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# Kalman filter for radio astronomy dynamic imaging based on empirical covariances

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**Abstract**—Traditional Kalman filters are not immediately suitable for the estimation of state parameters with sample covariance matrices as observations. This comes from the fact that the observations must be first expressed as a linear function of sample covariance matrices. The primary objective of this work is to enable the calculation of both the mean and covariance of the observation model noise, which is here correlated with the state vector. Any signal distribution is considered, revealing the multivariate kurtosis of the signal in the measurement noise covariance matrix. The proposed method is evaluated on simulated data representative of a dynamic radio astronomy framework. The results show that our method is capable of effectively tracking moving sources in complex scenes with theoretical guaranties when the signal multivariate kurtosis is known.

most widespread solution for nonlinear systems is to resort to system linearizations, leading to the so-called linearized or extended KF (EKF) [3]. In both cases, as well as for more advanced techniques such as sigma-point filters [4], the main assumption is a perfect system knowledge [5]: (i) known process and measurement functions, including their parameters, (ii) known inputs and (iii) noise statistics (i.e., first and second order moments for the KF and EKF). Thus, usage of KF may not be possible or of poor performance if one (or more) of the above requirements is not met [6].

This communication tackles the case where an analytical model of the measurement functions has never been established, as in dynamic imagery for radio astronomy where the measurement consists of a sample covariance matrix (SCM) based on a finite number of samples [7]. In the case of actual (asymptotic) covariance matrix as measurement, one would have directly obtained, in accordance with the van Cittert-Zernicke theorem [7], a strictly linear model: the covariance matrix being linked with the state (i.e. the radio astronomical image) using a discrete non-uniform Fourier transform matrix (i.e. the observation system response matrix). But in the considered case, the SCM, related to  $M$  antennas (i.e.  $M$  radio telescopes), cannot be expressed analytically in terms of the parameters of the sample observation model, even when linear, which a priori makes it unsuitable for designing a KF. However, this issue can be circumvented for instantaneous linear observations from multiple sources in the presence of additive noise, when the sources and noise are uncorrelated and the state to be estimated is the power of the sources.

Under the assumption of a deterministic dynamic state model (no state noise) and i.i.d. samples, we propose a statistical linearization method for the measurement model, allowing it to be expressed in terms of the state. This linearization is exact, which introduces an additive residual component, equivalent to a noise measurement, whose covariance matrix depends on the current state, a non-standard case for a state/measurement model. In this case, the covariance matrix of the equivalent noise measurement converges in probability to a limit whose analytic expression involves the normalized kurtosis [8] and then depends on whether the sources' distribution is heavy or light tailed. To support the discussion, the proposed method

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## 1. INTRODUCTION

The design and use of state estimation techniques is fundamental in a plethora of applications, such as robotics, tracking, guidance and navigation systems [1–3]. For a linear dynamic system, the Kalman filter (KF) is the best linear minimum mean square error (MSE) estimator. The

is evaluated on simulated data representative of a dynamic radio astronomy framework.

## 2. MEASUREMENT MODEL

Let us consider a network composed of  $M$  antennas receiving signals at consecutive short time integration (STI) intervals  $[t_k, t_k + \epsilon]$ ,  $k \geq 0$ . During the  $k$ -th STI interval, observations are i.i.d. realizations of a stochastic variable  $\mathbf{z}_k \in \mathbb{C}^{M \times 1}$  consisting of a linear mixture of signals coming from  $Q$  sources,  $\mathbf{s}_k \in \mathbb{C}^{Q \times 1}$ , in the presence of an additive noise  $\mathbf{n}_k$ :

$$\mathbf{z}_k = \mathbf{A}_k \mathbf{s}_k + \mathbf{n}_k, \quad (1)$$

with  $\mathbf{A}_k \in \mathbb{C}^{M \times Q}$  the system response matrix, and  $\mathbf{s}_k$  and  $\mathbf{n}_k$  being independent and centered complex circular random vectors. Especially, we consider the case where we only have access to the sample covariance matrix (SCM)

$$\hat{\mathbf{C}}_{\mathbf{z}_k} = \frac{1}{N} \sum_{n=1}^N \mathbf{z}_k(n) \mathbf{z}_k(n)^H, \quad (2)$$

where  $\mathbf{z}_k(n)$  is the  $n$ -th realization of  $\mathbf{z}_k$ ,  $N$  is the number of samples and the subscript  $H$  denotes the hermitian operator, i.e.  $\mathbf{z}_k^H \equiv (\mathbf{z}_k^*)^T$ . Asymptotically, i.e. when  $N$  tends to infinity,  $\hat{\mathbf{C}}_{\mathbf{z}_k}$  converges in probability to

$$\mathbf{C}_{\mathbf{z}_k} = \mathbb{E} [\mathbf{z}_k \mathbf{z}_k^H] = \mathbf{A}_k \mathbb{E} [\mathbf{s}_k \mathbf{s}_k^H] \mathbf{A}_k^H + \mathbf{C}_{\mathbf{n}_k}. \quad (3)$$

In practice,  $\mathbf{A}_k$  is known and  $\mathbf{C}_{\mathbf{n}_k}$  is known or can be measured at desired precision.

Assuming that all  $\mathbf{n}_k$  and  $\mathbf{s}_l$  for  $k, l \geq 1$  are independent and that the source signals  $s_k^q$ ,  $q \in [1, \dots, N]$ , are mutually independent, the goal is to estimate instantaneous intensities  $\mathbb{E}[\mathbf{s}_k \mathbf{s}_k^H] \equiv \text{diag}(\mathbf{x}_k)$  subject to

$$\mathbf{x}_k = \mathbf{F}_{k-1} \mathbf{x}_{k-1}, \quad (4)$$

where the state-transition matrix  $\mathbf{F}_{k-1}$  is known a priori.

