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# Exponential mixing of all orders for Arnol'd cat map lattices

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## Abstract

We show that the recently introduced classical Arnol'd cat map lattice field theories, which are chaotic, are exponentially mixing to all orders. Their mixing times are well-defined and are expressed in terms of the Lyapunov exponents, more precisely by the combination that defines the inverse of the Kolmogorov-Sinai entropy of these systems. We prove by an explicit recursive construction of correlation functions, that these exhibit  $l$ -fold mixing for any  $l = 3, 4, 5, \dots$ . This computation is relevant for Rokhlin's conjecture, which states that 2-fold mixing induces  $l$ -fold mixing for any  $l > 2$ . Our results show that 2-fold exponential mixing, while being necessary for any  $l$ -fold mixing to hold it is nevertheless not sufficient for Arnol'd cat map lattice field theories. The correspondence principle implies that these mixing times, also, control the scrambling of the underlying quantum system for short times.

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## 1 Introduction and Motivation

Fast information scrambling has occupied center stage in theoretical physics research in recent times. It forms an integral part of the so-called information loss paradox in quantum black holes and it is believed to be relevant for its long sought-after unitary resolution, thereby resolving the incompleteness of general relativity, using the principles of quantum mechanics [1].

Controversial issues of locality and unitarity between quantum theory and black hole physics have given rise to new concepts and principles in quantum gravity such as those of the holographic and complementarity principles and AdS/CFT correspondence, with black holes having been conjectured to be the fastest information scramblers in nature [2,3]. In essence they highlight that the no-cloning theorem of quantum information imposes constraints on the speed of information spreading within infalling matter on the black hole, due to the interaction with its horizon degrees of freedom. Its characteristic timescale has been estimated to be proportional to the logarithm of the entropy of the black hole. It is called the scrambling time, a conjectured lower bound to the time of information propagation in the universe, with the search for a microscopic many-body system that can saturate it being an area of topical research.

More specifically the ingredients of regular local field theories have been shown to be inadequate to accommodate fast information processing among the near horizon black hole microscopic degrees of freedom; this has led to the study of how nonlocality can be taken into account [2,4,5]. A further property, that has emerged as relevant is that of chaoticity. This has, in turn, motivated the study of chaos in matrix models [6], conformal field theories and a large class of SYK type of models [7] which have been dubbed chaotic field theories [8].

In a recent work we have studied the AdS<sub>2</sub> near horizon geometry of extremal black holes by means of a modular discretization thereof [9]. This discretization is relevant for probing the non-local features of the geometry using single particle excitations. Among such excitations the Arnol'd cat map stands out, since its isometries are those of the near horizon geometry and its dynamics can capture fast scrambling in a way that saturates the fast scrambling bound [10,11].

Furthermore, the model has been shown to admit a well defined continuum limit [12] showing how a discrete short distance near horizon spacetime can underlie an emergent continuum long distance classical limit.

In followup work we have taken a first step in implementing the conditions of nonlocality and strong chaos for many-body systems through the construction of lattice field theories, in which we replace the standard harmonic oscillator constituents with chaotic interacting constituents, namely, the so called Arnol'd cat map [13, 14]. The resulting evolution matrix of the  $n$ -body system is represented by elements of the symplectic group  $\text{Sp}_{2n}[\mathbb{Z}]$ . The model has been shown to exhibit strong classical chaos as well as non-locality, which leaves a direct imprint on the (spatiotemporal) chaotic benchmark of the model, its Kolmogorov-Sinai entropy.

In the present work we pursue the study of the chaotic properties of the many-body system, by elucidating its mixing properties, the classical avatars of scrambling. We will show that it exhibits fast exponential mixing of all orders in the number of observables. To this end we adapt an operator method developed by de Bièvre [15], firstly for the case of two observable functions in the computation of the decay rates of their correlation function. Moreover we generalize his method for any number of observables for the Arnol'd cat map lattice, deducing that it is a strongly  $l$ -fold mixing system, for any  $l = 2, 3, \dots$  and that the mixing time is  $1/S_{\text{K-S}}$ , where  $S_{\text{K-S}}$  is the Kolmogorov-Sinai entropy.

The role of deterministic chaos [16] in rendering mixing phenomena fast and “efficient” is well recognized and can be understood within the so-called ergodic hierarchy classification of hyperbolic dynamical systems [17]. Mixing is the classical analog of the fast scrambling of information, which is the rapid spreading of an initially localized perturbation, as it evolves towards a homogeneous stationary state.

Our results pertain, also, to the well known Rokhlin conjecture [18, 19] of whether 2-fold mixing implies multiple  $l$ -fold mixing and more generally mixing of all higher orders. We show, by explicit calculation, that 2-fold mixing is necessary but not sufficient for any  $l$ -fold ( $l > 2$ ) mixing to hold. We proceed, to this end, in steps, by establishing, first, both the 2-fold and  $l$ -fold mixing property of a single ACM. Subsequently we on the case of the extended  $n$ -body coupled Arnol'd cat map systems through the explicit computation of  $l$ -fold correlation functions. We find, by explicit calculation, that the correlation functions show exponential decay behavior and identify the mixing time as the inverse of the Kolmogorov-Sinai entropy. The reason that 2-fold mixing isn't sufficient for ensuring  $l$ -fold mixing can be understood from the recursive construction of the  $l + 1$ st correlation function, which is the sum of two terms, the first being the  $l$ -th correlation function, while the second is a remainder term that can be shown, also, to tend to zero for large times—with subleading behavior.

The plan of the paper is the following:

In section 2 we review the salient features of our previous paper [13], namely the construction of the evolution operator for  $n$  maps, the analytical computation of the spectrum of the Lyapunov exponents, from which we have obtained the Kolmogorov-Sinai entropy,  $S_{\text{K-S}}$ .

In sections 3-4 we discuss the definition of simple and multiple mixing of chaotic dynamical systems and we provide an explicit calculation of the corresponding mixing times for the case

of a single Arnol'd cat map.

In section 5 we extend the previous calculation to the case of  $n$ , symplectically interacting, Arnol'd cat maps, for any  $n$ . We find that the corresponding, simple, mixing time is equal to  $1/S_{K-S}$ .

Our conclusions and discussion for directions of further inquiry are set forth in section 6.

## 2 Interacting Arnol'd cat maps from symplectic couplings of $n, k$ -Fibonacci sequences

In this section we review the salient features of our previous paper [13], pertaining to the construction of the evolution operator for  $n$  interacting Arnol'd cat maps and the analytical computation of the spectrum of the Lyapunov exponents, from which we have obtained the Kolmogorov-Sinai entropy,  $S_{K-S}$ .

We consider a dynamical system of  $n$  degrees of freedom, whose space of states is the torus  $\mathbb{T}^{2n}$ , the  $2n$ -dimensional torus of radius 1. Therefore any state can be identified with a point  $\mathbf{x} \in \mathbb{T}^{2n}$ .

We shall describe the time evolution of this system by successive applications on a point  $\mathbf{x} \in \mathbb{T}^{2n}$  of elements,  $\mathbf{M}$ , of the symplectic group over the integers,  $\text{Sp}_{2n}[\mathbb{Z}]$ . The evolution is, thus, discrete in time. In equations,

$$\mathbf{x}_{m+1} = \mathbf{x}_m \mathbf{M} \bmod 1 \tag{2.1}$$

where  $\mathbf{x}_m = (\mathbf{q}_m, \mathbf{p}_m)$  describes the state of the system at the  $m$ -th time tick. Here the positions and momenta take values in  $\mathbb{T}^n$ . These equations describe the evolution of  $n$  "particles", from the tick  $m$  to the tick  $m + 1$ .

