The norm functor over schemes

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Abstract: We construct a globalization of Ferrand's norm functor over rings which generalizes it to the setting of a finite locally free morphism of schemes $T \to S$ of constant rank. It sends quasi-coherent modules over T to quasi-coherent modules over S. These functors restrict to the category of quasi-coherent algebras. We also assemble these functors into a norm morphism from the stack of quasi-coherent modules over a finite locally free of constant rank extension of the base scheme into the stack of quasi-coherent modules. This morphism also restricts to the analogous stacks of algebras. Restricting our attention to finite étale covers, we give a cohomological description of the norm morphism in terms of the Segre embedding. Using this cohomological description, we show that the norm gives an equivalence of stacks of algebras $A_1^2 \equiv D_2$, akin to the result shown in The Book of Involutions.

Keywords: Algebraic Groups, Azumaya algebras, Exceptional Isomorphism, Norm functor *MSC 2020: 16H05, 14F20, 20G10, 20G35*

Contents

Date: January 23, 2024.

The first author was supported by the project "Group schemes, root systems, and related representations" founded by the European Union - NextGenerationEU through Romania's National Recovery and Resilience Plan (PNRR) call no. PNRR-III-C9-2023-I8, Project CF159/31.07.2023, and coordinated by the Ministry of Research, Innovation and Digitalization (MCID) of Romania. The research of the second author was partially supported by an NSERC grant. The research of the third author was partially supported by the NSERC grants of the second author, Kirill Zainoulline, Mikhail Kotchetov, and Yorck Sommerhäuser.

Introduction

One of the coincidences in the theory of algebraic groups is the exceptional isomorphism between the Dynkin diagrams of type $A_1 + A_1$ and of type D_2. One way this manifests is as an isomorphism between split simply connected groups $\mathbf{SL}_2 \times \mathbf{SL}_2 \cong \mathbf{Spin}_4$, or between split adjoint groups $\mathbf{PGL}_2 \times \mathbf{PGL}_2 \cong \mathbf{PSO}_4$. However, due to the relationship between algebraic groups and algebras with involution, this also manifests as the following equivalence of groupoids shown in [KMRT, 15.B]. Let $\mathbb{F}$ be an arbitrary field and

(i) let A_1^2 be the groupoid of Azumaya algebras of degree 2 over a quadratic étale extension of $\mathbb{F}$ with $\mathbb{F}$–algebra isomorphisms as arrows, and
(ii) let D_2 be the groupoid of central simple $\mathbb{F}$–algebras of degree 4 equipped with quadratic pairs (see [KMRT, §5]) with $\mathbb{F}$–algebra isomorphisms respecting the quadratic pair as arrows.

Then, there is an equivalence of categories $A_1^2 \equiv D_2$. In particular, they show in [KMRT, 15.7] that a *norm functor* $\mathbf{N}\colon A_1^2 \to D_2$ provides this equivalence.

The norm functor used in [KMRT] is with respect to finite étale extensions of the base field. It is a generalization of the *corestriction* with respect to a finite separable field extension $\mathbb{K}/\mathbb{F}$ introduced by Riehm in [R]. Riehm's corestriction sends a central simple $\mathbb{K}$–algebra A of degree r to a central simple $\mathbb{F}$–algebra $\mathrm{cor}_{\mathbb{K}/\mathbb{F}}(A)$ of degree $r^{[\mathbb{K}:\mathbb{F}]}$ in such a way that the induced map on Brauer groups

$$\begin{aligned} \mathrm{Br}(\mathbb{K}) = H^2(\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}, \mathbb{K}), \mathbb{F}_{\mathrm{sep}}^{\times}) &\to H^2(\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}, \mathbb{F}), \mathbb{F}_{\mathrm{sep}}^{\times}) = \mathrm{Br}(\mathbb{F}) \\ [A] &\mapsto [\mathrm{cor}_{\mathbb{K}/\mathbb{F}}(A)] \end{aligned}$$

agrees with the usual corestriction in Galois cohomology. This was generalized by Knus and Ojanguren, who defined the norm functor of a finite étale extensions of rings in [KO], and theirs is the version used in [KMRT]. The norm functor was then extended further still by Ferrand in [Fer] to include the case of a finite locally free extension of rings. By [Fer, §5], his new construction agrees with the previous notions of norm functor. We review Ferrand's construction in Section 2.1.

In this paper we continue the "tradition" of extending the norm functor to new settings. We fix a base scheme S and work on the big fppf ringed site $(\mathfrak{Sch}_S, \mathcal{O})$ of schemes over S with the global sections functor $\mathcal{O}$. In fact, we extend the norm functor to a morphism of stacks over $\mathfrak{Sch}_S$ between certain stacks of quasi-coherent sheaves. We are able to do so because of the following key property of Ferrand's norm functor. For a finite locally free ring extension $R \to R'$, let $N_{R'/R} \colon \mathfrak{Mod}_{R'} \to \mathfrak{Mod}_R$ denote Ferrand's norm functor. If $R \to Q$ is any other ring homomorphism, thus making $Q \to R' \otimes_R Q$ a finite locally free extension as well, then for any R'–module M' there is an isomorphism

$$N_{R'/R}(M') \otimes_R Q \cong N_{(R' \otimes_R Q)/Q}(M' \otimes_R Q)$$

and this is functorial in M'. This compatibility with tensor products is exactly what allows the norm to be generalized to quasi-coherent sheaves as they are characterized by a similar condition, see Lemma C.1. Our construction is a general one which takes

(i) a family $\mathfrak{I}$ of affine morphisms in $\mathfrak{Sch}_S$ which is closed under arbitrary pullbacks and which allows descent (precisely, $\mathfrak{I}$ should be a substack of the stack of affine morphisms $\mathfrak{AffMor}$ as in Appendix C.8),

(ii) for each $h \colon U' \to U$ in $\mathfrak{I}$ with U and U' affine, a functor

$$\mathcal{F}_h \colon \mathfrak{Mod}_{\mathcal{O}(U')} \to \mathfrak{Mod}_{\mathcal{O}(U)},$$

(iii) and for every fiber product diagram in $\mathfrak{Sch}_S$

$$\begin{array}{ccc} V' & \longrightarrow & U' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ V & \longrightarrow & U \end{array}$$

where U, U', V, V' are all affine, a natural isomorphism of functors

$$\mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}_{h'}(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V'))$$

and assembles them into a morphism of stacks $\mathcal{F} \colon \mathfrak{QCoh}_{\mathfrak{I}} \to \mathfrak{QCoh}$. Here, $\mathfrak{QCoh}_{\mathfrak{I}}$ is the stack with objects $(T' \to T, \mathcal{M}')$ consisting of a morphism $T' \to T$ in $\mathfrak{I}$ and a quasi-coherent $\mathcal{O}|_{T'}$–module $\mathcal{M}'$, while $\mathfrak{QCoh}$ is the stack with objects $(X, \mathcal{M})$ consisting of a scheme $X \in \mathfrak{Sch}_S$ and a quasi-coherent $\mathcal{O}|_X$–module $\mathcal{M}$. For a review of quasi-coherent modules on $(\mathfrak{Sch}_S, \mathcal{O})$ and the details of this construction, see Appendix C.