### Linear discrete state-space model

In order to derive a Kalman filter for the estimation of  $\mathbf{x}_k \in \mathbb{R}_+^Q$ , we propose to look for a linear model based on (3), i.e. to apply a statistical linear fitting model as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k, \quad (5)$$

where measurements are the vectorized SCMs, which are concatenated with their complex conjugates for the sake of optimality, i.e.

$$\mathbf{y}_k = \begin{pmatrix} \text{vec}(\hat{\mathbf{C}}_{\mathbf{z}_k}) \\ \text{vec}(\hat{\mathbf{C}}_{\mathbf{z}_k}^*) \end{pmatrix}. \quad (6)$$

The observation models are

$$\mathbf{H}_k = \begin{bmatrix} \mathbf{A}_k^* * \mathbf{A}_k \\ \mathbf{A}_k * \mathbf{A}_k^* \end{bmatrix}, \quad (7)$$

with  $*$  being the column-wise Kronecker product, also called the Khatri–Rao product [9].

The observation residual is defined as

$$\mathbf{v}_k \equiv \mathbf{y}_k - \mathbf{H}_k \mathbf{x}_k. \quad (8)$$

The quantities of interest  $\mathbf{x}_k$  are involved in the observation noise model (8), which corresponds to a non-standard linear discrete state space (LDSS) model for which a Kalman filter is derived in the following section.

## 3. DESIGN OF THE KALMAN FILTER

### Kalman filter existence

For such a LDSS model, without state noise, one needs to verify that

$$\mathbf{C}_{\mathbf{v}_k, \mathbf{y}_l} = \mathbf{0}, \quad k \geq 2, \quad l < k, \quad (9)$$

in order to prove the existence of a Kalman filter [5]. The verification is straightforward since  $\mathbf{y}_k$  and  $\mathbf{y}_l$  are independent and  $\mathbf{x}_k$  is deterministic, i.e.

$$\mathbf{C}_{\mathbf{y}_k - \mathbf{H}_k \mathbf{x}_k, \mathbf{y}_l} = \mathbf{0}. \quad (10)$$

### Identification of first and second order statistics

The next step consists in evaluating the first and second order statistics of the observation noise.

*Mean of the observation noise*—Since  $\hat{\mathbf{C}}_{\mathbf{z}_k}$  is an unbiased estimate of  $\mathbf{C}_{\mathbf{z}_k}$ , one has that

$$\begin{aligned} \mathbb{E}[\mathbf{y}_k] &= \left[ \text{vec}(\mathbf{C}_{\mathbf{z}_k})^T, \text{vec}(\mathbf{C}_{\mathbf{z}_k}^*)^T \right]^T \\ &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k^a, \end{aligned} \quad (11)$$

where

$$\mathbf{v}_k^a = \left[ \text{vec}(\mathbf{C}_{\mathbf{n}_k})^T, \text{vec}(\mathbf{C}_{\mathbf{n}_k}^*)^T \right]^T. \quad (12)$$

From (8), one obtains that  $\mathbb{E}[\mathbf{v}_k] = \mathbb{E}[\mathbf{y}_k] - \mathbf{H}_k \mathbf{x}_k$  and therefore

$$\mathbb{E}[\mathbf{v}_k] = \mathbf{v}_k^a. \quad (13)$$

*Covariance of the observation noise*—As  $\mathbf{x}_k$  is deterministic, one obtains  $\mathbf{C}_{\mathbf{v}_k} = \mathbf{C}_{\mathbf{y}_k}$ , with

$$\mathbf{C}_{\mathbf{y}_k} = \frac{1}{N} \mathbf{C} \begin{pmatrix} \mathbf{z}_k^* \otimes \mathbf{z}_k \\ \mathbf{z}_k \otimes \mathbf{z}_k^* \end{pmatrix}, \quad (14)$$

where  $\otimes$  is the usual Kronecker product, such that  $\mathbf{z}_k^* \otimes \mathbf{z}_k = \text{vec}(\mathbf{z}_k \mathbf{z}_k^H)$ . Finally, one has that

$$\mathbf{C}_{\mathbf{v}_k} = \frac{1}{N} \begin{bmatrix} \mathbf{C}_{\mathbf{z}_k^* \otimes \mathbf{z}_k} & \mathbf{C}_{\mathbf{z}_k^* \otimes \mathbf{z}_k, \mathbf{z}_k \otimes \mathbf{z}_k^*} \\ \mathbf{C}_{\mathbf{z}_k^* \otimes \mathbf{z}_k, \mathbf{z}_k \otimes \mathbf{z}_k^*} & \mathbf{C}_{\mathbf{z}_k \otimes \mathbf{z}_k^*} \end{bmatrix}. \quad (15)$$