Since  $\mathbf{M} \in \text{Sp}_{2n}[\mathbb{Z}]$ , it preserves the symplectic structure,

$$\mathbf{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{2.2}$$

in the sense that

$$\mathbf{M}^T \mathbf{J} \mathbf{M} = \mathbf{J} \tag{2.3}$$

Upon decomposing  $\mathbf{M}$  into four blocks of  $n \times n$  matrices

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{2.4}$$

we obtain the constraints on the blocks,

$$\begin{aligned} \mathbf{A}^T \mathbf{D} - \mathbf{C}^T \mathbf{B} &= I_{n \times n} \\ \mathbf{A}^T \mathbf{C} &= \mathbf{C}^T \mathbf{A} \\ \mathbf{B}^T \mathbf{D} &= \mathbf{D}^T \mathbf{B} \end{aligned} \tag{2.5}$$

It is known that  $\text{Sp}_{2n}[\mathbb{Z}]$  is generated by the elements

$$\mathbf{M}_R = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}, \quad \mathbf{M}_L = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \quad \text{and} \quad \mathbf{D}_S = \begin{pmatrix} S^T & 0 \\ 0 & S^{-1} \end{pmatrix} \quad (2.6)$$

where  $R$  and  $L$  are symmetric matrices; all matrices have integer entries.

In what follows we are interested in those evolution matrices,  $\mathbf{M}$ , that have strictly positive eigenvalues—the reason is that this property is sufficient for the dynamics to be chaotic. Such symplectic matrices are called *hyperbolic* and the corresponding dynamical systems are called *hyperbolic* or *Anosov*.

It can be shown that the symplectic property implies that the eigenvalues come in pairs,  $(\rho_+^{(i)}, \rho_-^{(i)} = 1/\rho_+^{(i)} < 1)$ . For hyperbolic matrices this implies the existence of  $n$  planes, spanned by the corresponding eigenvectors: The dynamics is *expanding* along the eigenvectors,  $\mathbf{v}_+^{(i)}$ , corresponding to the eigenvalues  $\rho_+^{(i)}$  and *contracting* along the eigenvectors,  $\mathbf{v}_-^{(i)}$ , corresponding to the eigenvalues  $\rho_-^{(i)}$ .

We shall focus here on evolution matrices  $\mathbf{M}$  that are hyperbolic and symmetric, so the eigenvectors  $\mathbf{v}_\pm^{(i)}$  span orthogonal eigenspaces and  $\mathbf{v}_+^{(i)}$  and  $\mathbf{v}_-^{(i)}$  are orthogonal for each  $i = 1, 2, \dots, n$ .

This class of matrices includes, in particular, the generalization of the so-called Arnol'd cat map, defined by the  $2 \times 2$  matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad (2.7)$$

to the case of  $n$  such maps, in interaction.

Moreover, since the phase space is compact, this class of maps exhibits strong mixing as well as  $l$ -fold mixing of *any* order,  $l$ , as we shall show in the following. To this end we shall use the Lyapunov exponents and the Kolmogorov–Sinai entropy, that were calculated in closed form in [13].

In order to study the effects of coupling of  $n$  Arnol'd cat maps, each of which is defined on a lattice of  $n$  sites, we associate to each site a two-dimensional torus, with dynamics described by a single Arnol'd cat map. The total phase space of the system will be  $\mathbb{T}^{2n}$ , the  $2n$ -dimensional torus and the proposed dynamics will be described by appropriate elements of the symplectic group,  $\text{Sp}_{2n}[\mathbb{Z}]$ . The  $2n$ -dimensional symplectic maps will allow couplings of various degrees of locality and strength.

To begin with, we shall show that these maps can be constructed as iteration matrices of  $n$  coupled Fibonacci sequences.

We start from the relation between the ACM and the Fibonacci sequence of integers.

The Fibonacci sequence is one of the integer sequences, which has been studied, for a long time and there are journals dedicated to its properties and their applications.

The definition is given by the relations

$$\begin{aligned} f_0 &= 0; f_1 = 1 \\ f_{m+1} &= f_m + f_{m-1} \end{aligned} \quad (2.8)$$

which can be written in matrix form

$$\begin{pmatrix} f_m \\ f_{m+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} f_{m-1} \\ f_m \end{pmatrix} \quad (2.9)$$

The matrix  $\mathbf{A}$  is not a symplectic matrix, but it satisfies

$$\mathbf{A}^T \mathbf{J} \mathbf{A} = -\mathbf{J} \quad (2.10)$$

for  $n = 1$ .

We remark that the Arnol'd cat map, acting on the torus  $\mathbb{T}^2$ , can be written as

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \mathbf{A}^2 \quad (2.11)$$

Eq. (2.10) implies that  $[\mathbf{A}^2]^T \mathbf{J} \mathbf{A}^2 = \mathbf{J}$ , therefore that  $\mathbf{A}^2$  is symplectic. It's possible to generalize the Fibonacci sequence in the following way:

$$g_{m+1} = k g_m + g_{m-1} \quad (2.12)$$

with  $g_0 = 0$  and  $g_1 = 1$  and  $k$  is a positive integer. This is known as the “ $k$ -Fibonacci” sequence [20].

We may solve eq. (2.12) by  $g_m \equiv C \rho^m$ . The characteristic equation for  $\rho$  reads

$$\rho^2 - k\rho - 1 = 0 \Leftrightarrow \rho_{\pm}(k) = \frac{k \pm \sqrt{k^2 + 4}}{2} \quad (2.13)$$

and express  $g_m$  as a linear combination of the  $\rho_{\pm}$ , upon taking into account the initial conditions:

$$g_m = A_+ \rho_+(k)^m + A_- \rho_-(k)^m = \frac{\rho_+(k)^m - (-)^m \rho_+(k)^{-m}}{\sqrt{k^2 + 4}} \quad (2.14)$$

In matrix form

$$\begin{pmatrix} g_m \\ g_{m+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}}_{\mathbf{A}(k)} \begin{pmatrix} g_{m-1} \\ g_m \end{pmatrix} \quad (2.15)$$

Similarly as for the usual Fibonacci sequence, we may show, by induction, that

$$\mathbf{A}(k)^m = \begin{pmatrix} g_{m-1} & g_m \\ g_m & g_{m+1} \end{pmatrix} \quad (2.16)$$

We remark that  $\det \mathbf{A}(k)^m = (-)^m = g_{m+1} g_{m-1} - g_m^2$ , for  $m = 1, 2, \dots$  and that  $\lim_{m \rightarrow \infty} (g_{m+1}/g_m) = \rho_+(k)$ , which, for  $k = 1$ , is the golden ratio, for  $k = 2$  is the silver ratio and, for  $k > 2$  are generalizations thereof.

$\lambda_+(k) \equiv \log \rho_+(k)^2$  is the greatest Lyapunov exponent of the generalized evolution operator, i.e. the  $k$ -Arnol'd cat map,  $A(k)^2$  and is an increasing function of  $k$ .

