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# On the positive powers of $q$ -analogs of Euler series

Changgui ZHANG

ABSTRACT. The most simple and famous divergent power series coming from ODE may be the so-called Euler series  $\sum_{n \geq 0} (-1)^n n! x^{n+1}$ , that, as well as all its positive powers, is Borel-summable in any direction excepted the negative real half-axis (see [2] or [7]). By considering a family of linear  $q$ -difference operators associated with a given first order non-homogenous  $q$ -difference equation, it will be shown that the summability order of  $q$ -analogous counterparts of Euler series depends upon of the degree of power under consideration.

## 1. Introduction

It is well-known that, given any direction starting from the origin of the complex plane, the set of all Borel-summable series in this direction forms a differential algebra; see [7, Proposition 1.3.4.2] or [2, Theorem 36]. This is not the same for the summability with respect to the power series solutions of  $q$ -difference equations. Indeed, if  $q > 1$  and

$$(1.1) \quad \hat{E}_q(x) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} x^n,$$

the divergent power series  $\hat{E}_q(x)$  is  $Gq$ -summable of only one level but its square  $\hat{E}_q(x)^2$  is  $Gq$ -summable of two levels; see [11, §4.3.8] and [8, Théorème 2.2.1]. The first purpose of this paper is to obtain a linear  $q$ -difference equation for every positive power of the power series  $\hat{E}_q(x)$ . This will allow one to see the different level of summability for each of these corresponding power series.

In what follows, let  $q$  be a given nonzero complex number, and let  $\sigma_q$  be the associated  $q$ -difference operator defined by  $\sigma_q f(x) = f(qx)$ . By direct computation, one finds the series defined by (1.1) satisfies term by term the following linear  $q$ -difference equation :

$$(1.2) \quad (x\sigma_q + 1)y = 1.$$

Now, write (1.2) into the form  $x\sigma_q y = 1 - y$  and then square both sides. As  $(\sigma_q y)^2 = \sigma_q(y^2)$ , one obtains that  $x^2\sigma_q(y^2) - y^2 = 1 - 2y$ . By using again (1.2), one gets the following equation:

$$(x\sigma_q + 1)(x^2\sigma_q - 1)(y^2) = x + 1 - 2(x\sigma_q + 1)y,$$

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what implies that

$$(1.3) \quad (x\sigma_q + 1)(x^2\sigma_q - 1)\hat{E}_q(x)^2 = x - 1.$$

We shall explain how to obtain a linear  $q$ -difference equation for  $\hat{E}_q(x)^n$ . For doing that, we shall start by recalling some non-commutative rings of  $q$ -difference operators and then arrive at a  $n$ -th order such operator associated with any given  $n$ -th power of a power series solution of some first order non-homogenous  $q$ -difference equation. See Theorem 2.1 in § 2 and its generalisation stated in Theorem 4.3 in § 4. Section 3 is reserved to the proofs of Theorem 2.1 and two related Lemmas. In the last section, we shall apply, respectively, both Theorems 2.1 and 4.3 to the above-mentioned power series  $\hat{E}_q(x)$  and another  $q$ -analog  $\hat{E}(x; q)$  of the classic Euler series. At the end of the paper will be outlined some results about the summability of both  $\hat{E}_q(x)$  and  $\hat{E}(x; q)$  when  $q > 1$ ; see Theorem 5.3.

Throughout the present paper, we shall limit ourself to the  $q$ -summation method studied in our previous works [11] and [8], by means of a  $q$ -Laplace integral involving the multivalued function  $e^{\log^2 x/2 \ln q}$ . There exists another summation, using a Jacobi theta function; see [10]. Comparing both summations can be seen as part of Stokes analysis, and this would be interesting to find identities on the associated special functions.

## 2. Notation and statements

As usual, we will denote by  $\mathbb{C}\{x\}$  or  $\mathbb{C}[[x]]$  the integral domain constituting of the germs of holomorphic functions at  $x = 0$  in the complex plane or that of the power series of indeterminate  $x$ , respectively. Their respective corresponding field of fractions will be denoted by  $\mathbb{C}[x^{-1}]\{x\}$  or  $\mathbb{C}[x^{-1}][[x]]$ . One has  $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]$ , as well as  $\mathbb{C}[x^{-1}]\{x\} \subset \mathbb{C}[x^{-1}][[x]]$ , in a natural way via the classic Taylor or Laurent series.