For source signals  $\mathbf{s}_k$  following any distributions and Gaussian noises  $\mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\mathbf{n}_k})$ , the normalized kurtosis of the signals appears in expression (15). In particular, after dropping the index  $k$  in order to shorten expressions, (15) simplifies from identities:

$$\begin{aligned} \mathbf{C}_{\mathbf{z}^* \otimes \mathbf{z}} &= \mathbf{C}_{\mathbf{z}}^T \otimes \mathbf{C}_{\mathbf{z}} \\ &+ \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^H \rho_q x_q^2, \end{aligned} \quad (16)$$

$$\mathbf{C}_{\mathbf{z}^* \otimes \mathbf{z}, \mathbf{z} \otimes \mathbf{z}^*} = (\mathbf{C}_{\mathbf{z}}^T \otimes \mathbf{C}_{\mathbf{z}}) \mathbf{P} + \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^T \rho_q x_q^2 \quad (17)$$

where  $x_q$  is the  $q$ -th coordinate of  $\mathbf{x}$ ,  $\mathbf{a}_q$  is the  $q$ -th column of  $\mathbf{A}$ ,

$$\mathbf{P} = \sum_{m=1}^M \sum_{m'=1}^M (\mathbf{e}_m \otimes \mathbf{e}_{m'}) (\mathbf{e}_{m'} \otimes \mathbf{e}_m)^T, \quad (18)$$

with  $(\mathbf{e}_1, \dots, \mathbf{e}_M)$  the canonical basis of  $\mathbb{R}^M$ , and

$$\rho_q = \frac{\mathbb{E} \left[ (s_q^* s_q)^2 \right]}{x_q^2} - 2 \quad (19)$$

is the normalized multivariate kurtosis of  $s_q$  [8]. Sketch of proof of (16) and (17) are given in Appendices A and B.

For instance, for  $s_q$  such that  $Re(s_q)$  and  $Im(s_q)$  are independent and follows the same distribution  $\mathcal{D}$ ,

$$\rho_q = \frac{1}{2} (\text{kurtosis}(\mathcal{D}) - 3). \quad (20)$$

Since the kurtosis of a univariate real-valued Gaussian random variable is 3,  $\rho_q = 0$  for Gaussian source signals, but in general  $\rho_q \neq 0$ .

#### Kalman filter recursion

KF gives the best linear unbiased estimator (BLUE) in the MSE sense. At iteration  $k$ , the current estimate writes

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}), \quad (21)$$

where state and measurement predictions are

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}, \quad \hat{\mathbf{y}}_{k|k-1} = \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1} + \mathbf{v}_k^a, \quad (22)$$

and the optimal Kalman gain  $\mathbf{K}_k$  is computed with the recursion

$$\begin{cases} \mathbf{P}_{k|k-1} &= \mathbf{F}_{k-1} \mathbf{P}_{k-1|k-1} \mathbf{F}_{k-1}^H \\ \mathbf{S}_{k|k-1} &= \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^H + \hat{\mathbf{C}}_{\mathbf{v}_k} \\ \mathbf{K}_k &= \mathbf{P}_{k|k-1} \mathbf{H}_k^H (\mathbf{S}_{k|k-1})^{-1} \\ \mathbf{P}_{k|k} &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1} \end{cases} \quad (23)$$

where  $\mathbf{K}_k$  is the optimal Kalman gain,  $\mathbf{P}_{k|k-1}$ ,  $\mathbf{P}_{k|k}$  are respectively the predicted and a posteriori estimate error covariance matrix and  $\mathbf{S}_{k|k-1}$  is the innovation covariance matrix.

#### Kalman filter initialization

KF formalism supposes that the first and second order moments of  $\mathbf{x}_0$  and  $\mathbf{v}_k$  for  $k \geq 0$  are known, which in the case of a deterministic state vector  $\mathbf{x}_k$  amounts to know  $\mathbf{x}_0$ . However, in practice, the filter must be initialized with an estimate  $\hat{\mathbf{x}}_{0|0}$  and its corresponding mean square error  $\mathbf{P}_{0|0} = \mathbf{C}_{\hat{\mathbf{x}}_{0|0}}$ .

A plethora of estimators can be used [10], provided that the associated covariance is known. Among them, a linear unbiased estimator  $\hat{\mathbf{x}}_{0|0}$  is of particular interest since the unbiased property is conserved by the KF. It can be obtained

by a Distortionless Response Filter (DRF)  $\mathbf{K}_0$  verifying  $\hat{\mathbf{K}}_0 \mathbf{H}_0 = \mathbf{I}_Q$ , which would give

$$\hat{\mathbf{x}}_{0|0} = \mathbf{K}_0 (\mathbf{y}_0 - \mathbb{E}[\mathbf{y}_0]) \text{ and } \mathbf{P}_{0|0} = \mathbf{K}_0 \mathbf{C}_{\mathbf{v}_0} \mathbf{K}_0^H. \quad (24)$$

Among the DRFs, the Minimum Variance Distortionless Filter (MVDRF)

$$\mathbf{K}_0 = (\mathbf{H}_0^H \mathbf{C}_{\mathbf{v}_0}^{-1} \mathbf{H}_0)^{-1} \mathbf{H}_0^H \mathbf{C}_{\mathbf{v}_0}^{-1}, \quad (25)$$

which minimizes the covariance (hence the MSE), is used in this work.

#### Practical implementation and performance lower bound

It is clear from (24) that at the initialization step any DRF implementation supposes to know the observation noise covariance matrix  $\mathbf{C}_{\mathbf{v}_0}$ , which, due to the unconventional construction of the KF, depends on the current state  $\mathbf{x}_0$ . This also applies at each recursion step, for which  $\mathbf{C}_{\mathbf{v}_k}$  is supposed to be known. Hence, the KF must be computed by substituting  $\mathbf{C}_{\mathbf{v}_k}$  by an estimate  $\hat{\mathbf{C}}_{\mathbf{v}_k}$ . Indeed, it is suboptimal with respect to the MSE, i.e. compared to what can ideally achieve the KF when  $\mathbf{C}_{\mathbf{v}_k}$  is known. This implies that the considered innovation covariance matrices in (23) are estimates.

