



HAL
open science

On the positive powers of q -analogs of Euler series

Changgui Zhang

► **To cite this version:**

Changgui Zhang. On the positive powers of q -analogs of Euler series. Contemporary mathematics, 2023, 782, 10.1090/conm/782 . hal-04401440

HAL Id: hal-04401440

<https://hal.science/hal-04401440v1>

Submitted on 17 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the positive powers of q -analogs of Euler series

Changgui ZHANG

ABSTRACT. The most simple and famous divergent power series coming from ODE may be the so-called Euler series $\sum_{n \geq 0} (-1)^n n! x^{n+1}$, that, as well as all its positive powers, is Borel-summable in any direction excepted the negative real half-axis (see [2] or [7]). By considering a family of linear q -difference operators associated with a given first order non-homogenous q -difference equation, it will be shown that the summability order of q -analogous counterparts of Euler series depends upon of the degree of power under consideration.

1. Introduction

It is well-known that, given any direction starting from the origin of the complex plane, the set of all Borel-summable series in this direction forms a differential algebra; see [7, Proposition 1.3.4.2] or [2, Theorem 36]. This is not the same for the summability with respect to the power series solutions of q -difference equations. Indeed, if $q > 1$ and

$$(1.1) \quad \hat{E}_q(x) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} x^n,$$

the divergent power series $\hat{E}_q(x)$ is Gq -summable of only one level but its square $\hat{E}_q(x)^2$ is Gq -summable of two levels; see [11, §4.3.8] and [8, Théorème 2.2.1]. The first purpose of this paper is to obtain a linear q -difference equation for every positive power of the power series $\hat{E}_q(x)$. This will allow one to see the different level of summability for each of these corresponding power series.

In what follows, let q be a given nonzero complex number, and let σ_q be the associated q -difference operator defined by $\sigma_q f(x) = f(qx)$. By direct computation, one finds the series defined by (1.1) satisfies term by term the following linear q -difference equation :

$$(1.2) \quad (x\sigma_q + 1)y = 1.$$

Now, write (1.2) into the form $x\sigma_q y = 1 - y$ and then square both sides. As $(\sigma_q y)^2 = \sigma_q(y^2)$, one obtains that $x^2\sigma_q(y^2) - y^2 = 1 - 2y$. By using again (1.2), one gets the following equation:

$$(x\sigma_q + 1)(x^2\sigma_q - 1)(y^2) = x + 1 - 2(x\sigma_q + 1)y,$$

1991 *Mathematics Subject Classification*. Primary : 39B32 ; 40G10 ; 30E15 .

This work was partially supported by Labex CEMPI (Centre Européen pour les Mathématiques, la Physique et leurs Interaction).

what implies that

$$(1.3) \quad (x\sigma_q + 1)(x^2\sigma_q - 1)\hat{E}_q(x)^2 = x - 1.$$

We shall explain how to obtain a linear q -difference equation for $\hat{E}_q(x)^n$. For doing that, we shall start by recalling some non-commutative rings of q -difference operators and then arrive at a n -th order such operator associated with any given n -th power of a power series solution of some first order non-homogenous q -difference equation. See Theorem 2.1 in § 2 and its generalisation stated in Theorem 4.3 in § 4. Section 3 is reserved to the proofs of Theorem 2.1 and two related Lemmas. In the last section, we shall apply, respectively, both Theorems 2.1 and 4.3 to the above-mentioned power series $\hat{E}_q(x)$ and another q -analog $\hat{E}(x; q)$ of the classic Euler series. At the end of the paper will be outlined some results about the summability of both $\hat{E}_q(x)$ and $\hat{E}(x; q)$ when $q > 1$; see Theorem 5.3.

Throughout the present paper, we shall limit ourself to the q -summation method studied in our previous works [11] and [8], by means of a q -Laplace integral involving the multivalued function $e^{\log^2 x/2 \ln q}$. There exists another summation, using a Jacobi theta function; see [10]. Comparing both summations can be seen as part of Stokes analysis, and this would be interesting to find identities on the associated special functions.