Let  $R$  be one of the above-mentioned  $\mathbb{C}$ -vector spaces  $\mathbb{C}\{x\}$ ,  $\mathbb{C}[[x]]$ ,  $\mathbb{C}[x^{-1}]\{x\}$  and  $\mathbb{C}[x^{-1}][[x]]$ . By replacing  $x$  with  $qx$ , the  $q$ -difference operator  $\sigma_q$  acts as an automorphism over  $R$ . Let  $\sigma_q^k = \sigma_{q^k}$  for any integer  $k$ ; in particular,  $\sigma_q^0$  is simply the identity map on  $R$ . We will denote by  $R[\sigma_q]$  the set of the (linear)  $q$ -difference operators whose coefficients belong to  $R$ , *i.e.*  $L \in R[\sigma_q]$  if  $L = \sum_{k=0}^n a_k \sigma_q^k$ , where  $a_k \in R$ . By identifying each element of  $R$  with the corresponding multiplication map in  $R$ , the set  $R[\sigma_q]$  constitutes itself a non-commutative ring for the composition of operators. Given  $L_1, L_2 \in R[\sigma_q]$ , we will write  $L_1 L_2$  in stead of  $L_1 \circ L_2$ .

Furthermore, we will denote by  $\nu_0(f) \in \mathbb{Z} \cup \{\infty\}$  the valuation of any given element  $f$  of the field  $\mathbb{C}[x^{-1}][[x]]$  “at  $x = 0$ ”. This means that (i)  $\nu_0(f) = \infty$  iff  $f$  is the identically vanishing series, and (ii)  $\nu_0(f) = \nu \in \mathbb{Z}$  iff  $f \equiv cx^\nu \pmod{x^{\nu+1} \mathbb{C}[[x]]}$  for some nonzero complex number  $c$ .

Throughout the whole section,  $P$  will be some given nonzero power series belonging to the field  $\mathbb{C}[x^{-1}][[x]]$ . Define the associated family of power series  $\{P_n\}_{n \geq 0}$  as follows:  $P_0 = 0$ ,  $P_1 = P$ , and

$$(2.1) \quad P_n = \sum_{k=0}^{n-1} (-x\sigma_q)^k P \quad \text{for } n \geq 2.$$

By noticing that  $(-x\sigma_q)^k P = (-x)^k q^{k(k-1)/2} \sigma_q^k P$ , it follows from (2.1) that, for any pair  $(n, \ell) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{> 0}$ ,

$$\begin{aligned} P_{n+\ell} - P_n &= \sum_{k=n}^{n+\ell-1} (-x)^k q^{k(k-1)/2} \sigma_q^k P \\ &\equiv (-1)^n c q^{n\nu+n(n-1)/2} x^{\nu+n} \pmod{x^{\nu+n+1} \mathbb{C}[[x]]}, \end{aligned}$$

where  $c \neq 0$  and  $\nu = \nu_0(P) \in \mathbb{Z}$ . This implies that

$$(2.2) \quad P_m \neq P_n \quad \text{if} \quad m \neq n.$$

Given any positive integer  $n \geq 1$ , let  $\{L_{n,k}^P\}_{1 \leq k \leq n} \subset (\mathbb{C}[x^{-1}][[x]])[\sigma_q]$  be the associated family of  $q$ -difference operators defined in the following manner:

$$(2.3) \quad L_{n,1}^P = \frac{1}{P_1} (x^n \sigma_q - (-1)^n) = \frac{1}{P} (x^n \sigma_q - (-1)^n);$$

$$(2.4) \quad L_{n,k+1}^P = \frac{1}{P_{k+1}} (x^{n-k} \sigma_q - (-1)^{n-k}) L_{n,k}^P, \quad 1 \leq k < n.$$

Specifically, letting  $k = n$ , one gets the following form of the  $n$ -th order  $q$ -difference operator  $L_{n,n}^P \in (\mathbb{C}[x^{-1}][[x]])[\sigma_q]$ :

$$(2.5) \quad L_{n,n}^P = \frac{1}{P_n} (x\sigma_q + 1) \frac{1}{P_{n-1}} (x^2\sigma_q - 1) \cdots \frac{1}{P_1} (x^n \sigma_q - (-1)^n).$$

**THEOREM 2.1.** *Let  $n$  be an integer  $\geq 1$ , let  $P, f \in \mathbb{C}[x^{-1}][[x]]$  with  $P \neq 0$ , and suppose that the following  $q$ -difference equation is satisfied:*

$$(2.6) \quad x\sigma_q f + f = P.$$

Then:

$$(2.7) \quad L_{n,n}^P(f^n) = (-1)^{n(n-1)/2},$$

where  $L_{n,n}^P$  is the  $q$ -difference operator defined by (2.5).