In the following results, at initialization, the observation noise covariance matrix  $\hat{\mathbf{C}}_{\mathbf{v}_0}$  injected in the MVDRF initialization is built with a beamforming estimator

$$\hat{\mathbf{x}}_{0|0}^{\text{BF}} = \text{diag} \left( \mathbf{A}_k^H (\hat{\mathbf{C}}_{\mathbf{z}_0} - \mathbf{C}_{\mathbf{n}_0}) \mathbf{A}_k \right) \odot \text{diag} (\mathbf{A}_k^* \mathbf{A}_k)^{-1} \quad (26)$$

where  $\odot$  is the Hadamar product. Since  $(\hat{\mathbf{x}}_{0|0}^{\text{BF}})_q \geq (\mathbf{x}_0)_q$  for  $q = 1, \dots, Q$  by construction, one assumes that  $\hat{\mathbf{C}}_{\mathbf{v}_0} (\hat{\mathbf{x}}_{0|0}^{\text{BF}}) \geq \mathbf{C}_{\mathbf{v}_0}$ , which implies that the performance of the KF are upper bounded by  $\mathbf{P}_{0|0}$  [11] (i.e. the KF estimate  $\mathbf{P}_{0|0}$  is pessimistic).

At recursion  $k \geq 1$ , the current measurement noise covariance estimate is  $\hat{\mathbf{C}}_{\mathbf{v}_k} (\pi_{\mathbb{R}_+^Q} (\hat{\mathbf{x}}_{k|k-1}))$ , where  $\pi_{\mathbb{R}_+^Q}$  is the projector on  $\mathbb{R}_+^Q$ . The projector is required in order to conserve the structure of the estimated measurement noise covariance matrix since there is no non-negativity constraint on the estimation<sup>3</sup>. Likewise, estimation  $\hat{\mathbf{x}}_{k|k}$  fluctuates around the mean value  $\mathbf{x}_k$ , such that the estimation of the KF can include negative intensities. Typically, this corresponds to actual low intensities and those negative estimates are kept in the recursion. The projection is only applied on KF predictions which are used to construct  $\hat{\mathbf{C}}_{\mathbf{v}_k}$ .

An ideal KF, which corresponds to the best achievable performance, is given by considering the actual measurement noise covariance  $\hat{\mathbf{C}}_{\mathbf{v}_k} (\mathbf{x}_k) \equiv \mathbf{C}_{\mathbf{v}_k}$ . The error covariance matrix of the ideal KF is denoted  $\mathbf{P}_{k|k}^{\text{a}}$ .

In order to analyze the statistical behavior of the filter, the performance obtained by applying the designed KF on the

<sup>2</sup>While the proof of this result is not done yet, it is assumed that the result applies as a generalization of  $\hat{\mathbf{C}}_{\mathbf{z}_0} (\hat{\mathbf{x}}_{0|0}^{\text{BF}}) \geq \mathbf{C}_{\mathbf{z}_0}$ . The proof is let for future developments.

<sup>3</sup>A projection on  $\mathbb{R}_+^Q$  should also be applied on  $\hat{\mathbf{x}}_{0|0}^{\text{BF}}$ , but in practice one directly have that  $\hat{\mathbf{x}}_{0|0}^{\text{BF}} \in \mathbb{R}_+^Q$ .



actual signal model

$$\mathbf{P}_{k|k}^p = \mathbf{C}_{\mathbf{x}_k - \hat{\mathbf{x}}_k|k} \quad (27)$$

can be estimated by Monte Carlo simulations. In particular, one obtains that  $\mathbf{P}_{0|0}^a \leq \mathbf{P}_{0|0}^p \leq \mathbf{P}_{0|0}$ , which is not necessarily verified for  $k > 0$ .

#### 4. APPLICATION TO DYNAMIC RADIO ASTRONOMY IMAGING

In order to achieve a satisfactory resolution, modern radio astronomy observatories consist of a network of  $M$  antennas, possibly scattered around the globe [12, 13]. Considering their respective positions  $\mathbf{r}_m \in \mathbb{R}^2$ , wavelength  $\lambda$  and gain function  $g_m(\cdot)$  for  $m = 1, \dots, M$ , the transfer function associated to a band-limited signal modulated by a carrier frequency  $f_c = c/\lambda$  is given by

$$\mathbf{a}(\mathbf{u}) = \left( g_1(\mathbf{u}) e^{2j\pi \frac{\mathbf{r}_1^T \mathbf{u}}{\lambda}}, \dots, g_M(\mathbf{u}) e^{2j\pi \frac{\mathbf{r}_M^T \mathbf{u}}{\lambda}} \right)^T, \quad (28)$$