2. Notation and statements

As usual, we will denote by $\mathbb{C}\{x\}$ or $\mathbb{C}[[x]]$ the integral domain constituting of the germs of holomorphic functions at $x = 0$ in the complex plane or that of the power series of indeterminate x , respectively. Their respective corresponding field of fractions will be denoted by $\mathbb{C}[x^{-1}]\{x\}$ or $\mathbb{C}[x^{-1}][[x]]$. One has $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]$, as well as $\mathbb{C}[x^{-1}]\{x\} \subset \mathbb{C}[x^{-1}][[x]]$, in a natural way via the classic Taylor or Laurent series.

Let R be one of the above-mentioned \mathbb{C} -vector spaces $\mathbb{C}\{x\}$, $\mathbb{C}[[x]]$, $\mathbb{C}[x^{-1}]\{x\}$ and $\mathbb{C}[x^{-1}][[x]]$. By replacing x with qx , the q -difference operator σ_q acts as an automorphism over R . Let $\sigma_q^k = \sigma_{q^k}$ for any integer k ; in particular, σ_q^0 is simply the identity map on R . We will denote by $R[\sigma_q]$ the set of the (linear) q -difference operators whose coefficients belong to R , *i.e.* $L \in R[\sigma_q]$ if $L = \sum_{k=0}^n a_k \sigma_q^k$, where $a_k \in R$. By identifying each element of R with the corresponding multiplication map in R , the set $R[\sigma_q]$ constitutes itself a non-commutative ring for the composition of operators. Given $L_1, L_2 \in R[\sigma_q]$, we will write $L_1 L_2$ in stead of $L_1 \circ L_2$.

Furthermore, we will denote by $\nu_0(f) \in \mathbb{Z} \cup \{\infty\}$ the valuation of any given element f of the field $\mathbb{C}[x^{-1}][[x]]$ "at $x = 0$ ". This means that (i) $\nu_0(f) = \infty$ iff f is the identically vanishing series, and (ii) $\nu_0(f) = \nu \in \mathbb{Z}$ iff $f \equiv cx^\nu \pmod{x^{\nu+1} \mathbb{C}[[x]]}$ for some nonzero complex number c .

Throughout the whole section, P will be some given nonzero power series belonging to the field $\mathbb{C}[x^{-1}][[x]]$. Define the associated family of power series $\{P_n\}_{n \geq 0}$ as follows: $P_0 = 0$, $P_1 = P$, and

$$(2.1) \quad P_n = \sum_{k=0}^{n-1} (-x\sigma_q)^k P \quad \text{for } n \geq 2.$$

By noticing that $(-x\sigma_q)^k P = (-x)^k q^{k(k-1)/2} \sigma_q^k P$, it follows from (2.1) that, for any pair $(n, \ell) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{> 0}$,

$$\begin{aligned} P_{n+\ell} - P_n &= \sum_{k=n}^{n+\ell-1} (-x)^k q^{k(k-1)/2} \sigma_q^k P \\ &\equiv (-1)^n c q^{n\nu+n(n-1)/2} x^{\nu+n} \pmod{x^{\nu+n+1} \mathbb{C}[[x]]}, \end{aligned}$$

where $c \neq 0$ and $\nu = \nu_0(P) \in \mathbb{Z}$. This implies that

$$(2.2) \quad P_m \neq P_n \quad \text{if} \quad m \neq n.$$

Given any positive integer $n \geq 1$, let $\{L_{n,k}^P\}_{1 \leq k \leq n} \subset (\mathbb{C}[x^{-1}][[x]])[\sigma_q]$ be the associated family of q -difference operators defined in the following manner:

$$(2.3) \quad L_{n,1}^P = \frac{1}{P_1} (x^n \sigma_q - (-1)^n) = \frac{1}{P} (x^n \sigma_q - (-1)^n);$$

$$(2.4) \quad L_{n,k+1}^P = \frac{1}{P_{k+1}} (x^{n-k} \sigma_q - (-1)^{n-k}) L_{n,k}^P, \quad 1 \leq k < n.$$

Specifically, letting $k = n$, one gets the following form of the n -th order q -difference operator $L_{n,n}^P \in (\mathbb{C}[x^{-1}][[x]])[\sigma_q]$:

$$(2.5) \quad L_{n,n}^P = \frac{1}{P_n} (x\sigma_q + 1) \frac{1}{P_{n-1}} (x^2\sigma_q - 1) \cdots \frac{1}{P_1} (x^n \sigma_q - (-1)^n).$$

THEOREM 2.1. *Let n be an integer ≥ 1 , let $P, f \in \mathbb{C}[x^{-1}][[x]]$ with $P \neq 0$, and suppose that the following q -difference equation is satisfied:*

$$(2.6) \quad x\sigma_q f + f = P.$$

Then:

$$(2.7) \quad L_{n,n}^P(f^n) = (-1)^{n(n-1)/2},$$

where $L_{n,n}^P$ is the q -difference operator defined by (2.5).