Theorem 2.1 will be proved with the help of the following lemmas.

**LEMMA 2.2.** *Let  $(n, k) \in \mathbb{Z}^2$  such that  $1 \leq k \leq n$ , and let  $P, f$  be as in Theorem 2.1. For  $0 \leq j \leq k$ , define  $A_{k;j} \in \mathbb{C}[x^{-1}][[x]]$  by*

$$(2.8) \quad A_{k;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (P_j - P_\ell).$$

One has:

$$(2.9) \quad L_{n,k}^P(f^n) = (-1)^{k(2n-k+1)/2} \sum_{j=0}^k \frac{1}{A_{k;j}} (f - P_j)^n.$$

**LEMMA 2.3.** *Let  $n$  be a nonnegative integer,  $\mathfrak{C}$  an extension field of  $\mathbb{C}$ , and let  $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathfrak{C}^{n+1}$  such that  $\alpha_j \neq \alpha_\ell$  for  $j \neq \ell$ . Define the  $(n+1)$ -tuple  $(a_0, a_1, \dots, a_n) \in \mathfrak{C}^{n+1}$  as follows:*

$$a_j = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (\alpha_j - \alpha_\ell)$$

( $a_0 = 1$  if  $n = 0$ ). The following identity holds in the ring  $\mathfrak{C}[T]$  of the polynomial functions over  $\mathfrak{C}$ :

$$(2.10) \quad \sum_{j=0}^n \frac{1}{a_j} (T - \alpha_j)^n = (-1)^n .$$

### 3. Proofs

**Proof of Lemma 2.2.** To simplify the presentation, we will write  $L_{n,k}$  instead of  $L_{n,k}^P$ , and set  $\epsilon_{n,k} = (-1)^{k(2n-k+1)/2}$ . One has:

$$(3.1) \quad \epsilon_{n;k+1} = (-1)^{n-k} \epsilon_{n,k} .$$

Let us proceed by induction on  $k$ . For  $k = 1$ , since  $P_1 = P$ , one has:

$$\epsilon_{n,1} = (-1)^n; \quad A_{1;0} = -P, \quad A_{1;1} = P .$$

Thus, (2.3) implies that

$$L_{n,1}(f^n) = \frac{1}{P} (x^n \sigma_q(f^n) - (-1)^n f^n) .$$

At the same time, by considering (2.6), one gets that

$$x^n \sigma_q(f^n) = (x \sigma_q f)^n = (P - f)^n ;$$

In this way, it follows that

$$L_{n,1}(f^n) = \frac{\epsilon_{n,1}}{P} ((f - P)^n - f^n) ,$$

that is exactly what wanted in (2.9) for  $k = 1$ .

Now, suppose that equality (2.9) holds for some integer  $k$  between 1 and  $n - 1$ . By using relation (2.4), one can express  $L_{n,k+1}(f^n)$  as follows:

$$(3.2) \quad L_{n,k+1}(f^n) = \frac{\epsilon_{n,k}}{P_{k+1}} \sum_{j=0}^k (B_{n,k;j} - C_{n,k;j}) ,$$

where

$$(3.3) \quad B_{n,k;j} = x^{n-k} \sigma_q \left( \frac{1}{A_{k;j}} (f - P_j)^n \right)$$

and

$$(3.4) \quad C_{n,k;j} = \frac{(-1)^{n-k}}{A_{k;j}} (f - P_j)^n .$$

In view of (2.1), one finds that  $x \sigma_q P_j = P - P_{j+1}$ , hence:

$$x \sigma_q (f - P_j) = x \sigma_q f - x \sigma_q P_j = (P - f) - (P - P_{j+1}) = -(f - P_{j+1}) .$$