Unsurprisingly, Ferrand's norm functors, ranging over the family of finite locally free morphisms of a fixed degree d, satisfy the necessary conditions of the constructions in Appendix C. When $\mathfrak{I}$ is the family of finite locally free morphisms of degree d, we write $\mathfrak{QCoh}_{\mathfrak{I}} = \mathfrak{QCoh}^d_{\mathrm{flf}}$. Therefore, we obtain a stack morphism $N\colon \mathfrak{QCoh}^d_{\mathrm{flf}} \to \mathfrak{QCoh}$, as well as a functor $N_{T/S}\colon \mathfrak{QCoh}(T) \to \mathfrak{QCoh}(S)$ between categories of quasi-coherent $\mathcal{O}|_T$–modules and quasi-coherent $\mathcal{O}$–modules for any finite locally free morphism $T \to S$ of degree d. We verify this in Section 2.8. The functor $N_{T/S}$ of course has additional specific properties analogous to Ferrand's functor, most of which are in regards to polynomial laws. Since $f\colon T \to S$ is finite locally free, by [St, Tag 0BD2] there is a functor

$$\mathrm{norm}\colon f_*(\mathcal{O}|_T) \to \mathcal{O}$$

which arises from the determinant of left multiplication by elements of $f_*(\mathcal{O}|_T)$ on itself. Given a quasi-coherent $\mathcal{O}|_T$–module $\mathcal{M}$, we define a *normic polynomial law* to be a natural transformation $\phi\colon f_*(\mathcal{M}) \to \mathcal{N}$, where $\mathcal{N}$ is a quasi-coherent $\mathcal{O}$–module, such that

$$\phi(tm) = \mathrm{norm}(t)\phi(m)$$

holds for all sections $t \in f_*(\mathcal{O}|_T)(X)$ and $m \in f_*(\mathcal{M})(X)$ and for all $X \in \mathfrak{Sch}_S$. We show that the norm functor $N_{T/S}$ has the following properties.

A. **Theorem.** *Let $f\colon T \to S$ be a finite locally free morphism of schemes and let $N_{T/S}\colon \mathfrak{QCoh}(T) \to \mathfrak{QCoh}(S)$ be the norm functor.*

(i) *For every quasi-coherent $\mathcal{M}$ over T there exists a normic polynomial law $\nu_{\mathcal{M}}\colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$ such that the pair $(N_{T/S}(\mathcal{M}), \nu_{\mathcal{M}})$ is universal in the following sense; if $\nu'\colon f_*(\mathcal{M}) \to \mathcal{N}'$ is any other normic polynomial law into a quasi-coherent $\mathcal{O}$–module, then there is a unique $\mathcal{O}$–module morphism $\varphi\colon N_{T/S}(\mathcal{M}) \to \mathcal{N}'$ such that $\nu' = \varphi \circ \nu_{\mathcal{M}}$.*

(ii) *The universal property determines the image of $N_{T/S}$ on morphisms. If $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of quasi-coherent modules over T, then $\nu_{\mathcal{M}_2} \circ f_*(\varphi)$ is a normic polynomial law and $N_{T/S}(\varphi)$ is the unique $\mathcal{O}$–module morphism making the diagram*

$$\begin{array}{ccc} f_*(\mathcal{M}_1) & \xrightarrow{\nu_{\mathcal{M}_1}} & N_{T/S}(\mathcal{M}_1) \\ \downarrow{\scriptstyle f_*(\varphi)} & & \downarrow{\scriptstyle N_{T/S}(\varphi)} \\ f_*(\mathcal{M}_2) & \xrightarrow{\nu_{\mathcal{M}_2}} & N_{T/S}(\mathcal{M}_2) \end{array}$$

commute.

(iii) *The norm functor and universal normic polynomial law respects base change. If we have a fiber product diagram in* $\mathfrak{Sch}_S$

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

then there is an isomorphism of functors $N_{T'/S'} \circ g'^* \xrightarrow{\sim} g^* \circ N_{T/S}$ *and for any* $\mathcal{M} \in \mathfrak{QCoh}(T)$ *the diagram*

$$\begin{array}{ccc} f'_*(\mathcal{M}|_{T'}) & \xrightarrow{\nu_{(\mathcal{M}|_{T'})}} & N_{T'/S'}(\mathcal{M}|_{T'}) \\ \downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle \wr} \\ f_*(\mathcal{M})|_{S'} & \xrightarrow{(\nu_{\mathcal{M}})|_{S'}} & N_{T/S}(\mathcal{M})|_{S'} \end{array}$$

commutes.

(iv) $N_{T/S}(\mathcal{O}|_T) = \mathcal{O}$ *and* $\nu_{\mathcal{O}|_T} = \text{norm}\colon f_*(\mathcal{O}|_T) \to \mathcal{O}$.

(v) *If* $\mathcal{B}$ *is a quasi-coherent* $\mathcal{O}|_T$*–algebra, then* $N_{T/S}(\mathcal{B})$ *is naturally an* $\mathcal{O}$*–algebra and* $\nu_{\mathcal{B}}$ *is multiplicative. The norm also preserves algebra homomorphisms and thus restricts to the categories of quasi-coherent algebras.*

Further, if $T \to S$ *is finite étale of degree* d *we have the following.*

(vi) $N_{T/S}$ *sends (finite) locally free modules to (finite) locally free modules.*

(vii) $N_{T/S}$ *sends Azumaya algebras to Azumaya algebras.*

(viii) $N_{T/S}$ *sends locally free* $\mathcal{O}|_T$*–modules of constant rank* r *to locally free modules of constant rank* r^d *and it sends Azumaya algebras of constant degree* r *to Azumaya algebras of constant degree* r^d.

Properties (i) and (ii) are shown in Proposition 2.10 and Corollary 2.11 respectively. The isomorphism of functors in Property (iii) comes from Lemma C.9 and the statement about the universal normic laws is Corollary 2.12. Property (iv) follows from Example 2.14 after considering a sufficient localization. Property (v) is shown in Lemma 2.17. Finally, when $T \to S$ is étale, property (vi) follows from Example 2.14, property (vii) follows from property (viii), and property (viii) is Lemma 2.13 and Lemma 2.20.