Theorem 2.1 will be proved with the help of the following lemmas.

LEMMA 2.2. *Let $(n, k) \in \mathbb{Z}^2$ such that $1 \leq k \leq n$, and let P, f be as in Theorem 2.1. For $0 \leq j \leq k$, define $A_{k;j} \in \mathbb{C}[x^{-1}][[x]]$ by*

$$(2.8) \quad A_{k;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (P_j - P_\ell).$$

One has:

$$(2.9) \quad L_{n,k}^P(f^n) = (-1)^{k(2n-k+1)/2} \sum_{j=0}^k \frac{1}{A_{k;j}} (f - P_j)^n.$$

LEMMA 2.3. *Let n be a nonnegative integer, \mathfrak{C} an extension field of \mathbb{C} , and let $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathfrak{C}^{n+1}$ such that $\alpha_j \neq \alpha_\ell$ for $j \neq \ell$. Define the $(n+1)$ -tuple $(a_0, a_1, \dots, a_n) \in \mathfrak{C}^{n+1}$ as follows:*

$$a_j = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (\alpha_j - \alpha_\ell)$$

($a_0 = 1$ if $n = 0$). The following identity holds in the ring $\mathfrak{C}[T]$ of the polynomial functions over \mathfrak{C} :

$$(2.10) \quad \sum_{j=0}^n \frac{1}{a_j} (T - \alpha_j)^n = (-1)^n .$$

3. Proofs

Proof of Lemma 2.2. To simplify the presentation, we will write $L_{n,k}$ instead of $L_{n,k}^P$, and set $\epsilon_{n,k} = (-1)^{k(2n-k+1)/2}$. One has:

$$(3.1) \quad \epsilon_{n;k+1} = (-1)^{n-k} \epsilon_{n,k} .$$

Let us proceed by induction on k . For $k = 1$, since $P_1 = P$, one has:

$$\epsilon_{n,1} = (-1)^n; \quad A_{1;0} = -P, \quad A_{1;1} = P .$$

Thus, (2.3) implies that

$$L_{n,1}(f^n) = \frac{1}{P} (x^n \sigma_q(f^n) - (-1)^n f^n) .$$

At the same time, by considering (2.6), one gets that

$$x^n \sigma_q(f^n) = (x \sigma_q f)^n = (P - f)^n ;$$

In this way, it follows that

$$L_{n,1}(f^n) = \frac{\epsilon_{n,1}}{P} ((f - P)^n - f^n) ,$$

that is exactly what wanted in (2.9) for $k = 1$.

Now, suppose that equality (2.9) holds for some integer k between 1 and $n - 1$. By using relation (2.4), one can express $L_{n,k+1}(f^n)$ as follows:

$$(3.2) \quad L_{n,k+1}(f^n) = \frac{\epsilon_{n,k}}{P_{k+1}} \sum_{j=0}^k (B_{n,k;j} - C_{n,k;j}) ,$$

where

$$(3.3) \quad B_{n,k;j} = x^{n-k} \sigma_q \left(\frac{1}{A_{k;j}} (f - P_j)^n \right)$$

and

$$(3.4) \quad C_{n,k;j} = \frac{(-1)^{n-k}}{A_{k;j}} (f - P_j)^n .$$

In view of (2.1), one finds that $x \sigma_q P_j = P - P_{j+1}$, hence:

$$x \sigma_q (f - P_j) = x \sigma_q f - x \sigma_q P_j = (P - f) - (P - P_{j+1}) = -(f - P_{j+1}) .$$

Thus, it follows that

$$B_{n,k;j} = \frac{(x \sigma_q (f - P_j))^n}{x^k \sigma_q A_{k;j}} = \frac{(-1)^n}{x^k \sigma_q A_{k;j}} (f - P_{j+1})^n .$$