Coupled Fibonacci sequences have been considered in the literature, for instance in [21]. However, in these papers the possible applications to Hamiltonian dynamics, were not the topic of interest and moreover the corresponding maps were not symplectic.

After this review of the single Arnol'd cat map, we proceed to the study of how many such maps can interact, in a way such that the evolution matrix is an element of  $\text{Sp}_{2n}[\mathbb{Z}]$ . To this end, we shall, once more, use the correspondence between the Arnol'd cat maps and the Fibonacci sequences.

We start by coupling  $n = 2$  Fibonacci sequences,  $\{f_m\}$  and  $\{g_m\}$  (but we write the expressions in a way that generalizes immediately to arbitrary  $n$ ):

$$\begin{aligned} f_{m+1} &= a_1 f_m + b_1 f_{m-1} + c_1 g_m + d_1 g_{m-1} \\ g_{m+1} &= a_2 g_m + b_2 g_{m-1} + c_2 f_m + d_2 f_{m-1} \end{aligned} \quad (2.17)$$

where the  $a_i, b_i, c_i, d_i, i = 1, 2$  are integers,  $f_0 = 0 = g_0$  and  $f_1 = 1 = g_1$  are the initial conditions and  $m = 1, 2, 3, \dots$

In matrix form, these read

$$X_{m+1} \equiv \begin{pmatrix} f_m \\ g_m \\ f_{m+1} \\ g_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_1 & d_1 & a_1 & c_1 \\ d_2 & b_2 & c_2 & a_2 \end{pmatrix} \underbrace{\begin{pmatrix} f_{m-1} \\ g_{m-1} \\ f_m \\ g_m \end{pmatrix}}_{X_m} \quad (2.18)$$

Let us define the  $2 \times 2$  matrices

$$D \equiv \begin{pmatrix} b_1 & d_1 \\ d_2 & b_2 \end{pmatrix} \quad C \equiv \begin{pmatrix} a_1 & c_1 \\ c_2 & a_2 \end{pmatrix} \quad (2.19)$$

in terms of which the one-time-step evolution equation (2.18) can be written in block form as

$$X_{m+1} = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ D & C \end{pmatrix} X_m \quad (2.20)$$

In analogy with the case of a single Fibonacci sequence and its relation with the Arnol'd cat map, we impose the constraint (cf. (2.10))

$$\begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ D & C \end{pmatrix}^T J \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ D & C \end{pmatrix} = -J \quad (2.21)$$

This condition implies that

$$D = I_{n \times n} \quad C = C^T \quad (2.22)$$



Therefore  $a_1 = k_1$ ,  $a_2 = k_2$ ,  $c_1 = c_2 = c$ . This implies, in particular, that the coupling between the sequences is the same for both, in order for the square of the evolution matrix to be symplectic. As we shall explain below, the generalization of this property is that, for more sequences, the corresponding coupling matrix must be symmetric. This is important in order to define chains of interacting  $k$ -Arnol'd cat maps.

In terms of these parameters, the recursion relations take the form

$$\begin{aligned} f_{m+1} &= k_1 f_m + f_{m-1} + c g_m \\ g_{m+1} &= k_2 g_m + g_{m-1} + c f_m \end{aligned} \quad (2.23)$$

and can be identified as describing a particular coupling between a  $k_1$ - and a  $k_2$ -Fibonacci sequence. This particular coupling is determined by the condition that the square of the evolution matrix is an element of  $\text{Sp}_4[\mathbb{Z}]$ :

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & \mathbf{C} \end{pmatrix} \Rightarrow \mathbf{M} = \mathbf{A}^2 = \begin{pmatrix} 1 & \mathbf{C} \\ \mathbf{C} & 1 + \mathbf{C}^2 \end{pmatrix} \quad (2.24)$$

The generalization to a chain of  $n$   $k$ -Fibonacci sequences, with nearest-neighbor couplings, proceeds as follows: We choose two diagonal matrices, of positive integers,  $\mathbf{K}_{IJ} = K_I \delta_{IJ}$  and  $\mathbf{G}_{IJ} = G_I \delta_{IJ}$ , with  $I, J = 1, 2, \dots, n$ .

We now define the translation operator,  $\mathbf{P}$  along a closed chain, by  $\mathbf{P}_{I,J} = \delta_{I-1,J \bmod n}$ . The periodicity is expressed by the fact that  $\mathbf{P}^n = I_{n \times n}$ . Moreover,  $\mathbf{P}$  is orthogonal, since  $\mathbf{P}\mathbf{P}^T = I_{n \times n}$ .

Now we can define the coupling matrix for  $n$  sequences as

$$\mathbf{C} = \mathbf{K} + \mathbf{P}\mathbf{G} + \mathbf{G}\mathbf{P}^T \quad (2.25)$$

The corresponding  $2n \times 2n$  evolution matrix,  $\mathbf{A}$  is given by

$$\mathbf{A} = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ I_{n \times n} & \mathbf{C} \end{pmatrix} \quad (2.26)$$

and satisfies the relation  $\mathbf{A}^T \mathbf{J} \mathbf{A} = -\mathbf{J}$ . Its square,

$$\mathbf{M} = \mathbf{A}^2 = \begin{pmatrix} I_{n \times n} & \mathbf{C} \\ \mathbf{C} & I_{n \times n} + \mathbf{C}^2 \end{pmatrix} \quad (2.27)$$

therefore satisfies the relation  $\mathbf{M}^T \mathbf{J} \mathbf{M} = \mathbf{J}$ , showing that  $\mathbf{M} \in \text{Sp}_{2n}[\mathbb{Z}]$ . Since  $\mathbf{A}$  is symmetric, (from the property that  $\mathbf{C} = \mathbf{C}^T$ ),  $\mathbf{M}$  is positive definite and its eigenvalues come in pairs,  $(\rho, 1/\rho)$ , with  $\rho > 1$  (and the corresponding eigenvectors are orthogonal). This property implies that, for all matrices  $\mathbf{K}$  and  $\mathbf{G}$  this system of coupled maps is hyperbolic.

An important special case arises if we impose translation invariance along the chain, i.e.  $K_I = K$  and  $G_I = G$  for all  $I = 1, 2, \dots, n$ .

Let us now consider the case of the open chain. The only change involves the operator  $P$ , which, now, must be defined as  $P_{IJ} = \delta_{I-1,J}$ , for  $I, J = 1, 2, \dots, n$ . Due to the absence of the mod  $n$  operation, the “far non-diagonal” (upper right and lower left) elements are, now, zero. This express the property that the  $n$ -th Fibonacci is not coupled to the first one (and vice versa).

For both, closed or open, chains, we observe certain algebraic properties of the evolution matrix,  $A$ .

The  $k$ -Fibonacci sequence has the important property that the elements of the matrix  $A(k)^m$  are arranged in columns of consecutive pairs of the sequence. We shall show that this property can be generalized for  $n$  interacting  $k$ -Fibonacci sequences as follows:

**Theorem 1.** *The  $m$ -th power of the evolution matrix,  $A$  (cf. eq. (2.26)) can be written as*

$$A^m = \begin{pmatrix} C_{m-1} & C_m \\ C_m & C_{m+1} \end{pmatrix} \quad (2.28)$$

where  $C_0 = 0_{n \times n}$ ,  $C_1 = I_{n \times n}$  and  $C_{m+1} = CC_m + C_{m-1}$ , with  $m = 1, 2, 3, \dots$ . This matrix recursion relation generalizes to matrices the  $k$ -Fibonacci sequence for numbers. It holds for any matrix,  $C$  and, in particular for the (symmetric, integer) matrix  $C$ , defined by eq. (2.25). The solution to this matrix recursion relation is given in terms of the Fibonacci polynomials,  $F_m(x)$ , with argument  $x = C$ , i.e.  $C_m = F_m(C)$ . [13].