In Section 3 we give a description of maps on cohomology induced by the norm morphism. In particular, we restrict the norm morphism of stacks to various substacks, in fact subgerbes, which are equivalent to the gerbes of torsors for some semi-direct products of groups. As is discussed in Appendix B, the cohomology set $H^1(S, (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d)$, where $\mathbb{S}_d$ is the permutation group, classifies isomorphism classes of objects in the fiber over S of the gerbe whose objects are pairs $(T' \to T, \mathcal{M}')$ where $T' \to T$ is a degree d étale cover and $\mathcal{M}'$ is a locally free $\mathcal{O}|_{T'}$–module of constant rank r. Similarly, isomorphism classes in the fiber over S of the gerbe whose objects are pairs

$(T' \to T, \mathcal{A}')$, where now $\mathcal{A}'$ is an $\mathcal{O}|_{T'}$–Azumaya algebra of degree r, are classified by $H^1(S, (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d)$. Details on the definitions of these gerbes are given at the beginning of Section 3. We know by Theorem A(viii) that the norm will send such objects to $\mathcal{O}$–modules of rank r^d or Azumaya $\mathcal{O}$–algebras of degree r^d respectively. Isomorphism classes of these modules are classified by $H^1(S, \mathbf{GL}_{r^d})$ and isomorphisms of those Azumaya algebras are classified by $H^1(S, \mathbf{PGL}_{r^d})$. A general fact about stack morphisms, [Gir, III.2.5.3], states that the resulting maps on isomorphism classes will be induced from group homomorphisms $(\mathbf{GL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{GL}_{r^d}$ and $(\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{PGL}_{r^d}$. We show that these are the Segre embeddings. On the $\mathbf{GL}$ level, it sends elements $(A_1, \ldots, A_d) \in (\mathbf{GL}_r)^d$ to $A_1 \otimes \ldots \otimes A_d$ and elements of $\mathbb{S}_d$ to the corresponding permutation of the tensor factors of $\mathcal{O}^{(r^d)} \cong (\mathcal{O}^r)^{\otimes d}$. The map on the $\mathbf{PGL}$ level is defined similarly.

B. **Theorem.** *The map on cohomology sets induced by the norm functor, namely*

$$\begin{aligned} \widetilde{N} \colon H^1(S, (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d) &\to H^1(S, \mathbf{GL}_{r^d}) \\ [(T \to S, \mathcal{M})] &\mapsto [N_{T/S}(\mathcal{M})], \end{aligned}$$

agrees with the map induced by the Segre embedding $(\mathbf{GL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{GL}_{r^d}$. Furthermore, the behaviour of the norm functor on Azumaya algebras induces a map of cohomology sets

$$\begin{aligned} \widetilde{N_{\mathrm{alg}}} \colon H^1(S, (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d) &\to H^1(S, \mathbf{PGL}_{r^d}) \\ [(T \to S, \mathcal{A})] &\mapsto [N_{T/S}(\mathcal{A})]. \end{aligned}$$

which likewise agrees with the map induced by the Segre embedding $(\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{PGL}_{r^d}$.

Our cohomological description of the norm functor also extends beyond first cohomology. Over a scheme, Brauer classes of Azumaya algebras lie in the *Brauer-Grothendieck group* $\mathrm{Br}(S) = H^2(S, \mathbb{G}_m)$, where $\mathbb{G}_m$ is the multiplicative group. Since we are assuming $f \colon T \to S$ is finite étale, we know that the norm functor preserves Azumaya algebras and thus maps classes of Azumaya algebras in $\mathrm{Br}(T)$ to classes in $\mathrm{Br}(S)$. We show that this induced action is compatible with the trace morphism $\mathrm{tr} \colon \mathrm{Br}(T) \to \mathrm{Br}(S)$ induced by the trace morphism $f_*(\mathbb{G}_m|_T) \to \mathbb{G}_m$ of [SGA4, IX.5.1.2]. The following is Proposition 3.14.

C. **Proposition.** *Let $T \to S$ be a degree d étale cover. Let $\mathcal{B}$ be an Azumaya $\mathcal{O}|_T$–algebra of constant degree and let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra of constant degree. Denoting Brauer classes in square brackets, we have*

(i) $[N_{T/S}(\mathcal{B})] = \mathrm{tr}([\mathcal{B}]) \in \mathrm{Br}(S)$, *and*
(ii) $[N_{T/S}(\mathcal{A}|_T)] = d[\mathcal{A}] \in \mathrm{Br}(S)$.

Our goal in Section 4 is to show that the exceptional isomorphism discussed at the beginning of this introduction still occurs at the level of Azumaya algebras over schemes. In fact, we show that it holds as an equivalence of stacks. We consider the gerbes

(i) $\mathfrak{A}_1^2$ of quaternion algebras over a degree 2 étale extension, and
(ii) $\mathfrak{D}_2$ of degree 4 Azumaya algebras with a quadratic pair.

Quadratic pairs are a characteristic agnostic analogue of orthogonal involutions and thus are appropriate for discussing groups of type D and related objects over a general scheme. We include background on quadratic pairs in Section 1.5. We begin Section 4 by constructing a quadratic pair over $\mathbb{Z}$ which, after restriction, acts as the split object in $\mathfrak{D}_n$, the gerbe of Azumaya $\mathcal{O}$–algebras of degree $2n$ with quadratic pair. By restricting the Segre homomorphism to symplectic and orthogonal groups, we then show how the norm gives a morphism from the stack of Azumaya algebras of degree $2r$ with symplectic involution over an étale extension of degree d into the stack $\mathfrak{D}_{2^{d-1}r^d}$. By noting that when $r = 1$ and $d = 2$ we get an equivalence of categories, we obtain the following generalization of [KMRT, 15.7] to our setting.

D. **Theorem.** *There is an equivalence of stacks*

$$N\colon \mathfrak{A}_1^2 \to \mathfrak{D}_2$$
$$(T' \to T, \mathcal{B}) \mapsto \left(T, (N_{T'/T}(\mathcal{B}), \sigma_{N_{T'/T}(\mathcal{B})}, f_{N_{T'/T}(\mathcal{B})})\right)$$

given by the norm functor.

This is Theorem 4.9. If we focus on the fiber over S of this morphism, we get an equivalence of categories between the groupoid of quaternion algebras over a degree 2 étale extension of S into the groupoid of degree 4 Azumaya $\mathcal{O}$–algebras with quadratic pair, which is a more direct analogue of [KMRT, 15.7].

The organization of the paper is essentially outlined above. Section 1 recalls the common objects and some basic results we use throughout the paper. The important details of Ferrand's construction over rings are in Section 2 which also contains the construction of our new norm morphism/functor and the proofs of the many parts of Theorem A. Section 3 develops the cohomological interpretation of the newly constructed norm functor. Section 4 uses these tools to show Theorem D.

We also include three Appendices containing general technical lemmas. Appendix A considers a degree d étale cover $f\colon T \to S$ and then for any sheaf $\mathcal{F}$ on $\mathfrak{Sch}_S$ describes $f_*(\mathcal{F}|_T)$ in terms of twisting sheaves with torsors or in terms of Weil restrictions in the case $\mathcal{F}$ is representable. Appendix B gives a cohomological description of stacks related to semi-direct products of groups which appear frequently in Sections 3 and 4. Finally, as mentioned above, Appendix C contains a review of the properties of quasi-coherent module on $(\mathfrak{Sch}_S, \mathcal{O})$ as well as the details of the general construction we use to define the norm morphism.

1. Preliminaries

1.1. **Flat Sites.** Following the style of [CF] and [GNR], most of our objects will be sheaves on a category of schemes equipped with the fppf topology. We now review this setting, highlighting definitions of objects that are most important to us.