By using the expression of $A_{k;j}$ given in (2.8), one has :

$$x^k \sigma_q A_{k;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (x \sigma_q (P_j - P_\ell)) = (-1)^k \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (P_{j+1} - P_{\ell+1}) .$$

This is to say that

$$x^k \sigma_q A_{k;j} = \frac{(-1)^k}{P_{j+1}} A_{k+1,j+1}.$$

So, one can write (3.3) as follows:

$$(3.5) \quad B_{n,k;j} = \frac{(-1)^{n-k} P_{j+1}}{A_{k+1;j+1}} (f - P_{j+1})^n.$$

Rewrite (3.2) into the following form:

$$(3.6) \quad L_{n,k+1}(f^n) = \frac{\epsilon_{n,k}}{P_{k+1}} \left(\sum_{j=0}^{k-1} (B_{n,k;j} - C_{n,k;j+1}) + B_{n,k;k} - C_{n,k;0} \right).$$

Let $j < k$; one has:

$$A_{k+1,j+1} = (P_{j+1} - P_{k+1}) A_{k,j+1}.$$

Replace j by $j + 1$ in (3.4), and make use of (3.5). One gets that

$$B_{n,k;j} - C_{n,k;j+1} = \frac{(-1)^{n-k} P_{k+1}}{A_{k+1;j+1}} (f - P_{j+1})^n.$$

Furthermore, one sees that $A_{k+1;0} = -P_{k+1} A_{k;0}$. By letting $j = k$ and $j = 0$ in (3.5) and (3.4) respectively, one has:

$$B_{n,k;k} = \frac{(-1)^{n-k} P_{k+1}}{A_{k+1;k+1}} (f - P_{k+1})^n$$

and

$$C_{n,k;0} = -\frac{(-1)^{n-k} P_{k+1}}{A_{k+1;0}} (f - P_0)^n.$$

Thus, in view of (3.1), equality (3.6) implies that

$$L_{n,k+1}(f^n) = \epsilon_{n,k+1} \sum_{j=0}^{k+1} \frac{1}{A_{k+1;j}} (f - P_j)^n,$$

which corresponds to (2.9) in which k was replaced by $k + 1$. \square

Proof of Lemma 2.3. We give here two proofs.

Proof by induction - If $n = 0$, as $a_0 = 1$, equality (2.10) becomes evident. Suppose that (2.10) holds for some index $n \geq 0$, and consider an $(n + 2)$ -uplet $(\alpha_0, \dots, \alpha_{n+1})$ where, as before, $\alpha_j \neq \alpha_\ell$ if $j \neq \ell$. Set

$$\mathcal{P}(T) = \sum_{j=0}^{n+1} \frac{1}{\tilde{a}_j} (T - \alpha_j)^{n+1}, \quad \tilde{a}_j = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n+1}} (\alpha_j - \alpha_\ell).$$

Let k be an integer such that $0 \leq k \leq n + 1$. If one writes

$$a_{k;j} = \prod_{\substack{\ell \neq k, \ell \neq j \\ 0 \leq \ell \leq n+1}} (\alpha_j - \alpha_\ell),$$

one has $a_{k;j} = \tilde{a}_j / (\alpha_j - \alpha_k)$ for $j \neq k$, what implies that

$$(3.7) \quad \mathcal{P}(\alpha_k) = - \sum_{\substack{j \neq k \\ 0 \leq j \leq n+1}} \frac{1}{a_{k;j}} (\alpha_k - \alpha_j)^n.$$

By applying the induction hypothesis to the $(n+1)$ -uplet $(\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_{n+1})$ appeared in the right hand side of (3.7), one obtains that $\mathcal{P}(\alpha_k) = (-1)^{n+1}$. As \mathcal{P} is a polynomial with $\deg \mathcal{P} \leq n+1$, one finds that $\mathcal{P}(T) = (-1)^{n+1}$ identically.