*Proof.* The proof is by induction. For  $m = 1$  it is true, by definition. If we assume it holds for  $m > 1$ , then, by the relation  $A^{m+1} = A \cdot A^m$ , we immediately establish that it holds for  $m + 1$ .  $\square$

Having constructed a large class of symplectic many-body maps, that describe the dynamics of  $n$   $k$ -Arnol'd cat maps, we wish to understand their chaotic behavior, as they act on any initial condition of  $\mathbb{T}^{2n}$ , for  $n > 1$ .

These particular symplectic matrices generalize the chaotic behavior of one Arnol'd cat map, acting on  $\mathbb{T}^2$ , to that of  $n$  maps, acting on  $\mathbb{T}^{2n}$ . Indeed, starting with initial conditions  $\mathbf{x}_0 \in \mathbb{T}^{2n} \neq \mathbf{0}$ , which have irrational components, the evolution matrix  $M$ , as it acts on  $\mathbf{x}_0$ , defines an orbit,  $\mathbf{x}_m = \mathbf{x}_0 M^m$ , that will, in the limit  $m \rightarrow \infty$ , cover the whole torus  $\mathbb{T}^{2n}$ ; the full phase space is the attractor of the map. This means that the map is ergodic.

Furthermore, this map has the, additional, property of being *strongly chaotic*, which means that it has positive Kolmogorov-Sinai entropy. It is possible to tune the parameters  $K$  and  $G$  (in the translation invariant case) so that no Lyapunov exponent is equal to zero (which would imply the existence of a conservation law). In this case the map is *maximally* hyperbolic, i.e. it is an “Anosov C-system”.

Another quantitative measure of the “long time” chaotic behavior of the orbits of a hyperbolic map is provided by the properties of the time correlation functions of “observables”, i.e. functions on the phase space,  $\mathbb{T}^{2n}$ . These properties define the *mixing behavior* of the map and can be expressed by the fact that the connected correlation functions decay to zero, for long

times. This, thus, raises the problem of computing them and, in particular whether they decay exponentially, thereby defining the spectrum of mixing times.

On the other hand, another way to look at the chaotic properties of a dynamical system, is to consider, if possible, the full set of its unstable periodic orbits. For large periods, the corresponding periodic orbit should approach (this is called “shadowing”) the chaotic orbits, which fill the phase space [22]. Fortunately, for our system, of  $n$  interacting Arnol’d cat maps, the full set of unstable periodic orbits is produced by all the *rational* points of  $\mathbb{T}^{2n}$ , taken as initial conditions. So the problem reduces to finding periodic orbits, with “very large” periods. This problem is difficult, because the periods are random functions of the common-denominator,  $N$ , of the rational points, that are initial conditions [13].

In the following section we shall, thus, introduce the notion of mixing and the mixing time for ergodic systems and we shall present the method for obtaining a bound on the mixing time for the case of the single cat map; then we shall do the same calculation for two, symplectically coupled, cat maps, to see what are the issues that arise. Finally, we shall present the generalization to  $n$  symplectically coupled maps, where, in addition to the dependence on the coupling, the issue of the range becomes of interest.

### 3 The mixing time for one cat map

Mixing is the ubiquitous phenomenon of blending together distinct many body matter systems from an initial inhomogeneous state to a final homogeneous uniform configuration [23]. The mixing time is the characteristic transient period it takes for an initial local perturbation to delocalize and spread in a many-body system attaining a uniform and homogenized final state. What sets the scale of mixing time as well as its precise determination is both a conceptual and a computational challenge.

The development of mathematical methods associated with the Ergodic Theory have been brought into prominence [23,24] in the study of “Ergodic Mixing” through the study of stochasticity in measure-preserving dynamical systems.

For a discrete-time dynamical system,  $T : M \rightarrow M$ , which preserves a probability measure  $\mu$  a “strong mixing” condition can be formulated for two sets of points  $A$  and  $B$  on a constant energy surface  $E$ , as follows:

$$\mu(A \cap T^n(B)) \xrightarrow{n \rightarrow \infty} \mu(A) \mu(B) \quad (3.1)$$

Intuitively the mixing condition states that a dynamical system is strongly mixing whenever any two observable events  $A$  and  $B$  which occur at separate time instances specified by the action of  $T^n$  on  $B$  become independent in the infinite time separation limit  $n \rightarrow \infty$ .

Equivalently its standard diagnostic quantifier is the decay of the correlation function for any pair of observable functions  $f, g : M \rightarrow \mathbb{C}$

$$C_n(f, g^*) = \int f \circ T^n \cdot g^* d\mu - \left( \int f d\mu \right) \cdot \left( \int g^* d\mu \right) \xrightarrow{n \rightarrow \infty} 0 \quad (3.2)$$

The correlation function  $C_n(f, g^*)$  or the self-correlation for a single function  $C_n(f, f^*)$  fall off to zero either polynomially, exponentially (rapid mixing) or even super exponentially in the long time limit  $n \rightarrow \infty$  [25, 26].

We shall show explicitly that the Arnol'd cat map on  $\mathbb{T}^2$  is exponentially mixing and that its mixing time is given by the expression

$$\tau_{\text{mix}}(\text{Arnol'd}) = \frac{1}{\log \rho_+} \quad (3.3)$$

where  $\rho_+ = (3 + \sqrt{5})/2$ . This is, in fact equal to the inverse of the Kolmogorov–Sinai entropy of the system. This result is known and in the next section we shall generalize it to the case of symplectically coupled Arnol'd cat maps.

We start with some preparatory material: Any smooth and square integrable observable on  $\mathbb{T}^2$  has a uniformly convergent Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (3.4)$$

The set of all these observables defines a Hilbert space,  $H(\mathbb{T}^2)$ , with inner product

$$\langle f, g \rangle = \int_{\mathbb{T}^2} d^2 \mathbf{x} f(\mathbf{x}) g^*(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}} \cdot d_{\mathbf{k}}^* \quad (3.5)$$

where

$$g(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} d_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (3.6)$$

and the norm is defined by

$$\|f\|^2 = \int_{\mathbb{T}^2} d^2 \mathbf{x} f(\mathbf{x}) f^*(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} |c_{\mathbf{k}}|^2 \quad (3.7)$$

For any  $f, g \in H(\mathbb{T}^2)$  we have the Cauchy–Schwarz inequality

$$|\langle f, g \rangle| \leq \|f\| \|g\| \quad (3.8)$$

With these standard preliminaries, we proceed with the evaluation of the correlation functions for the Arnol'd cat map.

$$C_n(f, g^*) = \int d^2 \mathbf{x} f(T^n \mathbf{x}) g^*(\mathbf{x}) - \int d^2 \mathbf{x} f(\mathbf{x}) \int d^2 \mathbf{x} g^*(\mathbf{x}) = \langle f \circ T^n, g^* \rangle - \langle f \rangle \langle g^* \rangle \quad (3.9)$$

We follow the procedure sketched in ref. [15]. The idea is to choose as functions the eigenfunctions of a particular operator and show that, for these,  $|C_n(f, g^*)| \leq \text{const} \times e^{-n/\tau}$ .