Thus, it follows that

$$B_{n,k;j} = \frac{(x \sigma_q (f - P_j))^n}{x^k \sigma_q A_{k;j}} = \frac{(-1)^n}{x^k \sigma_q A_{k;j}} (f - P_{j+1})^n .$$

By using the expression of  $A_{k;j}$  given in (2.8), one has :

$$x^k \sigma_q A_{k;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (x \sigma_q (P_j - P_\ell)) = (-1)^k \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq k}} (P_{j+1} - P_{\ell+1}) .$$

This is to say that

$$x^k \sigma_q A_{k;j} = \frac{(-1)^k}{P_{j+1}} A_{k+1,j+1}.$$

So, one can write (3.3) as follows:

$$(3.5) \quad B_{n,k;j} = \frac{(-1)^{n-k} P_{j+1}}{A_{k+1;j+1}} (f - P_{j+1})^n.$$

Rewrite (3.2) into the following form:

$$(3.6) \quad L_{n,k+1}(f^n) = \frac{\epsilon_{n,k}}{P_{k+1}} \left( \sum_{j=0}^{k-1} (B_{n,k;j} - C_{n,k;j+1}) + B_{n,k;k} - C_{n,k;0} \right).$$

Let  $j < k$ ; one has:

$$A_{k+1,j+1} = (P_{j+1} - P_{k+1}) A_{k,j+1}.$$

Replace  $j$  by  $j + 1$  in (3.4), and make use of (3.5). One gets that

$$B_{n,k;j} - C_{n,k;j+1} = \frac{(-1)^{n-k} P_{k+1}}{A_{k+1;j+1}} (f - P_{j+1})^n.$$

Furthermore, one sees that  $A_{k+1;0} = -P_{k+1} A_{k;0}$ . By letting  $j = k$  and  $j = 0$  in (3.5) and (3.4) respectively, one has:

$$B_{n,k;k} = \frac{(-1)^{n-k} P_{k+1}}{A_{k+1;k+1}} (f - P_{k+1})^n$$

and

$$C_{n,k;0} = -\frac{(-1)^{n-k} P_{k+1}}{A_{k+1;0}} (f - P_0)^n.$$

Thus, in view of (3.1), equality (3.6) implies that

$$L_{n,k+1}(f^n) = \epsilon_{n,k+1} \sum_{j=0}^{k+1} \frac{1}{A_{k+1;j}} (f - P_j)^n,$$

which corresponds to (2.9) in which  $k$  was replaced by  $k + 1$ .  $\square$

**Proof of Lemma 2.3.** We give here two proofs.

*Proof by induction* - If  $n = 0$ , as  $a_0 = 1$ , equality (2.10) becomes evident. Suppose that (2.10) holds for some index  $n \geq 0$ , and consider an  $(n + 2)$ -uplet  $(\alpha_0, \dots, \alpha_{n+1})$  where, as before,  $\alpha_j \neq \alpha_\ell$  if  $j \neq \ell$ . Set

$$\mathcal{P}(T) = \sum_{j=0}^{n+1} \frac{1}{\tilde{a}_j} (T - \alpha_j)^{n+1}, \quad \tilde{a}_j = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n+1}} (\alpha_j - \alpha_\ell).$$

Let  $k$  be an integer such that  $0 \leq k \leq n + 1$ . If one writes

$$a_{k;j} = \prod_{\substack{\ell \neq k, \ell \neq j \\ 0 \leq \ell \leq n+1}} (\alpha_j - \alpha_\ell),$$

one has  $a_{k;j} = \tilde{a}_j / (\alpha_j - \alpha_k)$  for  $j \neq k$ , what implies that

$$(3.7) \quad \mathcal{P}(\alpha_k) = - \sum_{\substack{j \neq k \\ 0 \leq j \leq n+1}} \frac{1}{a_{k;j}} (\alpha_k - \alpha_j)^n.$$

By applying the induction hypothesis to the  $(n+1)$ -uplet  $(\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_{n+1})$  appeared in the right hand side of (3.7), one obtains that  $\mathcal{P}(\alpha_k) = (-1)^{n+1}$ . As  $\mathcal{P}$  is a polynomial with  $\deg \mathcal{P} \leq n+1$ , one finds that  $\mathcal{P}(T) = (-1)^{n+1}$  identically.