For a scheme X we denote by $\mathfrak{Sch}_X$ the big fppf site of X as in [SGA3, Exposé IV]. The objects of $\mathfrak{Sch}_X$ are schemes with a fixed structure morphism $Y \to X$, morphisms are scheme morphisms which respect the structure morphisms, and coverings in the site consist of families of the form $\{Y_i \to Y\}_{i\in I}$ which are jointly surjective and where each $Y_i \to Y$ is flat and locally of finite presentation. When given a cover, we denote $Y_{ij} = Y_i \times_Y Y_j$, as well as $Y_{ijk} = Y_i \times_Y Y_j \times_Y Y_k$, etc. Affine schemes will commonly be denoted with U or V and affine covers by $\{U_i \to Y\}_{i\in I}$ or likewise with V. Since Y need not be a separated scheme, an affine cover need not have its U_{ij}'s be affine.

1.2. **Remark.** In [St, Tag 021S], the Stacks project defines "a" big fppf site of X, instead of "the". This distinction, due to set theoretic considerations, is avoided in [SGA3] through the use of universes. We also simply use "the" big fppf site and similarly use "the" big affine fppf site introduced below.

We denote by $\mathfrak{Aff}_X$ the big affine fppf site of X as in [St, Tag 021S (2)]. It is the full subcategory of $\mathfrak{Sch}_X$ consisting of affine schemes over X and the covers are fppf covers of the form $\{U_i \to U\}_{i=1}^m$ where each U_i and U are affine schemes. We will frequently use the following lemma to discuss sheaves on all of $\mathfrak{Sch}_X$ by instead working with sheaves on $\mathfrak{Aff}_X$.

1.3. **Lemma.** *There is an equivalence of categories between sheaves on $\mathfrak{Sch}_X$ and sheaves on $\mathfrak{Aff}_X$ given by restricting a sheaf $\mathcal{F}\colon \mathfrak{Sch}_X \to \mathfrak{Sets}$ to the objects of $\mathfrak{Aff}_X$. Under this equivalence, intrinsic properties, such as being finite locally free or being quasi-coherent as recalled in Section 1.4, are preserved.*

Proof. The equivalence of categories (more precisely, equivalence of topoi) is [St, Tag 021V]. The properties of Section 1.4 are intrinsic properties by [St, Tag 03DM]. By the definition of intrinsic property at the beginning of [St, Tag 03DG], they are preserved under equivalences of topoi. □

If $Y \in \mathfrak{Sch}_X$, then $\mathfrak{Sch}_Y$ is naturally a subcategory of $\mathfrak{Sch}_X$ by composing the structure morphisms $Y' \to Y$ with the structure morphism $Y \to X$. For a sheaf $\mathcal{F}$ on $\mathfrak{Sch}_X$, we denote by $\mathcal{F}|_Y$ the restriction of the sheaf to $\mathfrak{Sch}_Y$. If $Y' \to Y$ is a morphism of X–schemes, then borrowing notation from [St], we use $t|_{Y'}$ to denote the image under $\mathcal{F}(Y) \to \mathcal{F}(Y')$ of a section $t \in \mathcal{F}(Y)$. This will also be referred to as the restriction of t to Y'. It will be clear from context which notion of restriction, for sheaves or for sections, is intended. By a slightly further abuse of notation, we may talk about a section $t \in \mathcal{F}$,

by which we mean a section $t \in \mathcal{F}(Y)$ for some $Y \in \mathfrak{Sch}_S$, i.e., t may be any section over any Y.

Given two schemes $Y_1, Y_2 \in \mathfrak{Sch}_X$, their presheaf of homomorphisms is the functor

$$\begin{aligned} \mathcal{H}om(Y_1, Y_2) \colon \mathfrak{Sch}_X &\to \mathfrak{Sets} \\ Z &\mapsto \mathrm{Hom}_Z(Y_1 \times_X Z, Y_2 \times_X Z) \end{aligned}$$

sending Z to the set of Z–scheme morphisms between the two fiber products. It is a sheaf by [St, Tag 040L]. The subsheaf of isomorphisms, denoted $\mathcal{I}som(Y_1, Y_2)$, is then also a sheaf.

1.4. **Ringed Sites.** We now fix a base scheme S. Unless otherwise stated, we assume that a ring is unital, commutative, and associative. The global sections functor

$$\begin{aligned} \mathcal{O} \colon \mathfrak{Sch}_S &\to \mathfrak{Rings} \\ X &\mapsto \mathcal{O}_X(X) \end{aligned}$$

where $\mathfrak{Rings}$ is the category of commutative rings, is a sheaf with respect to the fppf topology by [St, Tag 03DU]. It makes $(\mathfrak{Sch}_S, \mathcal{O})$ into a ringed site as in [St, Tag 03AD] and we call $\mathcal{O}$ the *structure sheaf*. If $X \in \mathfrak{Sch}_S$ is another scheme, then the structure sheaf of $\mathfrak{Sch}_X$ is $\mathcal{O}|_X$.

From [St, Tag 03CW], an $\mathcal{O}$*–module* is a sheaf $\mathcal{M} \colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ of abelian groups with a map of sheaves

$$\mathcal{O} \times \mathcal{M} \to \mathcal{M}$$

such that for each $X \in \mathfrak{Sch}_S$, the map $\mathcal{O}(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$ gives $\mathcal{M}(X)$ the structure of an $\mathcal{O}(X)$–module. A morphism of $\mathcal{O}$–modules is a morphism of sheaves such that the map on X points is $\mathcal{O}(X)$–linear for all $X \in \mathfrak{Sch}_S$. The notion of $\mathcal{O}|_X$–module on $\mathfrak{Sch}_X$ is analogous and likewise for the properties discussed below.

The *internal homomorphism* functor of two $\mathcal{O}$–modules $\mathcal{M}$ and $\mathcal{N}$ is

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N}) \colon \mathfrak{Sch}_S &\to \mathfrak{Ab} \\ T &\mapsto \mathrm{Hom}_{\mathcal{O}|_T}(\mathcal{M}|_T, \mathcal{N}|_T). \end{aligned}$$

It is another $\mathcal{O}$–module by [St, 03EM]. The internal endomorphisms of an $\mathcal{O}$–module $\mathcal{M}$ are denoted by $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{M})$, and the subsheaf of automorphisms is denoted $\mathcal{A}ut_{\mathcal{O}}(\mathcal{M})$.

If $g \colon X \to S$ is a morphism of schemes, then we have a direct image (or pushforward) functor g_* and a pull-back functor g^* as in [St, Tag 03D6]. These form an adjoint pair where g^* is left adjoint to g_*, [St, Tag 03D7]. That is, for each $\mathcal{O}|_X$–module $\mathcal{E}$ and each $\mathcal{O}$–module $\mathcal{F}$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}|_X - Mod}(g^*\mathcal{F}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O} - Mod}(\mathcal{F}, g_*\mathcal{E}).$$

In particular for $\mathcal{F} = \mathcal{O}$ we have

$$\mathrm{Hom}_{\mathcal{O}|_X-Mod}(\mathcal{O}|_X, \mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}-Mod}(\mathcal{O}, g_*\mathcal{E})$$

or in other words $\mathcal{E}(X) \xrightarrow{\sim} (g_*\mathcal{E})(S)$.