Let us recall the argument: The ACM has eigenvalues

$$\rho_{\pm} = \frac{3 \pm \sqrt{5}}{2} \quad (3.10)$$

and corresponding eigenvectors,

$$\mathbf{u}_{\pm} = \frac{1}{\sqrt{1 + \rho_{\pm}}} \begin{pmatrix} 1 \\ \rho_{\pm} - 1 \end{pmatrix} \quad (3.11)$$

$\mathbf{A}\mathbf{u}_{\pm} = \rho_{\pm}\mathbf{u}_{\pm}$ . Since  $\rho_+\rho_- = 1$  and they're real (since the matrix is symmetric)  $\rho_+ > 1$  and  $\rho_- = 1/\rho_+ < 1$ .

Now, let us define the operators

$$D_{\pm} \equiv -\frac{i}{2\pi} u_{\pm, I} \delta_{IJ} \partial_{x_J} \quad (3.12)$$

Since the torus,  $\mathbb{T}^2$ , is a compact manifold, the spectrum of the operators  $D_{\pm}$  is discrete and can be labeled by two integers,  $n_I$ . Indeed, it is straightforward to check that the functions

$$e_{\mathbf{k}}(\mathbf{x}) = e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (3.13)$$

with  $\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}$  and  $\mathbf{x} \in \mathbb{T}^2$  are eigenfunctions of  $D_{\pm}$ :

$$D_{\pm} e_{\mathbf{k}}(\mathbf{x}) = \mathbf{k} \cdot \mathbf{u}_{\pm} e_{\mathbf{k}}(\mathbf{x}) \quad (3.14)$$

Since the components of the  $\mathbf{u}_{\pm}$  are irrational numbers,  $\mathbf{k} \cdot \mathbf{u}_{\pm} \neq 0$  for any  $\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}$ , therefore the inverses,  $[D_{\pm}]^{-1}$ ,

$$[D_{\pm}]^{-1} e_{\mathbf{k}}(\mathbf{x}) = \frac{1}{\mathbf{k} \cdot \mathbf{u}_{\pm}} e_{\mathbf{k}}(\mathbf{x}) \quad (3.15)$$

are well-defined.

We proceed below with the details of the evaluation of the correlation function  $C_n(f, g^*)$  in a more explicit form.

First we split the Fourier sums of the functions  $f$  and  $g$  as follows:

$$\begin{aligned} C_n(f, g^*) &= \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}} d_{\mathbf{l}}^* \int_{\mathbb{T}^2} d^2 \mathbf{x} e^{2\pi i \mathbf{k} \cdot T^n \mathbf{x}} e^{-2\pi i \mathbf{l} \cdot \mathbf{x}} - c_0 d_0^* = \\ &= \sum_{\mathbf{k} \neq (0,0)} c_{\mathbf{k}} d_0^* \int_{\mathbb{T}^2} d^2 \mathbf{x} e^{2\pi i \mathbf{k} \cdot T^n \mathbf{x}} + \sum_{\mathbf{l} \neq (0,0)} c_0 d_{\mathbf{l}}^* \int_{\mathbb{T}^2} d^2 \mathbf{x} e^{-2\pi i \mathbf{l} \cdot \mathbf{x}} + \sum_{\mathbf{k} \neq (0,0), \mathbf{l} \neq (0,0)} c_{\mathbf{k}} d_{\mathbf{l}}^* \int_{\mathbb{T}^2} d^2 \mathbf{x} e^{2\pi i \mathbf{k} \cdot T^n \mathbf{x}} e^{-2\pi i \mathbf{l} \cdot \mathbf{x}} \end{aligned} \quad (3.16)$$

The first two terms, that involve sums over  $\mathbf{k} \neq (0, 0)$  or  $\mathbf{l} \neq (0, 0)$  are zero (since the integrals are  $\delta$ -functions on  $\mathbf{k} = (0, 0)$  and  $\mathbf{l} = (0, 0)$  respectively), therefore only the last term survives:

$$C_n(f, g^*) = \sum_{\mathbf{k} \neq (0,0), \mathbf{l} \neq (0,0)} c_{\mathbf{k}} d_{\mathbf{l}}^* \int_{\mathbb{T}^2} d^2 \mathbf{x} e^{2\pi i \mathbf{k} \cdot T^n \mathbf{x}} e^{-2\pi i \mathbf{l} \cdot \mathbf{x}} \quad (3.17)$$

Therefore, the operators  $D_{\pm} \equiv -(\mathbf{i}/(2\pi))\mathbf{u}_{\pm} \cdot \nabla_{\mathbf{x}}$  act on the function  $f(\mathbf{x}) = e^{2\pi\mathbf{i}\mathbf{k} \cdot (\mathbf{A}^n \mathbf{x})}$  as

$$D_{\pm} e^{2\pi\mathbf{i}\mathbf{k} \cdot (\mathbf{A}^n \mathbf{x})} = \mathbf{u}_{\pm} \cdot \mathbf{k} \rho_{\pm}^n e^{2\pi\mathbf{i}\mathbf{k} \cdot (\mathbf{A}^n \mathbf{x})}$$

as can be checked by direct calculation.

These calculations now lead to the following statement:

**Proposition 1.**

$$\int_{\mathbb{T}^2} d^2 \mathbf{x} D_{\pm} f(\mathbf{A}^n \mathbf{x}) D_{\pm}^{-1} g^*(\mathbf{x}) = - \int_{\mathbb{T}^2} d^2 \mathbf{x} f(\mathbf{A}^n \mathbf{x}) g^*(\mathbf{x})$$

which can be proved using integration by parts.

Since any function on the torus can be expanded in plane waves, we deduce that the correlation function of any two functions on the torus, that are sufficiently smooth for their Fourier expansions (and those of their derivatives) to converge, will show mixing behavior with the same mixing time.

Using Proposition 1, this can be established as follows:

$$C_n(f, g^*) = \lambda_{\pm}^n \int_{\mathbb{T}^2} d^2 \mathbf{x} \left( \sum_{\mathbf{k} \neq (0,0)} c_{\mathbf{k}} \mathbf{k} \cdot \mathbf{u}_{\pm} e^{2\pi\mathbf{i}\mathbf{k} \cdot T^n \mathbf{x}} \right) \left( \sum_{\mathbf{l} \neq (0,0)} d_{\mathbf{l}}^* \frac{1}{\mathbf{l} \cdot \mathbf{u}_{\pm}} e^{-2\pi\mathbf{i}\mathbf{l} \cdot \mathbf{x}} \right) \quad (3.18)$$

We now apply the Cauchy-Schwarz inequality:

$$|\langle f, g^* \rangle|^2 \leq \|f\|^2 \|g^*\|^2 \quad (3.19)$$

to deduce that

$$|C_n(f, g^*)|^2 \leq \rho_{\pm}^{2n} \int_{\mathbb{T}^2} d^2 \mathbf{x} \left| \sum_{\mathbf{k} \neq (0,0)} c_{\mathbf{k}} \mathbf{k} \cdot \mathbf{u}_{\pm} e^{2\pi\mathbf{i}\mathbf{k} \cdot \mathbf{x}} \right|^2 \int_{\mathbb{T}^2} d^2 \mathbf{x} \left| \sum_{\mathbf{l} \neq (0,0)} d_{\mathbf{l}}^* \frac{\mathbf{l} \cdot \mathbf{u}_{\pm}}{e} e^{-2\pi\mathbf{i}\mathbf{l} \cdot \mathbf{x}} \right|^2 \quad (3.20)$$

The integrals do not depend on  $n$ , so the best bound on how  $|C_n(f, g^*)|$  vanishes, as  $n \rightarrow \infty$ , is obtained in the form

$$|C_n(f, g^*)| \sim e^{-n \log \rho_{\pm}} = e^{-n \lambda_{\pm}} \quad (3.21)$$

as  $n \rightarrow \infty$ .