*Proof by Lagrange polynomials*¹ – Let $\mathbb{K} = \mathfrak{C}(T)$ be the field of the rational functions over \mathfrak{C} . Let X be a new indeterminate, and set

$$F(X) = (T - X)^n, \quad \Lambda_j(X) = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (X - \alpha_\ell).$$

By noticing that $F(X) \in \mathbb{K}[X]$ be such that $\deg F = n < n+1$, the Lagrange interpolation formula [3, Chap. IV, §2, p. 26] implies that

$$F(X) = \sum_{j=0}^n \frac{F(\alpha_j)}{a_j} \Lambda_j(X).$$

And now one compares the coefficients of X^n for both sides of the above equality. As that of Λ_j is equal to 1, one finds that

$$(-1)^n = \sum_{j=0}^n \frac{F(\alpha_j)}{a_j} = \sum_{j=0}^n \frac{(T - \alpha_j)^n}{a_j},$$

which is exactly the expected identity (2.10). \square

End of the proof of Theorem 2.1. In view of Lemma 2.2, letting $k = n$ in (2.9) yields that

$$L_{n,n}^P(f^n) = (-1)^{n(n+1)/2} \sum_{j=0}^n \frac{1}{A_{n;j}} (f - P_j)^n,$$

where

$$A_{n;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (P_j - P_\ell).$$

By (2.2), one knows that $P_j \neq P_\ell$ for $j \neq \ell$. Thus, by applying Lemma 2.3 with $\mathfrak{C} = \mathbb{C}[x^{-1}][[x]]$, $X = f$ and $\alpha_j = P_j$, one gets that

$$\sum_{j=0}^n \frac{1}{A_{n;j}} (f - P_j)^n = (-1)^n.$$

This permits to finish the proof of Theorem 2.1. \square

4. One generalisation of Theorem 2.1

Instead of (2.6), let us consider the following slightly more general q -difference equation:

$$(4.1) \quad \alpha \sigma_q y + y = \beta,$$

where both α and β are nonzero power series belonging to the field $\mathbb{C}[x^{-1}][[x]]$. In the same spirit as in (2.1), define the sequence $\{\beta_n\}_{n \geq 0}$ in $\mathbb{C}[x^{-1}][[x]]$ as follows:

$$(4.2) \quad \beta_0 = 0; \quad \beta_n = \sum_{k=0}^{n-1} (-\alpha \sigma_q)^k \beta, \quad n \geq 1.$$

¹We would like to thank our friend and collaborator J. Sauloy for this elegant proof.

PROPOSITION 4.1. *For any given positive integer n , the following conditions are equivalent.*

- (1) $\beta_j \neq 0$ for any positive integer j such that $0 < j \leq n$.
- (2) $\beta_j \neq \beta_\ell$ for any couple of integers (j, ℓ) such that $0 \leq \ell < j \leq n$.

PROOF. Indeed, by (4.2), it follows that

$$(4.3) \quad \beta_j - \beta_\ell = \sum_{k=\ell}^{j-1} (-\alpha \sigma_q)^k \beta = (-\alpha \sigma_q)^\ell \beta_{j-\ell}.$$

As $\alpha \neq 0$, the operator $(-\alpha \sigma_q)$ is an automorphism on the \mathbb{C} -vector space $\mathbb{C}[x^{-1}][[x]]$. So, this is the same for its ℓ -th power or iteration $(-\alpha \sigma_q)^\ell$. This implies the equivalence between the conditions stated in Proposition 4.1. \square

Given a nonzero $f \in \mathbb{C}[x^{-1}][[x]]$, one remembers that $\nu_0(f)$ denotes the valuation of f at $x = 0$, that is the lowest degree of the terms of f . By using (4.2), one obtains that, for $n > 0$:

$$\nu_0(\beta_n) = \begin{cases} \nu_0(\beta) & \text{if } \nu_0(\alpha) > 0; \\ (n-1)\nu_0(\alpha) + \nu_0(\beta) & \text{if } \nu_0(\alpha) < 0. \end{cases}$$

This gives the following statement for $\alpha, \beta \in \mathbb{C}[x^{-1}][[x]] \setminus \{0\}$.

REMARK 4.2. Both conditions considered in Proposition 4.1 are necessarily fulfilled if $\nu_0(\alpha) \neq 0$.