1.4.1. *Local Types of $\mathcal{O}$–modules.* We refer to [St, Tags 03DE, 03DL] for definitions of various properties of $\mathcal{O}$–modules. Since $S \in \mathfrak{Sch}_S$ is a final object, [St, Tag 03DN] applies and it suffices for us to define local conditions for an fppf-covering of S. Quasi-coherent modules are reviewed in Appendix C. Here we briefly review finite locally free modules.

We call an $\mathcal{O}$–module $\mathcal{M}$ *finite locally free* or *locally free of finite type* if for all $X \in \mathfrak{Sch}_S$, there is a covering $\{X_i \to X\}_{i\in I}$ such that for each $i \in I$, the restriction $\mathcal{M}|_{X_i}$ is a free $\mathcal{O}|_{X_i}$–module of finite rank. Explicitly, $\mathcal{M}|_{X_i} \cong \mathcal{O}|_{X_i}^{n_i}$ for some non-negative $n_i \in \mathbb{Z}$. If $n_i > 0$ for all $i \in I$ we say that $\mathcal{M}$ is of *finite positive rank*. If all $n_i = n$ for some integer n then we say $\mathcal{M}$ has *constant rank* n. Neither of these notions depend on the cover. If $\mathcal{M}$ is finite locally free, then so is $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$.

1.4.2. *$\mathcal{O}$–Algebras.* An *$\mathcal{O}$–algebra* is an abelian sheaf $\mathcal{B}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ together with sheaf morphisms

$$\mathcal{O} \to \mathcal{B} \text{ and } \mathcal{B} \otimes_{\mathcal{O}} \mathcal{B} \to \mathcal{B}$$

which makes $\mathcal{B}(X)$ into a $\mathcal{O}(X)$–algebra for all $X \in \mathfrak{Sch}_S$. It is unital, associative, commutative, etc., if each $\mathcal{B}(X)$ has that property. For an $\mathcal{O}$–module $\mathcal{M}$, the sheaf $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ is naturally an $\mathcal{O}$–algebra with multiplication coming from composition as usual.

An $\mathcal{O}$-algebra $\mathcal{A}$ is an *Azumaya $\mathcal{O}$–algebra* if it is finite locally free and it satisfies the following equivalent conditions.

(i) The enveloping morphism

$$\begin{aligned} \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\mathrm{op}} &\to \mathcal{E}nd_{\mathcal{O}}(\mathcal{A}) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism.

(ii) For any $U \in \mathfrak{Aff}_S$, we have that $\mathcal{A}(U)$ is an Azumaya $\mathcal{O}(U)$–algebra in the sense over rings such as in [Fo] or [Knu, III §5]. In particular, on $\mathfrak{Aff}_S$ an Azumaya $\mathcal{O}$–algebra is a sheaf of Azumaya algebras.

(iii) There exists a cover $\{X_i \to S\}_{i\in I}$ such that for each $i \in I$, $\mathcal{A}|_{X_i} \cong \mathcal{E}nd_{\mathcal{O}|_{X_i}}(\mathcal{M}_i)$ for a locally free $\mathcal{O}|_{X_i}$–module $\mathcal{M}$ of finite positive rank.

(iv) There exists a cover $\{X_i \to S\}_{i\in I}$ such that for each $i \in I$, $\mathcal{A}|_{X_i} \cong \mathrm{M}_{n_i}(\mathcal{O}|_{X_i})$ for some $0 < n_i \in \mathbb{Z}$.

Definition (i) above is from [Gro, 5.1], and definition (ii) is from [CF, 2.5.3.4].

Since an Azumaya $\mathcal{O}$–algebra is locally a matrix algebra, it locally has a notion of the trace $\mathrm{Tr}\colon \mathrm{M}_{n_i}(\mathcal{O}|_{X_i}) \to \mathcal{O}|_{X_i}$. These functions are compatible on X_{ij} and therefore glue into a global $\mathcal{O}$–linear map $\mathrm{Trd}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{O}$ called the *reduced trace* of $\mathcal{A}$. The local determinant maps are also compatible and glue into the *reduced norm* $\mathrm{Nrd}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{O}$, which is multiplicative.

By [CF, 2.5.3.6], whenever $\mathcal{A}|_X \cong \mathcal{O}|_X^n$ for some $X \in \mathfrak{Sch}_S$ and $n \in \mathbb{Z}$, the integer n will be a square. If $n = m^2$, then m is called the *degree* of $\mathcal{A}|_X$. If $\mathcal{A}$ is of constant rank m^2, we say it is of *constant degree* m.

1.5. **Quadratic Pairs.** Quadratic pairs are the characteristic independent analogue of orthogonal involutions and are used to study quadratic forms and groups of type D when 2 need not be invertible. We refer to [CF, §2.7] and [GNR]. Background on involutions and quadratic pairs over fields can be found in [KMRT] and background on involutions over rings in [Knu].

An *involution* (of the first kind) on an Azumaya $\mathcal{O}$–algebra is an $\mathcal{O}$–linear order 2 anti-automorphism, i.e., $\sigma\colon \mathcal{A} \to \mathcal{A}$ such that $\sigma^2 = \mathrm{Id}$ and $\sigma(ab) = \sigma(b)\sigma(a)$. We refer to such a pair $(\mathcal{A}, \sigma)$ as an Azumaya $\mathcal{O}$–algebra with involution. By [CF, 2.7.0.25], any Azumaya $\mathcal{O}$–algebra with involution will have a cover $\{X_i \to S\}_{i\in I}$ (which in fact can be an étale cover if one desires) over which $(\mathcal{A}|_{X_i}, \sigma|_{X_i}) \cong (\mathrm{M}_{n_i}(\mathcal{O}|_{X_i}), \eta_{b_i})$ where η_{b_i} is the adjoint involution of a regular bilinear form $b_i\colon \mathcal{O}|_{X_i}^{n_i} \times \mathcal{O}|_{X_i}^{n_i} \to \mathcal{O}|_{X_i}$. If the resulting b_i are all symmetric bilinear forms, i.e., $b_i(x, y) = b_i(y, x)$, then we call σ an *orthogonal involution*; if they are skew-symmetric, $b_i(x, y) = -b_i(y, x)$, then we call σ a *weakly-symplectic involution*; and if they are alternating, $b_i(x, x) = 0$, then we call σ a *symplectic involution*. These properties do not depend on the cover chosen and are not the only possibilities; for example, a bilinear form can be ε–symmetric for some $\varepsilon \in \mathcal{O}(X_i)$ with $\varepsilon^2 = 1$. Our terminology follows [CF] and [GNR] but differ slightly from [KMRT]. In our conventions, if $2 = 0 \in \mathcal{O}$, then orthogonal and weakly-symplectic coincide. In all cases, a symplectic involution is also weakly-symplectic and so we allow symplectic involutions to also be orthogonal.

