation of orthogonal rational functions and rational Gaussian quadrature formulas in order to insure numerical stability of our approach. Also, for an optimized choice of poles it might be interesting to consider condensers with the two (non convex) plates  $I_j$  and  $I \setminus I_j$ .

Also, on a long term, we also want to solve extremal problems for positive measures, which in addition requires to consider iterative balayage and  $F$ -functionals. Finally, we hope to be able to generalize some of our findings to the case of a union of curves and arcs instead of intervals.

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## A Some complements on Chebyshev Orthogonal rational functions

### A.1 Proof of Lemma 3.2 and Lemma 3.3

We start by showing the orthonormality as claimed in Lemma 3.2.

*Proof of Lemma 3.2.* With  $w = \phi_j(z)$ , consider the reciprocal conformal map

$$z = \psi_j(w) = \frac{a_j + b_j}{2} + \frac{b_j - a_j}{4} \left( w + \frac{1}{w} \right).$$

Then the change of variable  $x = \psi(w)$  gives for  $n, k \geq 1$

$$\begin{aligned} & \int_{I_j} t_{j,n}(x)t_{j,k}(x)d\omega_j(x) \\ &= \int_{|w|=1, \text{Im}(w)<0} t_{j,n}(\psi_j(w))t_{j,k}(\psi_j(w)) \frac{\psi_j'(w)dw}{\pi \sqrt{(\psi_j(w) - a_j)(b_j - \psi_j(w))}} \\ &= \frac{1}{2\pi i} \int_{|w|=1, \text{Im}(w)<0} [\varphi_{j,n}(w) + \varphi_{j,n}(1/w)] [\varphi_{j,k}(w) + \varphi_{j,k}(1/w)] \frac{dw}{w} \\ &= \frac{1}{2\pi i} \int_{|w|=1} \varphi_{j,n}(w)\varphi_{j,k}(w) \frac{w}{w} + \frac{1}{2\pi i} \int_{|w|=1} \varphi_{j,n}(w)\varphi_{j,k}(1/w) \frac{dw}{w} \\ &= \varphi_{j,n}(0)\varphi_{j,k}(0) + \delta_{j,k} = \delta_{j,k}, \end{aligned}$$

where in the last equality we used the Cauchy formula and the fact that  $w_{j,0} = 0$ , as well as the orthonormality of the  $\varphi_{j,n}$ . In the cases  $k = 0$  and/or  $n = 0$  one has a slightly different normalization, but a proof of orthonormality follows the same lines. Finally, as cited from Achieser by Meinardus [19, Thm??], the density of  $\{t_{j,n} : n = 0, 1, \dots\}$  in  $L^2(\eta_j)$  follows from  $\sum_j(1 - w_j) = \infty$ . A proof for the system  $\{u_{j,n} : n = 0, 1, \dots\}$  is similar, we omit details.  $\square$

Now we present a proof of Lemma 3.3.

*Proof of Lemma 3.3.* We have for  $n \geq 1$  and  $x = \psi(v) \notin I_j$ ,

$$\begin{aligned}
\mathcal{C}_j(t_{j,n})(x) &= \\
&= \int_{|w|=1, \text{Im}(w)<0} \frac{t_{j,n}(\psi_j(w))}{\psi_j(w) - x} \frac{\psi'_j(w)dw}{\pi\sqrt{(b_j - \psi_j(w))}} \\
&= \frac{1}{\sqrt{2}\pi i} \int_{|w|=1, \text{Im}(w)<0} \frac{\varphi_{j,n}(w) + \varphi_{j,n}(1/w)}{\psi_j(w) - x} \frac{dw}{w} \\
&= \frac{1}{\sqrt{2}\pi i} \int_{|w|=1} \frac{\varphi_{j,n}(w)}{\psi_j(w) - x} \frac{dw}{w} \\
&= \frac{4\sqrt{2}}{b_j - a_j} \frac{1}{v - 1/v} \left[ \frac{1}{2\pi i} \int_{|w|=1} \frac{\varphi_{j,n}(w)}{w - v} dw - \frac{1}{2\pi i} \int_{|w|=1} \frac{\varphi_{j,n}(w)}{w - 1/v} dw \right] \\
&= -\frac{4\sqrt{2}}{b_j - a_j} \frac{1}{v - 1/v} \varphi_{j,n}(1/v),
\end{aligned}$$

where in the last equality we used the Cauchy formula and the fact that  $|v| > 1$ . Thus our claim for  $\mathcal{C}_j(t_{j,n})$  for  $n \geq 1$  follows by observing that

$$v - \frac{1}{v} = \frac{4}{b_j - a_j} \sqrt{(x - b_j)(x - a_j)} = \frac{4}{b_j - a_j} \frac{\phi_j(x)}{\phi'_j(x)}.$$