THEOREM 4.3. *Let $n \in \mathbb{Z}_{>0}$, and let α, β and $f \in \mathbb{C}[x^{-1}][[x]]$. Suppose that f satisfies the q -difference equation in (4.1) and that $\beta_j \neq 0$ for any positive integer j such that $0 < j \leq n$. One has:*

$$(4.4) \quad L_n^{\alpha, \beta}(f^n) = (-1)^{n(n-1)/2},$$

where $L_n^{\alpha, \beta}$ is the n -th order q -difference operator defined by

$$(4.5) \quad L_n^{\alpha, \beta} = \frac{1}{\beta_n} (\alpha \sigma_q + 1) \frac{1}{\beta_{n-1}} (\alpha^2 \sigma_q - 1) \cdots \frac{1}{\beta_1} (\alpha^n \sigma_q - (-1)^n).$$

PROOF. Replace (k, P) with (n, β) in (2.8), and define:

$$A_{n;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (\beta_j - \beta_\ell), \quad 0 \leq j \leq n.$$

By taking into account Proposition 4.1, it follows that $A_{n;j} \neq 0$. Thus, one might proceed in the same way as for the proof of Theorem 2.1. We omit the details. \square

5. About the summability of the powers of q -Euler series

Let us come back to the power series $\hat{E}_q(x)$ defined by (1.1), which satisfies the q -difference equation stated in (1.2). Letting $P = 1$, Theorem 2.1 implies immediately the following result.

REMARK 5.1. Given any integer $n \geq 2$, the n -th power $\hat{E}_q(x)^n$ satisfies the following identity:

$$(5.1) \quad (x\sigma_q + 1) \frac{1}{P_{n-1}} (x^2\sigma_q - 1) \cdots (x^n\sigma_q - (-1)^n) \hat{E}_q(x)^n = (-1)^{n(n-1)/2} P_n,$$

where $P_k = 1 - x + \cdots + q^{(k-1)(k-2)}(-x)^{k-1}$ for $1 \leq k \leq n$.

In particular, when $n = 2$, as $P_2 = 1 - x$, (5.1) takes the form of (1.3).

Furthermore, it might be interesting to notice that, if $q = 1$, one has $\hat{E}_1(x) = \frac{1}{1+x}$ and, in this case, the above identity in (5.1) is simply equivalent to the following elementary relation:

$$\prod_{k=1}^n \frac{x^k - (-1)^k}{\sum_{j=0}^{k-1} (-x)^j} = (-1)^{n(n-1)/2} (x+1)^n.$$

This observation related with the specific case of $q = 1$ may be also made for Theorem 4.3, by assuming, for example, both α and β to belong to the field $\mathbb{C}[x^{-1}]\{x\}$ of the germs of meromorphic functions at $x = 0$ in \mathbb{C} or to the sub-field $\mathbb{C}(x)$ of the rational functions over \mathbb{C} .

Our next remark goes to another q -analog of the following so-called Euler series:

$$(5.2) \quad \hat{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}.$$

This power series is divergent for all $x \in \mathbb{C} \setminus \{0\}$ but Borel-summable in every direction excepted \mathbb{R}^- . It satisfies the following first order ODE:

$$(5.3) \quad (x\delta + 1)y = x, \quad \delta = x\partial_x = x \frac{d}{dx}.$$

Letting $Y = \hat{E}(x)^2$, one can check that

$$(5.4) \quad (x\delta + 1 - x)(x\delta + 2)Y = 2x^2.$$

Indeed, if $y = \hat{E}(x)$, one deduces from (5.3) that $x\delta(y^2) = 2xy\delta y = 2y(x-y) = 2xy - 2y^2$. This means that $(x\delta + 2)(y^2) = 2xy$. Thus, applying again (5.3) yields that $(x\delta + 1)\frac{1}{2x}(x\delta + 2)y^2 = x$. In this way, one gets (5.4), using the identity $(x\delta + 1)\frac{1}{2x} = \frac{1}{2x}(x\delta + 1 - x)$ in the non-commutative ring $\mathbb{C}(x)[\delta]$ of the differential operators over $\mathbb{C}(x)$.

Besides, the correspondances $n \leftrightarrow \frac{1-q^n}{1-q}$ and $n! \leftrightarrow \frac{(1-q)\dots(1-q^n)}{(1-q)^n}$ suggest one to consider the following q -analog of the Euler series:

$$(5.5) \quad \hat{E}(x; q) = x + \sum_{n \geq 1} (-1)^n \frac{(1-q)\dots(1-q^n)}{(1-q)^n} x^{n+1}.$$

This can be written in term of a basic hypergeometric series as follows:

$$\hat{E}(x; q) = x {}_2\phi_1(q, q; 0; q, -\frac{x}{1-q}) = x {}_2\phi_0(q^{-1}, q^{-1}; -; q^{-1}, \frac{x}{1-q^{-1}}).$$

See [6, (1.2.22), p. 4] for the general definition of ${}_r\phi_s(\dots; \dots; q, z)$.