Associated to an Azumaya $\mathcal{O}$–algebra with involution $(\mathcal{A}, \sigma)$ is the subsheaf of symmetric elements $\mathit{Sym}_{\mathcal{A},\sigma} \subseteq \mathcal{A}$, which is the kernel of the endomorphism $x \mapsto x - \sigma(x)$ on $\mathcal{A}$. It is an $\mathcal{O}$–module which is finite locally free if σ is orthogonal by [GNR, 3.3 (ii)]. A *quadratic pair* on $\mathcal{A}$ is a pair (σ, f) where

(i) $\mathcal{A}$ is an Azumaya $\mathcal{O}$–algebra,
(ii) σ is an orthogonal involution on $\mathcal{A}$, and
(iii) $f\colon \mathit{Sym}_{\mathcal{A},\sigma} \to \mathcal{O}$ is an $\mathcal{O}$–linear map such that $f(x+\sigma(x)) = \mathrm{Trd}_{\mathcal{A}}(x)$ for all $x \in \mathcal{A}$.

We also refer to $(\mathcal{A}, \sigma, f)$ as a *quadratic triple*. If $\frac{1}{2} \in \mathcal{O}$, then the third condition implies that $f = \frac{1}{2}\mathrm{Trd}_{\mathcal{A}}$ is the unique f making $(\mathcal{A}, \sigma, f)$ a quadratic triple, see [GNR, 4.3(a)]. The fact that a unique f exists when 2 is invertible is a reflection of the bijective correspondence between symmetric bilinear forms and quadratic forms in characteristic not 2. The connection between quadratic triples and quadratic forms is detailed in [GNR, 4.4] over schemes or [KMRT, 5.11] over fields.

1.6. **Algebraic Groups.** We review the groups we need from [CF].

If $\mathcal{B}$ is a unital associative $\mathcal{O}$–algebra which is finite locally free, then the functor of invertible elements

$$\begin{aligned}\mathbf{GL}_{1,\mathcal{B}}\colon \mathfrak{Sch}_S &\to \mathfrak{Grp}\\ X &\mapsto \mathcal{B}(X)^\times,\end{aligned}$$

is called the *general linear group* of $\mathcal{B}$. If $\mathcal{B} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$ we write $\mathbf{GL}_{1,\mathcal{B}} = \mathbf{GL}_n$. The *multiplicative group* is $\mathbb{G}_m = \mathbf{GL}_1$. The n^{th}*–roots of unity* are denoted $\boldsymbol{\mu}_n$ and are the kernel of the map $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^n$.

If $\mathcal{A}$ is an Azumaya $\mathcal{O}$–algebra, then the reduced norm is a group homomorphism $\mathrm{Nrd}_{\mathcal{A}}\colon \mathbf{GL}_{1,\mathcal{A}} \to \mathbb{G}_m$ and the kernel of this map, i.e., the group of norm 1 elements, is the *special linear group* $\mathbf{SL}_{\mathcal{A}}$, [CF, 3.5.0.91]. If $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$, we similarly write $\mathbf{SL}_n$.

The *projective general linear group* of an Azumaya $\mathcal{O}$–algebra $\mathcal{A}$ is the subsheaf of $\mathcal{A}ut_{\mathcal{O}}(\mathcal{A})$ consisting of algebra automorphisms. So we may write $\mathbf{PGL}_{\mathcal{A}} = \mathcal{A}ut_{\mathcal{O}\text{-alg}}(\mathcal{A})$. When $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$, we write $\mathbf{PGL}_n$. The canonical projection $\mathbf{GL}_{1,\mathcal{A}} \to \mathbf{PGL}_{\mathcal{A}}$ sends an element to its inner automorphism and this projection has kernel $\mathbb{G}_m$.

Orthogonal groups are defined in [CF, §4] for quadratic forms and quadratic triples. In particular, let $q\colon \mathcal{O}^{2n} \to \mathcal{O}$ be a regular quadratic form. By [GNR, 4.4 (i)] or [CF, 2.7.0.31], it has a corresponding quadratic triple $(\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)$ where σ_q is the adjoint involution. The *orthogonal groups* are then

$$\mathbf{O}_q(X) = \{x \in \mathbf{GL}_{2n}(X) \mid q \circ x = q\}, \text{ and}$$
$$\mathbf{O}_{\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q}(X) = \{x \in \mathbf{GL}_{2n}(X) \mid \sigma_q(x) = x^{-1}, f_q(x\_\_x^{-1}) = f_q\}$$

for all $X \in \mathfrak{Sch}_S$. We have that $\mathbf{O}_q \cong \mathbf{O}_{\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q}$ by [CF, 4.4.0.44]. The group $\mathbf{O}_q^+$ is a smooth affine S–group scheme which is the kernel of the map

$$\mathbf{O}_q \to \mathbb{Z}/2\mathbb{Z} \tag{1.6.1}$$

called the Dickson map in [Knu, IV, 5.1] and called the Arf map in [CF, 4.3.0.27]. An automorphism of a quadratic form induces an automorphism of the associated even Clifford algebra and the Dickson map sends the quadratic form automorphism to the induced map's restriction to the center of the even Clifford algebra. We use the terminology of [Knu].

The *projective orthogonal group* of a quadratic form is defined to be the automorphism group scheme $\mathbf{PGO}_{\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q} = \mathcal{A}ut(\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)$. This is also simply denoted by $\mathbf{PGO}_q$. An automorphism of $(\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)$ similarly induces an automorphism of the even Clifford algebra, and so there is also a Dickson map $\mathbf{PGO}_q \to \mathbb{Z}/2\mathbb{Z}$ whose kernel is denoted $\mathbf{PGO}_q^+$. The kernel of the canonical projection $\mathbf{O}_q^+ \to \mathbf{PGO}_q^+$ is the center of $\mathbf{O}_q^+$, which is isomorphic to $\boldsymbol{\mu}_2$.

We will also work with the *symplectic group* of an Azumaya $\mathcal{O}$–algebra with symplectic involution $(\mathcal{A}, \sigma)$. As in [CF, §7], this is

$$\mathbf{Sp}_{\mathcal{A},\sigma}(X) = \{x \in \mathcal{A}(X) \mid \sigma(x) = x^{-1}\}$$

for all $X \in \mathfrak{Sch}_S$. The *projective symplectic group* associated to $(\mathcal{A}, \sigma)$ is $\mathbf{PSp}_{\mathcal{A},\sigma} = \mathcal{A}\mathit{ut}(\mathcal{A}, \sigma)$: the automorphism group as an algebra with involution. The kernel of the canonical projection $\mathbf{Sp}_{\mathcal{A},\sigma} \to \mathbf{PSp}_{\mathcal{A},\sigma}$ is the center of $\mathbf{Sp}_{\mathcal{A},\sigma}$, which is isomorphic to $\boldsymbol{\mu}_2$.