For  $n = 0$ , the above computations remain valid if the result is divided by  $\sqrt{2}$ , as claimed in Lemma 3.3 for the Cauchy transform. The Hilbert transform can be obtained as a limiting process. Let

$$y \in [a_j, b_j], \quad \epsilon > 0, \quad x^+ = y + \epsilon i, \quad x^- = y - \epsilon i, \quad w^\pm = \phi_j(x^\pm).$$

Then for  $\epsilon \rightarrow 0$  we have that  $(\sqrt{(x - b_j)(x - a_j)})^+ = -(\sqrt{(x - b_j)(x - a_j)})^-$  and

$$\begin{aligned}
\mathcal{H}_j(t_{j,n})(y) &= -\frac{1}{\sqrt{2}} \lim_{\epsilon \rightarrow 0} \frac{\varphi_{j,n}(1/\phi_j(x^+)) - \varphi_{j,n}(1/\phi_j(x^-))}{(\sqrt{(x - b_j)(x - a_j)})^+} \\
&= \frac{1}{\sqrt{2}} \frac{\varphi_{j,n}(w) - \varphi_{j,n}(1/w)}{(w - 1/w)^{\frac{b_j - a_j}{4}}} = u_{j,n-1}(y).
\end{aligned}$$

For  $n = 0$  we trivially get  $\mathcal{H}_j(t_{j,n})(y) = \mathcal{H}_j(T_{j,0})(y) = 0$ .  $\square$

## A.2 Three-term recurrences and rational Gauss quadrature rules for Chebyshev ORF

According to [5, p. 261] and [6, Theorem 2.1], ORF  $(p_\ell)_{\ell \geq 0}$  with respect to a measure  $\sigma$  on the real line with  $p_\ell$  having real poles  $z_1, z_2, \dots, z_\ell$  outside the support of  $\sigma$  verify a three-term recurrence relation

$$\alpha_{n+1} \left(1 - \frac{z}{z_{n+1}}\right) p_{n+1}(z) + \alpha_n \left(1 - \frac{z}{z_{n-1}}\right) p_{n-1}(z) + \beta_n \left(1 - \frac{z}{z_n}\right) p_n(z) = z p_n(z)$$

for  $m \geq 0 \geq 0$ , with initial conditions  $p_0(x) = 1/\sqrt{\sigma(\mathbb{R})}$  and  $p_{-1}(x) = 0$ . Though the authors in [6] give explicit formulas for the  $\alpha_n$  and  $\beta_n$  in terms of suitable residuals, we prefer to give a direct proof revealing that our recurrence in fact is based on a similar recurrence for the Takenaka-Malmquist rational functions  $\varphi_{j,n}$ . Using the change of variables  $z = \psi_j(w)$ ,  $z_{j,n} = \psi_j(1/w_{j,n})$ , and shifting the variable  $z$  by  $\frac{a_j+b_j}{2}$ , we observe that  $\frac{2z-a_j-b_j}{b_j-a_j} = \frac{1}{2}(w + \frac{1}{w})$ , and

$$1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n} - \frac{a_j+b_j}{2}} = \frac{(w_{j,n} + \frac{1}{w_{j,n}}) - (w + \frac{1}{w})}{w_{j,n} + \frac{1}{w_{j,n}}} = \frac{(w - w_{j,n})(1 - ww_{j,n})}{w(1 + w_{j,n}^2)}. \quad (\text{A.1})$$

Applied to our Chebyshev ORF we obtain the following generalization of [6, Theorem 3.5] who consider a shifted sequence of poles for the  $u_{j,n}$ .

**Proposition A.1.** *We have for  $n \geq 0$  (which defines  $\varphi_{j,-1}$ )*

$$\begin{aligned} & \alpha_{j,n+1} \frac{(w - w_{j,n+1})(1 - ww_{j,n+1})}{w(1 + w_{j,n+1}^2)} \varphi_{j,n+1}(w) + \beta_{j,n} \frac{(w - w_{j,n})(1 - ww_{j,n})}{w(1 + w_{j,n}^2)} \varphi_{j,n}(w) \\ & + \alpha_{j,n} \frac{(w - w_{j,n-1})(1 - ww_{j,n-1})}{w(1 + w_{j,n-1}^2)} \varphi_{j,n-1}(w) = \frac{1}{2} \left(w + \frac{1}{w}\right) \varphi_{j,n}(w) \end{aligned}$$

with

$$\begin{aligned} \alpha_{j,n} &= \frac{1}{2} \frac{(1 + w_{j,n-1}^2)(1 + w_{j,n}^2)}{\sqrt{(1 - w_{j,n-1}^2)(1 - w_{j,n}^2)}} \frac{1}{1 - w_{j,n-1}w_{j,n}}, \\ \beta_{j,n} &= \frac{1}{2} \frac{1 + w_{j,n}^2}{1 - w_{j,n}^2} \frac{(1 - w_{j,n}^2)(w_{j,n-1} + w_{j,n+1}) + 2w_{j,n}(1 - w_{j,n-1}w_{j,n+1})}{(1 - w_{j,n-1}w_{j,n})(1 - w_{j,n}w_{j,n+1})}. \end{aligned}$$