Let $\delta_q = \frac{\sigma_q - 1}{q-1} = x\Delta_q$, where Δ_q is defined in [1, p. 488, (10.2.3)]. By observing that $n = \frac{\delta x^n}{x^n}$ and $\frac{1-q^n}{1-q} = \frac{\delta_q x^n}{x^n}$, a direct computation shows that $\hat{E}(x; q)$ satisfies the q -analog of (5.3) as follows: $(x\delta_q + 1)y = x$. If one writes

$$(5.6) \quad \alpha = \frac{x}{q-1-x}, \quad \beta = (q-1)\alpha,$$

it follows from the above that

$$(5.7) \quad (\alpha\sigma_q + 1)\hat{E}(x; q) = \beta.$$

REMARK 5.2. Let α be as in (5.6). The following identity holds in the field $\mathbb{C}[x^{-1}][[x]]$:

$$(5.8) \quad \left(\alpha \delta_q + \frac{1}{q-1-x} \right) \left(\alpha \delta_q - \frac{1}{x} + \frac{1}{q-1-x} \right) \hat{E}(x; q)^2 = \alpha (\sigma_q \alpha - 1).$$

Moreover, when $q \rightarrow 1$, (5.8) is reduced into the following equivalent form of (5.4):

$$(5.9) \quad \left(\delta + \frac{1}{x} \right) \left(\delta + \frac{2}{x} \right) \hat{E}(x)^2 = 2.$$

To obtain (5.8), one can apply (4.2) for $n = 1$ and $n = 2$, where α and β are defined by (5.6). This gives that

$$\beta_1 = (q-1)\alpha, \quad \beta_2 = \beta - \alpha \sigma_q \beta = (q-1)\alpha (1 - \sigma_q \alpha).$$

Thus, it follows from applying Theorem 4.3 to (5.7) that $\hat{E}(x; q)^2$ satisfies the following q -difference equation:

$$(5.10) \quad L \left(\hat{E}(x; q)^2 \right) = \alpha (\sigma_q \alpha - 1),$$

where

$$(5.11) \quad L = \frac{1}{(q-1)^2} (\alpha \sigma_q + 1) \left(\alpha \sigma_q - \frac{1}{\alpha} \right).$$

By replacing σ_q with $(q-1)\delta_q + 1$ in (5.11), one gets that

$$L = \left(\alpha \delta_q + \frac{1+\alpha}{q-1} \right) \left(\alpha \delta_q + \frac{\alpha^2 - 1}{(q-1)\alpha} \right).$$

As $\alpha + 1 = \frac{q-1}{q-1-x}$ and $\alpha^2 - 1 = \frac{(q-1)(2x-q+1)}{(q-1-x)^2}$, one deduces immediately (5.8) from (5.10). The limit equation form given in (5.9) can be obtained from (5.8) by noticing both limits $\alpha \rightarrow -1$ and $\delta_q \rightarrow \delta$ for $q \rightarrow 1$.

In the rest of this paper, we will suppose that $q > 1$ even if, in most cases, the hypothesis $|q| > 1$ may be really enough; see [9] and [10]. Let us recall some results about the summability of power series solutions of q -difference equations. Given any linear q -difference operator L of the following form:

$$L = \sum_{j=0}^n a_j \sigma_q^j \in \mathbb{C}\{x\}[\sigma_q], \quad a_0 a_n \neq 0,$$

one defines its associated Newton polygon $\mathcal{NP}(L)$ as being the convex hull of the set $\{(j, m) : m \geq \nu_0(a_j)\}$ in the strip $[0, n] \times [0, +\infty)$. By following [8, (3.1.2)] and [11, Proposition 5.14], one knows that, if $\mathcal{NP}(L)$ admits only integer slopes, saying $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$, then there exist $\nu \in \mathbb{Z}$, $(h_1, h_2, \dots, h_n) \in (1 + x\mathbb{C}\{x\})^n$ and $(c_0, c_1, \dots, c_n) \in (\mathbb{C} \setminus \{0\})^{n+1}$ such that

$$(5.12) \quad L = c_0 x^\nu h_1 (x^{\kappa_1} \sigma_q + c_1) h_2 (x^{\kappa_2} \sigma_q + c_2) \dots h_n (x^{\kappa_n} \sigma_q + c_n).$$