*Proof by Lagrange polynomials*<sup>1</sup> – Let  $\mathbb{K} = \mathfrak{C}(T)$  be the field of the rational functions over  $\mathfrak{C}$ . Let  $X$  be a new indeterminate, and set

$$F(X) = (T - X)^n, \quad \Lambda_j(X) = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (X - \alpha_\ell).$$

By noticing that  $F(X) \in \mathbb{K}[X]$  be such that  $\deg F = n < n+1$ , the Lagrange interpolation formula [3, Chap. IV, §2, p. 26] implies that

$$F(X) = \sum_{j=0}^n \frac{F(\alpha_j)}{a_j} \Lambda_j(X).$$

And now one compares the coefficients of  $X^n$  for both sides of the above equality. As that of  $\Lambda_j$  is equal to 1, one finds that

$$(-1)^n = \sum_{j=0}^n \frac{F(\alpha_j)}{a_j} = \sum_{j=0}^n \frac{(T - \alpha_j)^n}{a_j},$$

which is exactly the expected identity (2.10).  $\square$

**End of the proof of Theorem 2.1.** In view of Lemma 2.2, letting  $k = n$  in (2.9) yields that

$$L_{n,n}^P(f^n) = (-1)^{n(n+1)/2} \sum_{j=0}^n \frac{1}{A_{n;j}} (f - P_j)^n,$$

where

$$A_{n;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (P_j - P_\ell).$$

By (2.2), one knows that  $P_j \neq P_\ell$  for  $j \neq \ell$ . Thus, by applying Lemma 2.3 with  $\mathfrak{C} = \mathbb{C}[x^{-1}][[x]]$ ,  $X = f$  and  $\alpha_j = P_j$ , one gets that

$$\sum_{j=0}^n \frac{1}{A_{n;j}} (f - P_j)^n = (-1)^n.$$

This permits to finish the proof of Theorem 2.1.  $\square$

#### 4. One generalisation of Theorem 2.1

Instead of (2.6), let us consider the following slightly more general  $q$ -difference equation:

$$(4.1) \quad \alpha \sigma_q y + y = \beta,$$

where both  $\alpha$  and  $\beta$  are nonzero power series belonging to the field  $\mathbb{C}[x^{-1}][[x]]$ . In the same spirit as in (2.1), define the sequence  $\{\beta_n\}_{n \geq 0}$  in  $\mathbb{C}[x^{-1}][[x]]$  as follows:

$$(4.2) \quad \beta_0 = 0; \quad \beta_n = \sum_{k=0}^{n-1} (-\alpha \sigma_q)^k \beta, \quad n \geq 1.$$

<sup>1</sup>We would like to thank our friend and collaborator J. Sauloy for this elegant proof.

PROPOSITION 4.1. *For any given positive integer  $n$ , the following conditions are equivalent.*

- (1)  $\beta_j \neq 0$  for any positive integer  $j$  such that  $0 < j \leq n$ .
- (2)  $\beta_j \neq \beta_\ell$  for any couple of integers  $(j, \ell)$  such that  $0 \leq \ell < j \leq n$ .

PROOF. Indeed, by (4.2), it follows that

$$(4.3) \quad \beta_j - \beta_\ell = \sum_{k=\ell}^{j-1} (-\alpha \sigma_q)^k \beta = (-\alpha \sigma_q)^\ell \beta_{j-\ell}.$$

As  $\alpha \neq 0$ , the operator  $(-\alpha \sigma_q)$  is an automorphism on the  $\mathbb{C}$ -vector space  $\mathbb{C}[x^{-1}][[x]]$ . So, this is the same for its  $\ell$ -th power or iteration  $(-\alpha \sigma_q)^\ell$ . This implies the equivalence between the conditions stated in Proposition 4.1.  $\square$

Given a nonzero  $f \in \mathbb{C}[x^{-1}][[x]]$ , one remembers that  $\nu_0(f)$  denotes the valuation of  $f$  at  $x = 0$ , that is the lowest degree of the terms of  $f$ . By using (4.2), one obtains that, for  $n > 0$ :

$$\nu_0(\beta_n) = \begin{cases} \nu_0(\beta) & \text{if } \nu_0(\alpha) > 0; \\ (n-1)\nu_0(\alpha) + \nu_0(\beta) & \text{if } \nu_0(\alpha) < 0. \end{cases}$$

This gives the following statement for  $\alpha, \beta \in \mathbb{C}[x^{-1}][[x]] \setminus \{0\}$ .