## 4 Multiple mixing for the single Arnol'd cat map

We now proceed to extend the notion of mixing to any number of observables in both the ergodic theory of measure preserving transformations as well as to their corresponding diagnostic

quantifiers of decay of higher order correlation functions of observables. This was pioneered by Rokhlin [18] and has since been the focus of considerable activity.

Let  $T$  be the measure preserving transformation of a dynamical system  $\Gamma$  on a phase space  $\Sigma$ . The generalized 3-strong mixing condition for any three observables or measurable sets  $A$ ,  $B$ , and  $C$  take the form

$$\lim_{m,n \rightarrow \infty} \mu(A \cap T^m B \cap T^{m+n} C) = \mu(A) \mu(B) \mu(C) \quad (4.1)$$

The strong mixing condition for the more general case of  $l$ -fold mixing takes the form

$$\lim_{\alpha_1, \dots, \alpha_k \rightarrow \infty} \mu\left(A_1 \cap T^{\alpha_1} A_2 \dots T^{\sum_{i=1}^{k-1} \alpha_i} A_k\right) = \prod_{i=1}^k \mu(A_i) \quad (4.2)$$

In the equivalent language of correlations functions we define the  $l$ -fold mixing correlation function of  $l+1$  observables,  $f_i(\mathbf{x}), i = 1, 2, \dots, l+1, l = 1, 2, \dots, \mathbf{x} \in \Sigma$  as follows:

$$C_{n_1, n_2, \dots, n_l}(f_1, \dots, f_{l+1}) = \int_{\Sigma} d\mu(\mathbf{x}) f_1(\mathbf{x}) f_2(T^{n_1} \mathbf{x}) f_3(T^{n_1+n_2} \mathbf{x}) \dots f_{l+1}(T^{n_1+n_2+\dots+n_l} \mathbf{x}) - \prod_{i=1}^{l+1} \int_{\Sigma} d\mu(\mathbf{x}) f_i(\mathbf{x}) \quad (4.3)$$

where  $n_1, n_2, \dots, n_l = 1, 2, \dots$

We say that the dynamical system  $(\Gamma, T, \Sigma)$  exhibits  $l$ -fold mixing iff

$$C_{n_1, n_2, \dots, n_l}(f_1, \dots, f_{l+1}) \rightarrow 0 \quad (4.4)$$

as  $(n_1, n_2, \dots, n_l) \rightarrow \infty$ , for observables  $\{f_i(\mathbf{x})\}_{i=1, \dots, l+1}$  which are smooth enough and square integrable, as well as all their derivatives.

We are interested in the case when  $T$  is the Arnol'd cat map and  $\Sigma = \mathbb{T}^2$ , the two-dimensional torus with radii equal to 1 and measure  $d\mu(\mathbf{x}) = d^2\mathbf{x}$ . In this section we shall show that the ACM exhibits  $l$ -fold mixing for every integer  $l = 1, 2, \dots$  and compute the corresponding mixing times, generalizing the calculations of the previous section.

To this end, we expand the  $f_i(\mathbf{x})$  in Fourier series,

$$f_i(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}}^{(i)} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (4.5)$$

and extract the constant part:

$$f_i(\mathbf{x}) = c_{\mathbf{0}}^{(i)} + \tilde{f}_i(\mathbf{x}) \quad (4.6)$$

where

$$c_{\mathbf{0}}^{(i)} = \int_{\mathbb{T}^2} d^2\mathbf{x} f_i(\mathbf{x}) \quad (4.7)$$

and

$$\tilde{f}_i(\mathbf{x}) = \sum_{\mathbf{k} \neq (0,0)} c_{\mathbf{k}}^{(i)} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (4.8)$$

We apply this decomposition only to  $f_1(\mathbf{x})$  and obtain the following expression for  $C_{n_1, n_2, \dots, n_l}(f_1, \dots, f_{l+1})$ :

$$\begin{aligned} C_{n_1, n_2, \dots, n_l}(f_1, \dots, f_{l+1}) &= c_{\mathbf{0}}^{(1)} \int_{\Sigma} d^2 \mathbf{x} f_2(T^{n_1} \mathbf{x}) \cdots f_{l+1}(T^{n_1 + \dots + n_l} \mathbf{x}) + \\ &\int d^2 \mathbf{x} \tilde{f}_1(\mathbf{x}) f_2(T^{n_1} \mathbf{x}) \cdots f_{l+1}(T^{n_1 + \dots + n_l} \mathbf{x}) - \prod_{i=1}^{l+1} c_{\mathbf{0}}^{(i)} \end{aligned} \quad (4.9)$$

The first integral in this expression can be rewritten, upon performing the change of variables,  $\mathbf{x} \rightarrow \mathbf{y} = T^{n_1} \mathbf{x}$ , that leaves the measure invariant, as

$$\begin{aligned} \int_{\Sigma} d^2 \mathbf{x} f_2(T^{n_1} \mathbf{x}) \cdots f_{l+1}(T^{n_1 + \dots + n_l} \mathbf{x}) &= \int_{\Sigma} d^2 \mathbf{x} f_2(\mathbf{x}) \cdots f_{l+1}(T^{n_2 + \dots + n_l} \mathbf{x}) = \\ &C_{n_2, n_3, \dots, n_l}(f_2, f_3, \dots, f_{l+1}) + \prod_{i=2}^{l+1} c_{\mathbf{0}}^{(i)} \end{aligned} \quad (4.10)$$

whence we deduce the ‘‘recurrence relation’’

$$C_{n_1, \dots, n_l}(f_1, f_2, \dots, f_{l+1}) = c_{\mathbf{0}}^{(1)} C_{n_2, n_3, \dots, n_l}(f_2, f_3, \dots, f_{l+1}) + \int d^2 \mathbf{x} \tilde{f}_1(\mathbf{x}) f_2(T^{n_1} \mathbf{x}) \quad (4.11)$$

What is noteworthy is that the product of the constant terms has been eliminated.

If we now assume that  $C_{n_2, \dots, n_l}(f_2, \dots, f_{l+1})$ —the  $l-1$ st order correlation function, tends to 0 as  $n_2, \dots, n_l \rightarrow \infty$ , it remains to prove that the last term in eq. (4.11), also, tends to 0, as  $n_1, n_2, \dots, n_l \rightarrow \infty$ .