Also, we have for  $n \geq 1$  with  $u_{j,-1}(z) = 0$

$$\begin{aligned} & \alpha_{j,n+1} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n+1} - \frac{a_j+b_j}{2}}\right) u_{j,n}(z) + \beta_{j,n} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n} - \frac{a_j+b_j}{2}}\right) u_{j,n-1}(z) \\ & + \alpha_{j,n} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n-1} - \frac{a_j+b_j}{2}}\right) u_{j,n-2}(z) = \frac{2z - a_j - b_j}{b_j - a_j} u_{j,n-1}(z). \end{aligned}$$

Finally setting  $\beta'_{j,0} = w_{j,1}$ ,  $\alpha'_{j,0} = 0$ ,  $\alpha'_{j,1} = \sqrt{2}\alpha_{j,1}$  and for  $n \geq 1$  setting  $\beta'_{j,n} = \beta_{j,n}$ ,  $\alpha'_{j,n+1} = \alpha_{j,n+1}$  we have for all  $n \geq 1$ ,

$$\begin{aligned} & \alpha'_{j,n+1} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n+1} - \frac{a_j+b_j}{2}}\right) t_{j,n+1}(z) + \beta'_{j,n} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n} - \frac{a_j+b_j}{2}}\right) t_{j,n}(z) \\ & + \alpha'_{j,n} \left(1 - \frac{z - \frac{a_j+b_j}{2}}{z_{j,n-1} - \frac{a_j+b_j}{2}}\right) t_{j,n-1}(z) = \frac{2z - a_j - b_j}{b_j - a_j} t_{j,n}(z). \end{aligned}$$

*Proof.* Notice that

$$\begin{aligned} & \alpha_{j,n+1} \frac{(w - w_{j,n+1})(1 - ww_{j,n+1})}{w(1 + w_{j,n+1}^2)} \frac{\varphi_{j,n+1}(w)}{\varphi_{j,n}(w)} \\ & = \alpha_{j,n+1} \frac{(w - w_{j,n+1})(1 - ww_{j,n+1})}{w(1 + w_{j,n+1}^2)} \sqrt{\frac{1 - w_{j,n+1}^2}{1 - w_{j,n}^2}} \frac{w - w_{j,n}}{1 - ww_{j,n+1}} \\ & = \frac{1}{2} \frac{1 + w_{j,n}^2}{1 - w_{j,n}^2} \frac{(w - w_{j,n+1})(w - w_{j,n})}{w(1 - w_{j,n}w_{j,n+1})}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \alpha_{j,n} \frac{(w - w_{j,n-1})(1 - ww_{j,n-1})}{w(1 + w_{j,n-1}^2)} \frac{\varphi_{j,n-1}(w)}{\varphi_{j,n}(w)} = \\ & = \alpha_{j,n} \frac{(w - w_{j,n-1})(1 - ww_{j,n-1})}{w(1 + w_{j,n-1}^2)} \sqrt{\frac{1 - w_{j,n-1}^2}{1 - w_{j,n}^2}} \frac{1 - ww_{j,n}}{w - w_{j,n-1}} \\ & = \frac{1}{2} \frac{1 + w_{j,n}^2}{1 - w_{j,n}^2} \frac{(1 - ww_{j,n-1})(1 - ww_{j,n})}{w(1 - w_{j,n-1}w_{j,n})}. \end{aligned}$$

Adding both terms gives with  $c = \frac{1+w_{j,n}^2}{1-w_{j,n}^2} \frac{1-w_{j,n-1}w_{j,n}^2w_{j,n+1}}{(1-w_{j,n-1}w_{j,n})(1-w_{j,n}w_{j,n+1})}$

$$\begin{aligned} h(w) &= \frac{1 + w_{j,n}^2}{2w_{j,n}} + \frac{c}{2} \left( \left(w + \frac{1}{w}\right) - \left(w_{j,n} + \frac{1}{w_{j,n}}\right) \right) \\ &= \frac{1}{2} \left(w + \frac{1}{w}\right) + \frac{c-1}{2} \left( \left(w + \frac{1}{w}\right) - \left(w_{j,n} + \frac{1}{w_{j,n}}\right) \right) \\ &= \frac{1}{2} \left(w + \frac{1}{w}\right) - \beta_{j,n} \frac{(w - w_{j,n})(1 - ww_{j,n})}{w(1 + w_{j,n}^2)}, \end{aligned}$$