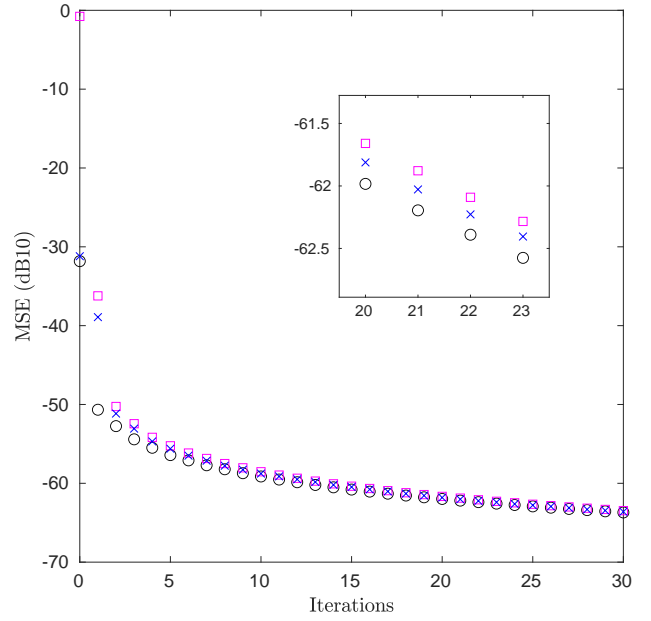
where  $\mathbf{u} \in \mathbb{R}^2$  is the direction of arrival, i.e. the unit vector oriented towards the source of the signal, and  $c$  is the speed of light in a vacuum. Given an image of  $Q$  pixels, there are respectively  $Q$  direction of arrivals  $\mathbf{u}_q$  defining the array response matrix  $\mathbf{A} = (\mathbf{a}(\mathbf{u}_1), \dots, \mathbf{a}(\mathbf{u}_Q))$  [7, 14]. During the  $k$ -th STI interval, signals from the different sources are complex circular, mutually independent. Hereinafter, they are concatenated in a vector  $\mathbf{s}_k$ . The signal received by the network of antennas  $\mathbf{z}_k$  is modeled as (1), where  $\mathbf{n}_k$  is the measurement noise of the interferometer, which is complex circular, Gaussian distributed and independent of the source signals  $\mathbf{s}_k$ . The observed visibility matrix, i.e. the SCM  $\hat{\mathbf{C}}_{\mathbf{z}_k}$ , is computed as in (2) from  $N$  independent and identically distributed realizations of  $\mathbf{z}_k$ .

The proposed formalization of the KF provides an estimate of the time-varying power of the sources based on  $\hat{\mathbf{C}}_{\mathbf{z}_{k'}}$  measurements for  $0 \leq k' \leq k$  in the case of a deterministic state model (4) and which can be computed iteratively.

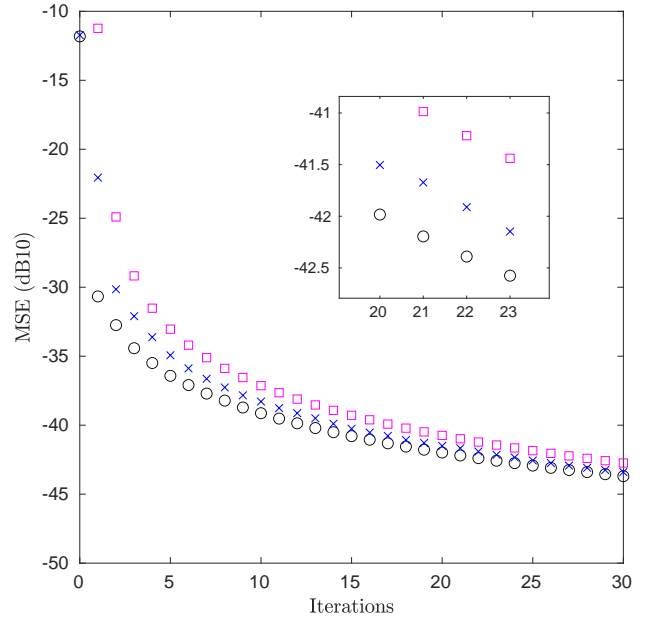
##### Results

The considered network corresponds to the Karl G. Jansky Very Large Array (VLA) observatory composed of 27 independent radio telescopes divided in three branches and located in New Mexico [13]. The method evaluation is performed on a synthetic image composed of  $Q = 22 \times 22$  pixels, with a state-transition model  $\mathbf{F}_k$  being a rotation matrix which corresponds to a rotation of the image by a fixed angle of 90 deg between each STI interval. Without loss of generality, all the gains  $g_m$  are considered equal to 1. As typically admitted in radio astronomy [7],  $N = 10^5$  samples of bivariate signals  $\mathbf{s}_k$  and Gaussian noises  $\mathbf{n}_k$  are considered for the SCM construction. A constant normalized noise covariance matrix is considered as  $\mathbf{C}_{\mathbf{n}_k} \equiv \mathbf{I}_M$ . Signals have independent real and imaginary parts following a Laplace distribution, i.e.  $\rho_q = 3/2$ .

Performances of the KF are evaluated in terms of estimation MSE and illustrated in Figure 1. The assumed KF performance obtained from (23) trace  $(\mathbf{P}_{k|k})$ , the true KF performance trace  $(\mathbf{P}_{k|k}^p)$  (computed by Monte Carlo simulations) and lower bound on KF performance trace  $(\mathbf{P}_{k|k}^a)$



(a)