REMARK 4.2. Both conditions considered in Proposition 4.1 are necessarily fulfilled if  $\nu_0(\alpha) \neq 0$ .

THEOREM 4.3. *Let  $n \in \mathbb{Z}_{>0}$ , and let  $\alpha, \beta$  and  $f \in \mathbb{C}[x^{-1}][[x]]$ . Suppose that  $f$  satisfies the  $q$ -difference equation in (4.1) and that  $\beta_j \neq 0$  for any positive integer  $j$  such that  $0 < j \leq n$ . One has:*

$$(4.4) \quad L_n^{\alpha, \beta}(f^n) = (-1)^{n(n-1)/2},$$

where  $L_n^{\alpha, \beta}$  is the  $n$ -th order  $q$ -difference operator defined by

$$(4.5) \quad L_n^{\alpha, \beta} = \frac{1}{\beta_n} (\alpha \sigma_q + 1) \frac{1}{\beta_{n-1}} (\alpha^2 \sigma_q - 1) \cdots \frac{1}{\beta_1} (\alpha^n \sigma_q - (-1)^n).$$

PROOF. Replace  $(k, P)$  with  $(n, \beta)$  in (2.8), and define:

$$A_{n;j} = \prod_{\substack{\ell \neq j \\ 0 \leq \ell \leq n}} (\beta_j - \beta_\ell), \quad 0 \leq j \leq n.$$

By taking into account Proposition 4.1, it follows that  $A_{n;j} \neq 0$ . Thus, one might proceed in the same way as for the proof of Theorem 2.1. We omit the details.  $\square$

## 5. About the summability of the powers of $q$ -Euler series

Let us come back to the power series  $\hat{E}_q(x)$  defined by (1.1), which satisfies the  $q$ -difference equation stated in (1.2). Letting  $P = 1$ , Theorem 2.1 implies immediately the following result.

REMARK 5.1. Given any integer  $n \geq 2$ , the  $n$ -th power  $\hat{E}_q(x)^n$  satisfies the following identity:

$$(5.1) \quad (x\sigma_q + 1) \frac{1}{P_{n-1}} (x^2\sigma_q - 1) \cdots (x^n\sigma_q - (-1)^n) \hat{E}_q(x)^n = (-1)^{n(n-1)/2} P_n,$$

where  $P_k = 1 - x + \cdots + q^{(k-1)(k-2)}(-x)^{k-1}$  for  $1 \leq k \leq n$ .

In particular, when  $n = 2$ , as  $P_2 = 1 - x$ , (5.1) takes the form of (1.3).

Furthermore, it might be interesting to notice that, if  $q = 1$ , one has  $\hat{E}_1(x) = \frac{1}{1+x}$  and, in this case, the above identity in (5.1) is simply equivalent to the following elementary relation:

$$\prod_{k=1}^n \frac{x^k - (-1)^k}{\sum_{j=0}^{k-1} (-x)^j} = (-1)^{n(n-1)/2} (x+1)^n.$$

This observation related with the specific case of  $q = 1$  may be also made for Theorem 4.3, by assuming, for example, both  $\alpha$  and  $\beta$  to belong to the field  $\mathbb{C}[x^{-1}]\{x\}$  of the germs of meromorphic functions at  $x = 0$  in  $\mathbb{C}$  or to the sub-field  $\mathbb{C}(x)$  of the rational functions over  $\mathbb{C}$ .

Our next remark goes to another  $q$ -analog of the following so-called Euler series:

$$(5.2) \quad \hat{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}.$$

This power series is divergent for all  $x \in \mathbb{C} \setminus \{0\}$  but Borel-summable in every direction excepted  $\mathbb{R}^-$ . It satisfies the following first order ODE:

$$(5.3) \quad (x\delta + 1)y = x, \quad \delta = x\partial_x = x \frac{d}{dx}.$$

Letting  $Y = \hat{E}(x)^2$ , one can check that

$$(5.4) \quad (x\delta + 1 - x)(x\delta + 2)Y = 2x^2.$$

Indeed, if  $y = \hat{E}(x)$ , one deduces from (5.3) that  $x\delta(y^2) = 2xy\delta y = 2y(x-y) = 2xy - 2y^2$ . This means that  $(x\delta + 2)(y^2) = 2xy$ . Thus, applying again (5.3) yields that  $(x\delta + 1)\frac{1}{2x}(x\delta + 2)y^2 = x$ . In this way, one gets (5.4), using the identity  $(x\delta + 1)\frac{1}{2x} = \frac{1}{2x}(x\delta + 1 - x)$  in the non-commutative ring  $\mathbb{C}(x)[\delta]$  of the differential operators over  $\mathbb{C}(x)$ .