Furthermore, let $K^+ = \mathbb{Z}_{>0} \cap \{\kappa_j : 1 \leq j \leq n\}$, and define \vec{k} as follows:

$$(5.13) \quad \vec{k} = \begin{cases} \emptyset & \text{if } K^+ = \emptyset \\ (k_1, \dots, k_m) & \text{if } K^+ = \{\kappa_\ell : 1 \leq \ell \leq m\} \text{ and } k_1 < \dots < k_m. \end{cases}$$

By considering [8, § 3.3.5], it follows that any power series $\hat{f} \in \mathbb{C}[[x]]$ such that $L(\hat{f}) \in \mathbb{C}\{x\}$ remains convergent or is Gq -summable of order $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}_{>0}^n$ depending on whether $\vec{k} = \emptyset$ or not.

THEOREM 5.3. *Given any integer $n \geq 2$, the n -th power $\hat{E}_q(x)^n$ is Gq -summable of order $(1, 2, \dots, n)$. And this is the same for $\hat{E}(x; q)^n$.*

PROOF. This follows from taking into account (5.1) or applying Theorem 4.3 to (5.7) with the help of Remark 4.2. \square

Applying [2, Theorem 36] or [7, Proposition 1.3.4.2] implies that every power $\hat{E}(x)^n$ of the Euler series is Borel-summable at the same level, contrarily to what happens in the case of their respective q -analog $\hat{E}(x; q)^n$; see Theorem 5.3. Furthermore, thanks to [4, Theorem 3.15], one knows that the Gq -sum of $\hat{E}(x; q)$ tends toward the Borel-sum of $\hat{E}(x)$ when $q \rightarrow 1^+$. In our coming work [5], it would be shown that, given one generic direction d and two power series \hat{f}_1 and \hat{f}_2 such that $L_j(\hat{f}_j) \in \mathbb{C}\{x\}$ for some $L_j \in \mathbb{C}\{x\}[\sigma_q]$, where $j = 1$ or 2 , the Gq -sum of $\hat{f}_1 \hat{f}_2$ along d is equal to the product of the Gq -sum of \hat{f}_1 along d with that of \hat{f}_2 . In this way, one would obtain that the Gq -sum of $\hat{E}_q(x)^n$ or $\hat{E}(x; q)^n$ can be really expressed as the n -th power of that of $\hat{E}_q(x)$ or $\hat{E}(x; q)$, respectively.

References

1. G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, volume 71, Cambridge University Press, 1999.
2. W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext. Springer-Verlag, New York, 2000.
3. N. Bourbaki, *Éléments de mathématique XI, Première partie, Livre II, Chapitres IV et V*, Paris Hermann, 1950.
4. L. Di Vizio and C. Zhang, On q -summation and confluence, *Ann. Inst. Fourier* 59 (2009), 347-392.
5. T. Dreyfus and C. Zhang, A q -summation process that is a morphism of difference ring, work in progress, 2021.
6. G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, 96 (2nd ed.), Cambridge University Press, 2004.
7. B. Malgrange, Sommatation des séries divergentes, *Exposition. Math.* **13** (1995), no. 2-3, 163-222.
8. F. Marotte and C. Zhang, Multisommabilité des séries entières solutions formelles d'une équation aux q -différences linéaire analytique, *Ann. Inst. Fourier* 50 (2000), 1859-1890.
9. J. Roques, On classical irregular q -difference equations, *Compositio Mathematica*, 148 (2012), 1624-1644.
10. J.-P. Ramis, J. Sauloy and C. Zhang, Local analytic classification of q -difference equations, *Astérisque* 355 (2013).
11. C. Zhang, Développements asymptotiques q -Gevrey et séries Gq -sommables, *Ann. Inst. Fourier* 49 (1999), 227-261.

LABORATOIRE P. PAINLEVÉ (UMR – CNRS 8524), DÉPARTEMENT MATH., FST, UNIVERSITÉ DE LILLE, CITÉ SCIENTIFIQUE, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

Email address: changgui.zhang@univ-lille.fr