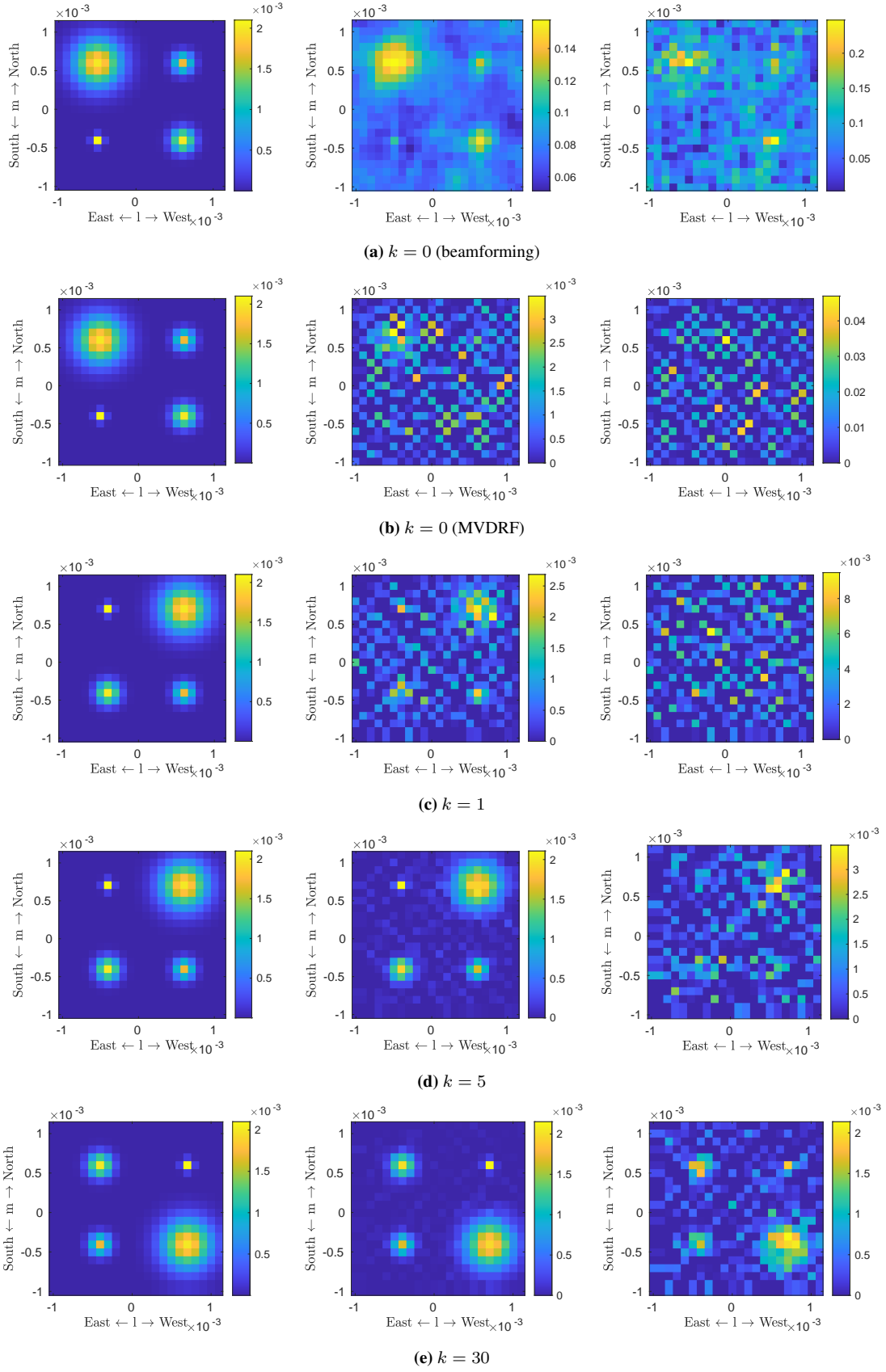


(b)

**Figure 1:** MSE lower bound (black circles), MSE predicted by KF (pink boxes) and MSE actually achieved (blue crosses) of the proposed FK, for (a)  $N = 10^5$  and (b)  $N = 10^3$  samples.

(computed by KF with  $\hat{\mathbf{C}}_{\mathbf{v}_k}(\mathbf{x}_k) \equiv \mathbf{C}_{\mathbf{v}_k}$ ) are given with respect to the number of iterations. It shows that the assumed and true KF performances, i.e. trace  $(\mathbf{P}_{k|k})$  and trace  $(\mathbf{P}_{k|k}^p)$  respectively, converge towards each other along time. As mentioned before, there is no formal proof that the true KF performance is bounded by the assumed performance, although it is the case for the presented results.

Actual images and KF estimates are displayed in Figure 2. As the KF has no non-negativity constraint, some estimated pixels may take negative values. In practice, a threshold must



**Figure 2:** Actual images (left) and KF estimates  $\pi_{\mathbb{R}^Q}(\hat{\mathbf{x}}_{k|k})$  at  $k = 0, 1, 5$  and  $30$  (**(b)**, **(c)**, **(d)** and **(e)** respectively) for  $N = 10^5$  (middle) and  $N = 10^3$  (right) samples. The beamforming estimation **(a)** is used to initialize  $\hat{\mathbf{C}}_{\mathbf{v}_0}$  for the MVDRF estimation **(b)**.

be set on the estimated image to select “physical”/admissible pixels or pixels with sufficiently high values, before using estimates as such for other purpose. Experimentation shows that a projection of  $\hat{\mathbf{x}}_{k|k}$  on  $\mathbb{R}_+^Q$  at each step  $k$  decreases the performance of the KF as it introduces a bias.

Experimentation shows that the method works for smaller sample size  $N$  (but the performance decreases) even if the limits are not analytically established. Figures 1 and 2 present the results for  $N = 10^3$ . Notably, the KF can reach the same performance with a lower sample size but with an increasing iteration number. For  $N = 10^5$ , the true KF MSE is below  $-50$  db10 for  $k \geq 3$ , while it is for  $k \geq 130$  with  $N = 10^3$ .

## 5. CONCLUSION AND PERSPECTIVES

This work presents a formalization of the Kalman filter for the dynamic estimation of signal source power based on empirical covariance measurements, specifically for any signal distribution. It shows that the observation noise covariance matrix expression involves the multivariate kurtosis of the source signals. An application on simulated data representative of a dynamic radio astronomy imaging framework was presented, highlighting the applicability of the proposed filter. Results show that, knowing the kurtosis of the source signals, one can compute a lower performance bound towards which the filter performance converges along iterations. While such configuration is not always realistic, i.e. no state noise and a known permutation matrix as state-transition model, it illustrates the applicability of the proposed filter as well as its performance in a simplified case for which standard KF formalism does not apply. The cases of any noise distribution, unknown state-transition model and additive state noise, as well as the limit of applicability with respect to the sample size  $N$ , is left for future work.