To this end we shall employ the method of the previous section, making use of the self-adjoint differential operator,

$$D_{\mathbf{u}} = \frac{1}{2\pi i} \mathbf{u} \cdot \boldsymbol{\partial} \quad (4.12)$$

where  $\boldsymbol{\partial} = (\partial_{x_1}, \partial_{x_2})$  and  $\mathbf{u} = (u_1, u_2)$  is the eigenvector of the ACM that corresponds to the largest eigenvalue of the ACM. The inverse differential operator,  $[D_{\mathbf{u}}]^{-1}$  acts on the complex functions of the torus that don’t have a constant term. This is the reason it’s useful to extract it by writing the function  $f_1(\mathbf{x})$  as  $f_1(\mathbf{x}) = c_{\mathbf{0}}^{(1)} + \tilde{f}_1(\mathbf{x})$ .

We notice, now, that

$$I_{n_1, \dots, n_l} \equiv \int_{\mathbb{T}^2} d^2 \mathbf{x} \tilde{f}_1(\mathbf{x}) f_2(T^{n_1} \mathbf{x}) \cdots f_{l+1}(T^{n_1 + \dots + n_l} \mathbf{x}) = - \int_{\mathbb{T}^2} d^2 \mathbf{x} \left( [D_{\mathbf{u}}]^{-1} \tilde{f}_1(\mathbf{x}) \right) D_{\mathbf{u}}(f_2 \cdots f_{l+1}) \quad (4.13)$$

where

$$D_{\mathbf{u}} f_i(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}}^{(i)} (\mathbf{k} \cdot \mathbf{u}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (4.14)$$



for  $i = 2, \dots, l+1$  and

$$D_{\mathbf{u}}^{-1} \tilde{f}_1(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}}^{(1)} \frac{1}{\mathbf{k} \cdot \mathbf{u}} e^{2\pi i \mathbf{k} \cdot T^n \mathbf{x}} \quad (4.15)$$

as well as

$$D_{\mathbf{u}} f_i(T^n \mathbf{x}) = \lambda^n \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{Z}} c_{\mathbf{k}}^{(i)} (\mathbf{k} \cdot \mathbf{u}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (4.16)$$

We can now evaluate the action of  $D_{\mathbf{u}}$  on the product  $f_2 \cdots f_{l+1}$ ; we find

$$D_{\mathbf{u}} [f_2(T^{n_1+n_2} \mathbf{x}) f_3(T^{n_1+n_2+n_3} \mathbf{x}) \cdots f_{l+1}(T^{n_1+n_2+\cdots+n_{l+1}} \mathbf{x})] = \sum_{m=2}^{l+1} F_{l,m}(\mathbf{x}) D_{\mathbf{u}} f_m(T^{n_1+n_2+\cdots+n_m} \mathbf{x}) \quad (4.17)$$

where

$$F_{l,m}(\mathbf{x}) = \prod_{i=2, i \neq m}^{l+1} f_i(T^{n_1+n_2+\cdots+n_i} \mathbf{x}) \quad (4.18)$$

These imply that eq. (4.19) can be written as

$$D_{\mathbf{u}} [f_2(T^{n_1+n_2} \mathbf{x}) f_3(T^{n_1+n_2+n_3} \mathbf{x}) \cdots f_{l+1}(T^{n_1+n_2+\cdots+n_{l+1}} \mathbf{x})] = \sum_{m=2}^{l+1} F_{lm}(\mathbf{x}) \rho^{n_1+n_2+\cdots+n_m} f'_m(T^{n_1+n_2+\cdots+n_m} \mathbf{x}) \quad (4.19)$$

In order to obtain the desired result, that the integral in eq. (4.13) does vanish—and, what's much more interesting, how does it vanish—in the limit  $n_2, n_3, \dots, n_l \rightarrow \infty$ , we apply the triangle and the Cauchy–Scharz inequalities repeatedly:

$$|I_{n_1, n_2, \dots, n_l}|^2 \leq \|D_{\mathbf{u}}^{-1} \tilde{f}_1\|^2 \|D_{\mathbf{u}} [f_2 \cdots f_{l+1}]\|^2 \leq \|D_{\mathbf{u}}^{-1} \tilde{f}_1\|^2 \sum_{m=2}^{l+1} \rho^{2(n_1+\cdots+n_m)} \|F_{lm}(\mathbf{x})\|^2 \|f'_m\|^2 \quad (4.20)$$

The norm of  $F_{lm}$  can be bounded as

$$\|F_{lm}(\mathbf{x})\|^2 \leq \prod_{i=2, i \neq m}^{l+1} \|f \circ T^{n_1+n_2+\cdots+n_{i-1}} \mathbf{x}\|^2 \quad (4.21)$$

We notice that due to the measure preserving properties of  $T$  the various factors do not depend on  $n_1, n_2, \dots, n_l$ :

$$\|f \circ T^{n_1+n_2+\cdots+n_{i-1}} \mathbf{x}\|^2 = \int d^2 \mathbf{x} |f_i(T^{n_1+n_2+\cdots+n_{i-1}} \mathbf{x})|^2 = \int d^2 \mathbf{x} |f_i(\mathbf{x})|^2 \quad (4.22)$$

The same occurs for the  $\|f'_m\|^2$ .

Collecting together all terms we obtain the effective bound using the eigenvector  $\mathbf{u}_-$ , corresponding to the eigenvalue  $\rho_- < 1$ , therefore

$$|I_{n_1, \dots, n_l}|^2 \leq \sum_{m=1}^{l+1} d_m \rho_-^{2(n_1+n_2+\dots+n_m)} \quad (4.23)$$

where  $d_m$  contain all numerical factors.

To leading order, therefore, we find that

$$|I_{n_1, \dots, n_l}|^2 \rightarrow d_1 \rho_-^{2n_1} = d_1 e^{-2n_1 \log \lambda_+} \quad (4.24)$$

thereby completing the induction hypothesis.

That this term, indeed, vanishes, in the long time limit, guarantees the “sufficient” part of Rokhlin’s conjecture, that 2-mixing induces  $l$ -fold mixing for any  $l > 2$ .

## 5 The mixing time for $n$ symplectically coupled Arnol’d cat maps

It is, now, interesting to examine the mixing properties of the system of  $n > 1$  coupled Arnol’d cat maps. It is known that hyperbolic (Anosov) linear maps on compact phase spaces exhibit strong, as well as  $l$ -fold (for all  $l = 1, 2, \dots$ ), mixing. In this section, therefore, we shall focus on the effects the coupling has on the mixing properties of such maps, generalizing the calculation for the single map, that was studied in section 3 (the case of  $l$ -fold mixing is a straightforward generalization).

The idea is to use the differential operators,

$$D_{\mathbf{u}}^i = \frac{1}{2\pi i} \mathbf{u}^i \cdot \boldsymbol{\partial} = \frac{1}{2\pi i} u_a^i \partial_a \quad (5.1)$$

that are the generalization for  $n$  degrees of freedom of the differential operators (4.12) that were used for computing the mixing time of one cat map. Here  $\partial_a = \partial/\partial \mathbf{x}_a$  and  $a = 1, 2, \dots, 2n$  labels the point on the  $2n$ -dimensional torus.  $\mathbf{u}^i$  is an eigenvector of the evolution operator

$$\mathbf{M} = \begin{pmatrix} I & \mathbf{C} \\ \mathbf{C} & I + \mathbf{C}^2 \end{pmatrix} \quad (5.2)$$

where the symmetric,  $n \times n$  integer matrix  $\mathbf{C}$  is parametrized as

$$\mathbf{C} = K + G(\mathbf{P} + \mathbf{P}^T) \quad (5.3)$$

for the case of the closed chain of  $n$  maps and  $K$  and  $G$  are diagonal matrices.