with

$$\beta_{j,n} = \frac{c-1}{2} \frac{1 + w_{j,n}^2}{w_{j,n}} = \frac{1}{2} \frac{1 + w_{j,n}^2}{1 - w_{j,n}^2} \frac{(1 - w_{j,n}^2)(w_{j,n-1} + w_{j,n+1}) + 2w_{j,n}(1 - w_{j,n-1}w_{j,n+1})}{(1 - w_{j,n-1}w_{j,n})(1 - w_{j,n}w_{j,n+1})}$$

as claimed above. Notice that the recurrence for the  $\varphi_{j,n}(w)$  is also true for the  $\varphi_{j,n}(1/w)$  and thus for  $\varphi_{j,n}(w) - \varphi_{j,n}(1/w)$  and the  $u_{j,n-1}$  with  $u_{j,-1} = 0$ . The same observation is true for  $\varphi_{j,n}(w) + \varphi_{j,n}(1/w)$  and thus the  $t_{j,n}$ , at but we have to take care that there is a different normalization factor for  $t_{j,0}$  (and  $n \geq 1$ ), leading to a modified factor  $\alpha_{j,1}$ . The coefficient  $\beta'_{j,0}$  has to be computed such that the claimed recurrence is also true for  $n = 0$  with  $T_{j,-1} = 0$ , we omit details.  $\square$

We are now prepared to state the well-known link between Gauss quadrature rules for the measure  $\eta_j$  and eigenelements of a tridiagonal matrix pencil. For simplifying writing, it will be suitable to consider the change of variables

$$y = \frac{2z - a_j - b_j}{b_j - a_j}, \quad y_{j,n} = \frac{2z_{j,n} - a_j - b_j}{b_j - a_j}$$

so that our recurrence for the  $u_{j,n}$  of Proposition A.1 becomes

$$\alpha_{j,n+1} \left(1 - \frac{y}{y_{j,n+1}}\right) u_{j,n}(z) + \beta_{j,n} \left(1 - \frac{y}{y_{j,n}}\right) u_{j,n-1}(z) + \alpha_{j,n} \left(1 - \frac{y}{y_{j,n-1}}\right) u_{j,n-2}(z) = y u_{j,n-1}(z)$$

for all  $n \geq 1$ . In the setting of Proposition 3.4 and thus  $w_{j,N_j^{rat}+1} = 0$  implying that  $z_{j,N_j^{rat}+1} = \infty = y_{j,N_j^{rat}+1}$ , we obtain

$$\left(u_{j,0}(z), \dots, u_{j,N_j^{rat}-1}(z)\right) (yJ_j - H_j) = \left(0, \dots, 0, \alpha_{j,N_j^{rat}+1} u_{j,N_j^{rat}}(z)\right)$$

with tridiagonal matrices

$$H_j = \begin{bmatrix} \beta_{j,1} & \alpha_{j,2} & 0 & \cdots & 0 \\ \alpha_{j,2} & \beta_{j,2} & \alpha_{j,3} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \alpha_{j,N_j^{rat}} \\ 0 & \cdots & 0 & \alpha_{j,N_j^{rat}} & \beta_{j,N_j^{rat}} \end{bmatrix}, \quad J_j = I + \text{diag}\left(\frac{1}{y_{j,1}}, \dots, \frac{1}{y_{j,N_j^{rat}}}\right) H_j,$$

which allows us to conclude as in [7, Theorem 3.1, §4 & §5] that the zeros  $x_{j,h}^{rat}$  of  $u_{j,N_j^{rat}}$  and the eigenvalues  $\lambda_h$  of the symmetric (but in general full) matrix  $H_j J_j^{-1}$  are related through

$$\lambda_h = \frac{2x_{j,h}^{rat} - a_j - b_j}{b_j - a_j}, \quad (\text{A.2})$$

with the matrix of corresponding left eigenvectors given by the orthogonal matrix  $M_j^{rat}$  defined in (3.1).

Similarly, for the rational Gauss quadrature in  $L^2(\omega_j)$ , we have to consider the tridiagonal pencil

$$\left(t_{j,0}(z), \dots, t_{j,N_j^{rat}}(z)\right) (yJ_j - H_j) = \left(0, \dots, 0, \alpha_{j,N_j^{rat}+1} t_{j,N_j^{rat}+1}(z)\right)$$

with tridiagonal matrices

$$H_j = \begin{bmatrix} \beta'_{j,0} & \alpha'_{j,1} & 0 & \cdots & 0 \\ \alpha'_{j,1} & \beta_{j,1} & \alpha_{j,2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \alpha_{j,N_j^{rat}} \\ 0 & \cdots & 0 & \alpha_{j,N_j^{rat}} & \beta_{j,N_j^{rat}} \end{bmatrix}, \quad J_j = I + \text{diag}\left(0, \frac{1}{y_{j,1}}, \dots, \frac{1}{y_{j,N_j^{rat}}}\right) H_j.$$