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## APPENDICES

### A. VECTORIZED SCM COVARIANCE

This section provides expression of  $\mathbf{C}_{\mathbf{z}_k^* \otimes \mathbf{z}_k}$  in order to implement (14). To proceed, one starts to write

$$\mathbf{C}_{\mathbf{z}^* \otimes \mathbf{z}} = \mathbf{R}_{\mathbf{z}^* \otimes \mathbf{z}} - \mathbf{m}_{\mathbf{z}^* \otimes \mathbf{z}} \mathbf{m}_{\mathbf{z}^* \otimes \mathbf{z}}^H, \quad (29)$$

where the subscript  $k$  is dropped in order to simplify expressions, with

$$\mathbf{R}_{\mathbf{z}^* \otimes \mathbf{z}} = \mathbb{E} \left[ (\mathbf{z}^* \otimes \mathbf{z}) (\mathbf{z}^* \otimes \mathbf{z})^H \right], \quad (30)$$

and

$$\mathbf{m}_{\mathbf{z}^* \otimes \mathbf{z}} = \mathbb{E} [\mathbf{z}^* \otimes \mathbf{z}]. \quad (31)$$

The latter is developed from the general form of  $\mathbf{z}$

$$\mathbf{z} = \sum_{q=1}^Q \mathbf{a}_q s_q + \mathbf{n} \quad (32)$$

as

$$\mathbf{m}_{\mathbf{z}^* \otimes \mathbf{z}} = \mathbf{vec}(\mathbf{C}_{\mathbf{z}}), \quad (33)$$

where

$$\begin{aligned} \mathbf{vec}(\mathbf{C}_{\mathbf{z}}) &= \mathbf{vec} \left( \sum_{q=1}^Q x_q \mathbf{a}_q \mathbf{a}_q^H + \mathbf{vec}(\mathbf{C}_{\mathbf{n}}) \right) \\ &= \sum_{q=1}^Q x_q \mathbf{a}_q^* \otimes \mathbf{a}_q + \mathbf{vec}(\mathbf{C}_{\mathbf{n}}). \end{aligned} \quad (34)$$

The former is developed from

$$\begin{aligned} \mathbf{z}^* \otimes \mathbf{z} &= \sum_{q=1}^Q \sum_{q'=1}^Q \mathbf{a}_q^* \otimes \mathbf{a}_{q'} (s_q^* s_{q'}) + \mathbf{n}^* \otimes \mathbf{n} \\ &\quad + \sum_{q'=1}^Q (s_{q'} \mathbf{n}^*) \otimes \mathbf{a}_{q'} + \sum_{q=1}^Q \mathbf{a}_q^* \otimes (s_q \mathbf{n}), \end{aligned} \quad (35)$$

which, given that  $\mathbf{s}$ ,  $\mathbf{n}$  are independent and composed of respectively  $Q$  and  $M$  circular random variables, leads to

$$\begin{aligned} \mathbf{R}_{\mathbf{z}^* \otimes \mathbf{z}} &= \sum_{q=1}^Q \sum_{q'=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_q^* \otimes \mathbf{a}_{q'})^H x_q x_{q'} \\ &\quad + \sum_{q=1}^Q \sum_{q'=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_{q'}^* \otimes \mathbf{a}_{q'})^H x_q x_{q'} \\ &\quad + \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^H \left( \mathbb{E} \left[ (s_q^* s_q)^2 \right] - 2x_q^2 \right) \\ &\quad + \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \otimes \mathbf{a}_q \right) \mathbf{vec}(\mathbf{C}_{\mathbf{n}})^H \\ &\quad + \left( \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \otimes \mathbf{a}_q \right) \mathbf{vec}(\mathbf{C}_{\mathbf{n}})^H \right)^H \\ &\quad + \mathbf{C}_{\mathbf{n}}^* \otimes \left( \sum_{q=1}^Q x_q \mathbf{a}_q \mathbf{a}_q^H \right) + \left( \sum_{q=1}^Q x_q \mathbf{a}_q \mathbf{a}_q^H \right)^* \otimes \mathbf{C}_{\mathbf{n}} \\ &\quad + \mathbb{E} \left[ (\mathbf{n}^* \otimes \mathbf{n}) (\mathbf{n}^* \otimes \mathbf{n})^H \right]. \end{aligned} \quad (36)$$

Since  $\mathbf{n}$  is Gaussian, one obtains that  $\mathbf{C}_{\mathbf{n}^* \otimes \mathbf{n}} = \mathbf{C}_{\mathbf{n}}^T \otimes \mathbf{C}_{\mathbf{n}}$  (see Eq. (21) of [15]), and then

$$\begin{aligned} \mathbf{C}_{\mathbf{z}^* \otimes \mathbf{z}} &= \mathbf{R}_{\mathbf{z}^* \otimes \mathbf{z}} - \mathbf{vec}(\mathbf{C}_{\mathbf{z}}) \mathbf{vec}(\mathbf{C}_{\mathbf{z}})^H \\ &= \sum_{q=1}^Q \sum_{q'=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_q^* \otimes \mathbf{a}_{q'})^H x_q x_{q'} \\ &\quad + \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^H \rho_q x_q^2 \\ &\quad + \mathbf{C}_{\mathbf{n}}^* \otimes \left( \sum_{q=1}^Q \sigma_q^2 \mathbf{a}_q \mathbf{a}_q^H \right) \\ &\quad + \left( \sum_{q=1}^Q \sigma_q^2 \mathbf{a}_q \mathbf{a}_q^H \right)^* \otimes \mathbf{C}_{\mathbf{n}} + \mathbf{C}_{\mathbf{n}}^T \otimes \mathbf{C}_{\mathbf{n}}. \end{aligned} \quad (37)$$