Finally, we also work with torsors for various groups using the definition of [CF, 2.2.2.2] or [St, Tag 03AH]. If $\mathbf{G}\colon \mathfrak{Sch}_S \to \mathfrak{Grp}$ is a group sheaf, a $\mathbf{G}$*–torsor* is a sheaf $\mathcal{P}\colon \mathfrak{Sch}_S \to \mathfrak{Sets}$ with a map of sheaves $\mathcal{P} \times \mathbf{G} \to \mathcal{P}$ such that

(i) $\mathcal{P}(X) \times \mathbf{G}(X) \to \mathcal{P}(X)$ gives a simply transitive right $\mathbf{G}(X)$–action on $\mathcal{P}(X)$ for all $X \in \mathfrak{Sch}_S$, and
(ii) there exists a cover $\{X_i \to S\}_{i\in I}$ such that $\mathcal{P}(X_i) \neq \emptyset$ for all $i \in I$.

The sheaf $\mathbf{G}$ itself, viewed simply as a sheaf of sets with right action given by right multiplication, is called the *trivial torsor*. We will work with non-abelian cohomology as in [Gir, 2.4.2], defining $H^1(S, \mathbf{G})$ to be the set of isomorphism classes of $\mathbf{G}$–torsors over S for any group sheaf $\mathbf{G}$.

If $\mathcal{P}$ is a $\mathbf{G}$–torsor and $X \in \mathfrak{Sch}_S$, then it is clear that $\mathcal{P}|_X$ is a $\mathbf{G}|_X$–torsor. When we wish to emphasize the morphism $g\colon X \to S$ we will also write $g^*(\mathcal{P}) = \mathcal{P}|_X$, mirroring the pullback notation for $\mathcal{O}$–modules and algebras.

1.7. **Contracted Products.** Given a group sheaf $\mathbf{G}$, a sheaf of sets $\mathcal{X}$ with a right action of $\mathbf{G}$, and a sheaf of sets $\mathcal{Y}$ with a left action of $\mathbf{G}$, the contracted product as defined in [CF, 2.2.2.9] is denoted $\mathcal{X} \wedge^{\mathbf{G}} \mathcal{Y}$. It is the sheaf associated to the presheaf $(\mathcal{X} \times \mathcal{Y})/\sim$ where the equivalence relation $\sim$ identifies elements (xg, y) and (x, gy) for all appropriate sections $x \in \mathcal{X}, y \in \mathcal{Y}$, and $g \in \mathcal{G}$. If $\mathcal{X}$ also has a left action of $\mathcal{G}$, then $\mathcal{X} \wedge^{\mathbf{G}} \mathcal{Y}$ has an inherited left action, and similarly if $\mathcal{Y}$ also has a right action then so does $\mathcal{X} \wedge^{\mathbf{G}} \mathcal{Y}$.

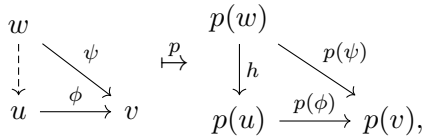
In particular, if $\varphi\colon \mathbf{G} \to \mathbf{H}$ is a group sheaf homomorphism and $\mathcal{P}$ is a $\mathbf{G}$–torsor, then giving $\mathbf{H}$ a left action of $\mathbf{G}$ through φ and a right action on itself by multiplication, the contracted product $\mathcal{P} \wedge^{\mathbf{G}} \mathbf{H}$ will have a right $\mathbf{H}$–action and by [CF, 2.2.2.12] will be an $\mathbf{H}$–torsor. This is also called the *twist of* $\mathbf{H}$ *by* $\mathcal{P}$.

Furthermore, if $\mathcal{P}$ is a $\mathbf{G}$–torsor and if $\mathcal{Y}$ has additional structure, say it is a sheaf of groups, rings, modules, etc., and the left action of $\mathbf{G}$ is by automorphisms which respect that structure, then the contracted product $\mathcal{P} \wedge^{\mathbf{G}} \mathcal{Y}$ will also be a sheaf of groups, rings, modules, etc. This has been formalized in [CF, 2.1], which we invite the interested reader to consult. The contracted product $\mathcal{P} \wedge^{\mathbf{G}} \mathcal{Y}$ will also be locally isomorphic to $\mathcal{Y}$. Indeed, over any cover $\{Y_i \to S\}_{i\in I}$ which trivializes $\mathcal{P}$, we will have $(\mathcal{P} \wedge^{\mathbf{G}} \mathcal{Y})|_{Y_i} \cong \mathcal{Y}|_{Y_i}$.

1.8. **Stacks.** We make use of stacks over the fppf site $\mathfrak{Sch}_S$ defined in Section 1.1 though readers familiar with stacks will recognize that the following discussion also holds over a general site. We recall this notion from [Gir] (where the French word for "stack" is "*champ*"), [Ols], and [St].

First is the notion of a fibered category. We take the following definition from [Ols, 3.1], specifying all notions to be over the site $\mathfrak{Sch}_S$. Alternatively, this is [St, Tag 02XJ], where they use the terminology "strongly cartesian" where we use "cartesian".

1.9. **Definition.** Let $\mathfrak{F}$ be a category with a functor $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$. We call a morphism $\phi\colon u \to v$ in $\mathfrak{F}$ *cartesian* if whenever we have another morphism $\psi\colon w \to v$ such that $p(w)$ factors through $p(\phi)$, diagrammatically



then there exists a unique morphism $\lambda\colon w \to u$ in $\mathfrak{F}$, taking the place of the dashed arrow above, such that the diagram commutes and $p(\lambda) = h$.

The category $\mathfrak{F}$, or more precisely the data $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$, is called a *fibered category over* $\mathfrak{Sch}_S$ if for every $x \in \mathfrak{F}$ and morphism $f\colon Y \to p(x)$ in $\mathfrak{Sch}_S$, there exists a cartesian morphism $\phi\colon y \to x$ such that $p(\phi) = f$, and therefore also $p(y) = Y$. The object y may be denoted $y = f^*(x)$ and called a *pullback* of x along f. We will call the functor p the *structure functor*.

In the above definition we say "a" pullback, instead of "the" pullback, since pullbacks need not be unique. However, they are unique up to unique isomorphism by [Ols, 3.1]. If the morphism $f\colon Y \to p(x)$ is clear from context we may write $f^*(x) = x|_Y$. By [Vis, 3.4], compositions of cartesian morphisms are cartesian and isomorphisms in $\mathfrak{F}$ are also cartesian.

A morphism from a fibered category $(\mathfrak{F}, p_{\mathfrak{F}})$ over $\mathfrak{Sch}_S$ to another fibered category $(\mathfrak{G}, p_{\mathfrak{G}})$ over $\mathfrak{Sch}_S$ is a functor $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ such that $p_{\mathfrak{G}} \circ \varphi = p_{\mathfrak{F}}$ is an equality of functors into $\mathfrak{Sch}_S$ and φ sends cartesian morphisms to cartesian morphisms.