As for the single map, we are interested in bounding the correlation function

$$C_r(f, g^*) = \int_{\mathbb{T}^{2n}} d^{2n} \mathbf{x} f(M^r \mathbf{x}) g^*(\mathbf{x}) - \int_{\mathbb{T}^{2n}} d^{2n} \mathbf{x} f(\mathbf{x}) \int d^{2n} g^*(\mathbf{x}) \quad (5.4)$$

for  $f, g \in L^2(\mathbb{T}^{2n})$ , which have continuous *and* square integrable partial derivatives.

We proceed as follows: We observe that to get the strong mixing property as well as the rate of convergence (the mixing time), it is enough to use a straightforward generalization of the method used in for the single map, *viz.* we write

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{k} \in \mathbb{Z}^{2n}} c_{\mathbf{k}}^f e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \equiv c_{\mathbf{0}}^f + \tilde{f}(\mathbf{x}) \\ g(\mathbf{x}) &= \sum_{\mathbf{k} \in \mathbb{Z}^{2n}} c_{\mathbf{k}}^g e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \equiv c_{\mathbf{0}}^g + \tilde{g}(\mathbf{x}) \end{aligned} \quad (5.5)$$

whence we, immediately, find that

$$\begin{aligned} C_r(f, g^*) &= c_{\mathbf{0}}^{(g)^*} \int d^{2n} \mathbf{x} f(M^r \mathbf{x}) + \int d^{2n} \mathbf{x} \tilde{g}(\mathbf{x})^* f(M^r \mathbf{x}) - c_{\mathbf{0}}^{(f)} c_{\mathbf{0}}^{(g)^*} = \\ &= c_{\mathbf{0}}^{(g)^*} \int d^{2n} \mathbf{x} \tilde{f}(M^r \mathbf{x}) + \int d^{2n} \mathbf{x} \tilde{g}^*(\mathbf{x}) f(M^r \mathbf{x}) = \int d^{2n} \mathbf{x} \tilde{g}^*(\mathbf{x}) f(M^r \mathbf{x}) \end{aligned} \quad (5.6)$$

Now we carry out the same procedure as for the case of one map, by acting now with the product of all the  $D_u^{(i)}$ ,  $i = 1, \dots, n$ . Upon integrating by parts as for the case of one map, we find, this time, that

$$\begin{aligned} C_r(f, g^*) &= (-)^n \int d^{2n} \mathbf{x} [D_{u_-}^{(1)}]^{-1} \dots [D_{u_-}^{(n)}]^{-1} \tilde{g}(\mathbf{x})^* D_{u_-}^{(1)} \dots D_{u_-}^{(n)} f(M^r \mathbf{x}) = \\ &= (-)^n \int d^{2n} \mathbf{x} \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{\prod_{i=1}^n \mathbf{u}_-^{(i)} \cdot \mathbf{k}} c_{\mathbf{k}}^{g^*} e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} \prod_{i=1}^n \lambda_-^{(i)} \sum_{\mathbf{l} \in \mathbb{Z}^{2n}} c_{\mathbf{l}}^f \mathbf{u}_-^{(i)} \cdot \mathbf{l} e^{2\pi i \mathbf{l} \cdot M^r \mathbf{x}} \end{aligned} \quad (5.7)$$

Using the Cauchy-Schwarz inequality, we obtain the bound

$$|C_r(f, g^*)| \leq \left( \prod_{i=1}^n \rho_-^{(i)} \right)^r \|\tilde{g}\| \|\tilde{f}\| \quad (5.8)$$

where

$$\begin{aligned} \tilde{g} &= \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{\prod_{i=1}^n \mathbf{u}_-^{(i)} \cdot \mathbf{k}} c_{\mathbf{k}}^g e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \\ \tilde{f} &= \sum_{\mathbf{l} \neq \mathbf{0}} \left( \prod_{i=1}^n \mathbf{u}_-^{(i)} \cdot \mathbf{l} \right) c_{\mathbf{l}}^f e^{2\pi i \mathbf{l} \cdot \mathbf{x}} \end{aligned} \quad (5.9)$$

Finally, we observe that the product over the eigenvalues can be written as

$$\left( \prod_{i=1}^n \rho_-^{(i)} \right)^r = e^{-r \sum_{i=1}^n \log \rho_+^{(i)}} = e^{-r S_{K-S}} \quad (5.10)$$

where  $S_{K-S}$  is the Kolmogorov–Sinai entropy of the system of  $n$  coupled maps.

Therefore, we conclude that

$$|C_r(f, g^*)| \sim e^{-rS_{K-S}}, \quad (5.11)$$

as  $r \rightarrow \infty$ , and this allows us to identify the mixing time of the system of  $n$  coupled ACM maps with  $1/S_{K-S}$ :

$$\tau_{\text{mixing}}(\text{ACM lattice}) = \frac{1}{S_{K-S}} \quad (5.12)$$

The steps of this computation can be straightforwardly generalized to the case of so-called “ $l$ -fold mixing”, as was done for the single cat map in the previous section.

These bounds hold even for the case of observables with square integrable first order partial derivatives. In the case that higher order derivatives are square integrable, we can use correspondingly higher powers of the differential operators  $D_{u_-}^{(i)}$  and obtain, correspondingly, better bounds.

## 6 Conclusions and outlook

In this work we have studied, in detail, the large time asymptotic behavior of the correlation functions of observables, that describe the mixing properties of a special class of automorphisms, those of coupled Arnol’d cat maps, which are both symplectic and hyperbolic, on toroidal phase spaces and have computed the mixing time, in closed form. We have shown that the mixing time is given by  $1/S_{K-S}$ , where  $S_{K-S}$  is the Kolmogorov-Sinai entropy.

In the literature numerical studies of nonlinear systems with many degrees of freedom have shown similar dependence of the mixing time on the Kolmogorov-Sinai entropy; in our case we have an analytic result, which can be used to study the problem of relaxation of chaotic systems, subject to localized initial perturbations.

At the classical level it seems that there isn’t any bound on how small the mixing time of the system can be; on the other hand, at the quantum level, studies of many-body quantum systems and the evolution of localized perturbations of black hole horizons seem to indicate that such a bound does, in fact, exist and is given by  $\log S_{\text{BH}}$ , where  $S_{\text{BH}}$  is the black hole entropy. Since the correspondance principle enforces that the short time behavior of the relaxation processes for quantum systems is given by the classical mixing properties of the system, we expect that our work is relevant for understanding the crossover of mixing from the quantum to the classical system.

Our calculation is, also, relevant for the so-called Rokhlin conjecture, for which we clarify the conditions under which it may hold.

Our results allow us to focus on the conditions for physical systems, that can minimize the mixing time, in the classical limit and set the stage for addressing how the mixing time of classical many-body systems is related to the scrambling time of the corresponding quantum systems. This is of relevance for understanding, on the one hand, transport properties of novel quantum materials, as well as the properties of quantum black holes [2, 3, 27]. To this end it is

necessary to construct the corresponding unitary evolution operators for coupled Arnol'd cat maps, going beyond our previous work [9], for the case of the single Arnol'd cat map.

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