On the other hand, one may remark that

$$\begin{aligned} \mathbf{C}_z^T \otimes \mathbf{C}_z &= \sum_{q=1}^Q \sum_{q'=1}^{Q'} x_q x_{q'} (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_q^* \otimes \mathbf{a}_{q'})^H \\ &+ \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \mathbf{a}_q^T \right) \otimes \mathbf{C}_n \\ &+ \mathbf{C}_n^T \otimes \left( \sum_{q=1}^Q x_q \mathbf{a}_q \mathbf{a}_q^H \right) + \mathbf{C}_n^T \otimes \mathbf{C}_n \end{aligned} \quad (38)$$

from which the final result is deduced, i.e.

$$\begin{aligned} \mathbf{C}_{z^* \otimes z, z \otimes z^*} &= \mathbf{C}_z^T \otimes \mathbf{C}_z \\ &+ \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^H \rho_q x_q^2. \end{aligned} \quad (39)$$

## B. VECTORIZED SCM PSEUDO-COVARIANCE

This section outlines the proof of (17), which follows the same structure as Appendix A. Starting from

$$\mathbf{C}_{z^* \otimes z, z \otimes z^*} = \mathbf{R}_{z^* \otimes z, z \otimes z^*} - \text{vec}(\mathbf{C}_z) \text{vec}(\mathbf{C}_z)^T, \quad (40)$$

we suppose that (33) and (34) are known. From (35), one obtains that

$$\begin{aligned} \mathbf{R}_{z^* \otimes z, z \otimes z^*} &= \sum_{q=1}^Q \sum_{q'=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_{q'}^* \otimes \mathbf{a}_{q'})^T x_q x_{q'} \\ &+ \sum_{q=1}^Q \sum_{q'=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_{q'}^* \otimes \mathbf{a}_q)^T x_q x_{q'} \\ &+ \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^T \rho_q x_q^2 \\ &+ \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \otimes \mathbf{a}_q \right) \text{vec}(\mathbf{C}_n)^T \\ &+ \text{vec}(\mathbf{C}_n) \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \otimes \mathbf{a}_q \right)^T \\ &+ \sum_{q=1}^Q x_q E \left[ (\mathbf{n}^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{n})^T \right] \\ &+ \sum_{q=1}^Q x_q E \left[ (\mathbf{a}_q^* \otimes \mathbf{n}) (\mathbf{n}^* \otimes \mathbf{a}_q)^T \right] \\ &+ E \left[ (\mathbf{n}^* \otimes \mathbf{n}) (\mathbf{n}^* \otimes \mathbf{n})^T \right], \end{aligned} \quad (41)$$

which leads to

$$\begin{aligned} \mathbf{C}_{z^* \otimes z, z \otimes z^*} &= \sum_{q=1}^Q \sum_{q'=1, q' \neq q}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_{q'}^* \otimes \mathbf{a}_q)^T x_q x_{q'} \\ &+ \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^T \rho_q x_q^2 \\ &+ \sum_{q=1}^Q x_q E \left[ (\mathbf{n}^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{n})^T \right] \\ &+ \sum_{q=1}^Q x_q E \left[ (\mathbf{a}_q^* \otimes \mathbf{n}) (\mathbf{n}^* \otimes \mathbf{a}_q)^T \right] \\ &+ \mathbf{C}_{n^* \otimes n, n \otimes n^*}. \end{aligned} \quad (42)$$

After a few developments, noticing that

$$\begin{aligned} \mathbf{C}_z^T \otimes \mathbf{C}_z &= \sum_{q=1}^Q \sum_{q'=1}^Q x_q x_{q'} (\mathbf{a}_q^* \otimes \mathbf{a}_{q'}) (\mathbf{a}_q^* \otimes \mathbf{a}_{q'})^H \\ &+ \left( \sum_{q=1}^Q x_q \mathbf{a}_q^* \mathbf{a}_q^T \right) \otimes \mathbf{C}_n \\ &+ \mathbf{C}_n^T \otimes \left( \sum_{q=1}^Q x_q \mathbf{a}_q \mathbf{a}_q^H \right) + \mathbf{C}_n^T \otimes \mathbf{C}_n, \end{aligned} \quad (43)$$

and

$$(\mathbf{a}_q^* \otimes \mathbf{a}_{q'})^H \mathbf{P} = (\mathbf{a}_{q'}^* \otimes \mathbf{a}_q)^T, \quad (44)$$

this results in

$$\begin{aligned} \mathbf{C}_{z^* \otimes z, z \otimes z^*} &= \left( \mathbf{C}_{z|x_k}^T \otimes \mathbf{C}_{z|x_k} \right) \mathbf{P} \\ &+ \sum_{q=1}^Q (\mathbf{a}_q^* \otimes \mathbf{a}_q) (\mathbf{a}_q^* \otimes \mathbf{a}_q)^T \rho_q x_q^2, \end{aligned} \quad (45)$$

where we used that  $\mathbf{C}_{n^* \otimes n, n \otimes n^*} = (\mathbf{C}_n^T \otimes \mathbf{C}_n) \mathbf{P}$  for a Gaussian noise  $\mathbf{n}$ .

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