Besides, the correspondances  $n \leftrightarrow \frac{1-q^n}{1-q}$  and  $n! \leftrightarrow \frac{(1-q)\dots(1-q^n)}{(1-q)^n}$  suggest one to consider the following  $q$ -analog of the Euler series:

$$(5.5) \quad \hat{E}(x; q) = x + \sum_{n \geq 1} (-1)^n \frac{(1-q)\dots(1-q^n)}{(1-q)^n} x^{n+1}.$$

This can be written in term of a basic hypergeometric series as follows:

$$\hat{E}(x; q) = x {}_2\phi_1(q, q; 0; q, -\frac{x}{1-q}) = x {}_2\phi_0(q^{-1}, q^{-1}; -; q^{-1}, \frac{x}{1-q^{-1}}).$$

See [6, (1.2.22), p. 4] for the general definition of  ${}_r\phi_s(\dots; \dots; q, z)$ .

Let  $\delta_q = \frac{\sigma_q - 1}{q-1} = x\Delta_q$ , where  $\Delta_q$  is defined in [1, p. 488, (10.2.3)]. By observing that  $n = \frac{\delta x^n}{x^n}$  and  $\frac{1-q^n}{1-q} = \frac{\delta_q x^n}{x^n}$ , a direct computation shows that  $\hat{E}(x; q)$  satisfies the  $q$ -analog of (5.3) as follows:  $(x\delta_q + 1)y = x$ . If one writes

$$(5.6) \quad \alpha = \frac{x}{q-1-x}, \quad \beta = (q-1)\alpha,$$

it follows from the above that

$$(5.7) \quad (\alpha\sigma_q + 1)\hat{E}(x; q) = \beta.$$

REMARK 5.2. Let  $\alpha$  be as in (5.6). The following identity holds in the field  $\mathbb{C}[x^{-1}][[x]]$ :

$$(5.8) \quad \left( \alpha \delta_q + \frac{1}{q-1-x} \right) \left( \alpha \delta_q - \frac{1}{x} + \frac{1}{q-1-x} \right) \hat{E}(x; q)^2 = \alpha (\sigma_q \alpha - 1).$$

Moreover, when  $q \rightarrow 1$ , (5.8) is reduced into the following equivalent form of (5.4):

$$(5.9) \quad \left( \delta + \frac{1}{x} \right) \left( \delta + \frac{2}{x} \right) \hat{E}(x)^2 = 2.$$

To obtain (5.8), one can apply (4.2) for  $n = 1$  and  $n = 2$ , where  $\alpha$  and  $\beta$  are defined by (5.6). This gives that

$$\beta_1 = (q-1)\alpha, \quad \beta_2 = \beta - \alpha \sigma_q \beta = (q-1)\alpha (1 - \sigma_q \alpha).$$

Thus, it follows from applying Theorem 4.3 to (5.7) that  $\hat{E}(x; q)^2$  satisfies the following  $q$ -difference equation:

$$(5.10) \quad L \left( \hat{E}(x; q)^2 \right) = \alpha (\sigma_q \alpha - 1),$$

where

$$(5.11) \quad L = \frac{1}{(q-1)^2} (\alpha \sigma_q + 1) \left( \alpha \sigma_q - \frac{1}{\alpha} \right).$$

By replacing  $\sigma_q$  with  $(q-1)\delta_q + 1$  in (5.11), one gets that

$$L = \left( \alpha \delta_q + \frac{1+\alpha}{q-1} \right) \left( \alpha \delta_q + \frac{\alpha^2 - 1}{(q-1)\alpha} \right).$$

As  $\alpha + 1 = \frac{q-1}{q-1-x}$  and  $\alpha^2 - 1 = \frac{(q-1)(2x-q+1)}{(q-1-x)^2}$ , one deduces immediately (5.8) from (5.10). The limit equation form given in (5.9) can be obtained from (5.8) by noticing both limits  $\alpha \rightarrow -1$  and  $\delta_q \rightarrow \delta$  for  $q \rightarrow 1$ .

In the rest of this paper, we will suppose that  $q > 1$  even if, in most cases, the hypothesis  $|q| > 1$  may be really enough; see [9] and [10]. Let us recall some results about the summability of power series solutions of  $q$ -difference equations. Given any linear  $q$ -difference operator  $L$  of the following form:

$$L = \sum_{j=0}^n a_j \sigma_q^j \in \mathbb{C}\{x\}[\sigma_q], \quad a_0 a_n \neq 0,$$

one defines its associated Newton polygon  $\mathcal{NP}(L)$  as being the convex hull of the set  $\{(j, m) : m \geq \nu_0(a_j)\}$  in the strip  $[0, n] \times [0, +\infty)$ . By following [8, (3.1.2)] and [11, Proposition 5.14], one knows that, if  $\mathcal{NP}(L)$  admits only integer slopes, saying  $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$ , then there exist  $\nu \in \mathbb{Z}$ ,  $(h_1, h_2, \dots, h_n) \in (1 + x\mathbb{C}\{x\})^n$  and  $(c_0, c_1, \dots, c_n) \in (\mathbb{C} \setminus \{0\})^{n+1}$  such that

$$(5.12) \quad L = c_0 x^\nu h_1 (x^{\kappa_1} \sigma_q + c_1) h_2 (x^{\kappa_2} \sigma_q + c_2) \dots h_n (x^{\kappa_n} \sigma_q + c_n).$$

Furthermore, let  $K^+ = \mathbb{Z}_{>0} \cap \{\kappa_j : 1 \leq j \leq n\}$ , and define  $\vec{k}$  as follows:

$$(5.13) \quad \vec{k} = \begin{cases} \emptyset & \text{if } K^+ = \emptyset \\ (k_1, \dots, k_m) & \text{if } K^+ = \{\kappa_\ell : 1 \leq \ell \leq m\} \text{ and } k_1 < \dots < k_m. \end{cases}$$

By considering [8, § 3.3.5], it follows that any power series  $\hat{f} \in \mathbb{C}[[x]]$  such that  $L(\hat{f}) \in \mathbb{C}\{x\}$  remains convergent or is  $Gq$ -summable of order  $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}_{>0}^m$  depending on whether  $\vec{k} = \emptyset$  or not.

**THEOREM 5.3.** *Given any integer  $n \geq 2$ , the  $n$ -th power  $\hat{E}_q(x)^n$  is  $Gq$ -summable of order  $(1, 2, \dots, n)$ . And this is the same for  $\hat{E}(x; q)^n$ .*

**PROOF.** This follows from taking into account (5.1) or applying Theorem 4.3 to (5.7) with the help of Remark 4.2.  $\square$

Applying [2, Theorem 36] or [7, Proposition 1.3.4.2] implies that every power  $\hat{E}(x)^n$  of the Euler series is Borel-summable at the same level, contrarily to what happens in the case of their respective  $q$ -analog  $\hat{E}(x; q)^n$ ; see Theorem 5.3. Furthermore, thanks to [4, Theorem 3.15], one knows that the  $Gq$ -sum of  $\hat{E}(x; q)$  tends toward the Borel-sum of  $\hat{E}(x)$  when  $q \rightarrow 1^+$ . In our coming work [5], it would be shown that, given one generic direction  $d$  and two power series  $\hat{f}_1$  and  $\hat{f}_2$  such that  $L_j(\hat{f}_j) \in \mathbb{C}\{x\}$  for some  $L_j \in \mathbb{C}\{x\}[\sigma_q]$ , where  $j = 1$  or  $2$ , the  $Gq$ -sum of  $\hat{f}_1 \hat{f}_2$  along  $d$  is equal to the product of the  $Gq$ -sum of  $\hat{f}_1$  along  $d$  with that of  $\hat{f}_2$ . In this way, one would obtain that the  $Gq$ -sum of  $\hat{E}_q(x)^n$  or  $\hat{E}(x; q)^n$  can be really expressed as the  $n$ -th power of that of  $\hat{E}_q(x)$  or  $\hat{E}(x; q)$ , respectively.

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