Given a fibered category $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$, for any scheme $X \in \mathfrak{Sch}_S$ we denote by $\mathfrak{F}(X)$ the subcategory of objects $x \in \mathfrak{F}$ such that $p(x) = X$, and morphisms $\phi\colon x \to x'$ such that $p(\phi) = \mathrm{Id}_X$. This is called the *fiber over* X. A morphism of fibered categories $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ induces functors $\varphi_X\colon \mathfrak{F}(X) \to \mathfrak{G}(X)$ between the corresponding fibers. We take [Ols, 3.1.10] as our definition and say that φ is an *equivalence of fibered categories* if every φ_X is an equivalence of categories.

A fibered category is called *fibered in groupoids* if $\mathfrak{F}(X)$ is a groupoid for all $X \in \mathfrak{Sch}_S$. A convenient fact about categories fibered in groupoids is the following.

1.10. **Lemma.** [Vis, 3.22] *If $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ is a fibered category, then $\mathfrak{F}$ is fibered in groupoids if and only if every morphism in $\mathfrak{F}$ is a cartesian morphism.*

In particular, this means that if $\mathfrak{F}$ is a fibered category and $\mathfrak{G}$ is fibered in groupoids, then any functor $\mathfrak{F} \to \mathfrak{G}$ which is compatible with the structure functors into $\mathfrak{Sch}_S$ will be a morphism of fibered categories.

Following [St, Tag 02ZB] or [Vis, §3.7] (or [Ols, 3.4.7] when $\mathfrak{F}$ is fibered in groupoids), given two objects $x, x' \in \mathfrak{F}(X)$, we define their *internal homomorphism functor*

$$\begin{aligned}\mathcal{H}om(x,x') = \mathcal{H}om_{\mathfrak{F}}(x,x')\colon \mathfrak{Sch}_X &\to \mathfrak{Sets}\\ Y &\mapsto \mathrm{Hom}_{\mathfrak{F}(Y)}(x|_Y, x'|_Y).\end{aligned}$$

Independence from the choice of pullbacks in shown in [Vis, §3.7]. The restriction maps for this functor are defined as follows. If $f\colon Y' \to Y$ is a morphism of X–schemes and $\varphi\colon x|_Y \to x'|_Y$ is a morphism in $\mathfrak{F}(Y)$, we have a diagram in $\mathfrak{F}$

$$\begin{array}{ccc} x|_{Y'} & \dashrightarrow & x'|_{Y'} \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_{x'}} \\ x|_Y & \xrightarrow{\varphi} & x'|_Y \end{array}$$

where ϕ_x and $\phi_{x'}$ are cartesian morphisms. Since φ is a morphism in $\mathfrak{F}(Y)$, we have $p(\varphi) = \mathrm{Id}_Y$ and so tautologically $p(\varphi \circ \phi_x)$ factors through $p(\phi_{x'}) = p(\phi_x)$. Therefore by Definition 1.9 there is a unique morphism in $\mathfrak{F}(Y')$, which we denote $\varphi|_{Y'}$ or $f^*(\varphi)$ when emphasizing f, that takes the place of the dashed arrow. These assignments $\varphi \mapsto \varphi|_{Y'}$ give the restriction maps of the functor $\mathcal{H}om(x,x')$. When $x = x'$ we write $\mathcal{E}nd(x) = \mathcal{H}om(x,x)$. The subfunctors of invertible elements will be denoted $\mathcal{I}som(x,x')$ and $\mathcal{A}ut(x)$ respectively. If $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ is a morphism of fibered categories, it induces natural transformations of functors $\varphi_{x,x'}\colon \mathcal{H}om_{\mathfrak{F}}(x,x') \to \mathcal{H}om_{\mathfrak{G}}(\varphi(x), \varphi(x'))$ in the canonical way, and this restricts to the $\mathcal{I}som$ subfunctors.

We take our definition of stack from [St, Tag 026F]. This is slightly more general than the definition in [Ols, 4.6.1], which requires that stacks be fibered in groupoids.

1.11. **Definition.** An *S–stack*, or simply a *stack* when S is clear from context, is a fibered category $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ such that the following hold.

(i) For any $X \in \mathfrak{Sch}_S$ and objects $x, x' \in \mathfrak{F}(X)$, the functor

$$\mathcal{H}om(x,x')\colon \mathfrak{Sch}_X \to \mathfrak{Sets}$$

is an fppf sheaf on $\mathfrak{Sch}_X$.

(ii) For any fppf covering $\{X_i \to X\}_{i\in I}$ of $\mathfrak{Sch}_S$, objects $x_i \in \mathfrak{F}(X_i)$, and isomorphisms $\psi_{ij}\colon x_i|_{X_{ij}} \xrightarrow{\sim} x_j|_{X_{ij}}$ in $\mathfrak{F}$ satisfying the cocycle condition

$$\psi_{jk}|_{X_{ijk}} \circ \psi_{ij}|_{X_{ijk}} = \psi_{ik}|_{X_{ijk}},$$

there exists an object $x \in \mathfrak{F}(X)$ and isomorphisms $\alpha_i\colon x_i \xrightarrow{\sim} x|_{X_i}$ in $\mathfrak{F}(X_i)$ such that

$$\psi_{ij} = (\alpha_j)|_{X_{ij}}^{-1} \circ (\alpha_i)|_{X_{ij}}.$$

Intuitively, this means that a stack is a fibered category which allows gluing (or descent) of local objects along glueing data (or descent data). Condition (i) in Definition 1.11 implies that that the functors $\mathcal{E}nd(x)$, $\mathcal{I}som(x,x')$, and $\mathcal{A}ut(x)$ are sheaves as well.

Most of the stacks we will consider consist of categories of sheaves with certain properties, and are therefore all variations (or more precisely, substacks as defined in [Vis, 3.29, 4.18]) of the following example.

1.12. **Example.** Let $\mathfrak{Sh}_S$, usually abbreviated to just $\mathfrak{Sh}$ when S is clear, be the category whose

(i) objects are pairs $(X, \mathcal{F}\colon \mathfrak{Sch}_X \to \mathfrak{Sets})$ consisting of a scheme $X \in \mathfrak{Sch}_S$ and a sheaf $\mathcal{F}$ on the fppf site of schemes over X, and whose
(ii) morphisms are pairs $(g,\varphi)\colon (X',\mathcal{F}') \to (X,\mathcal{F})$ where $g\colon X' \to X$ is a morphism of S–schemes and $\varphi\colon \mathcal{F}' \to \mathcal{F}|_{X'}$ is a morphism of sheaves on $\mathfrak{Sch}_{X'}$. If $(h,\psi)\colon (X'',\mathcal{F}'') \to (X',\mathcal{F}')$ is another morphism, then composition is given by

$$(g,\varphi)\circ(h,\psi) = (g\circ h, \varphi|_{X''}\circ\psi).$$

Equipping $\mathfrak{Sh}$ with the structure functor $p\colon \mathfrak{Sh} \to \mathfrak{Sch}_S$ sending $(X,\mathcal{F}) \mapsto X$ and $(g,\varphi) \mapsto g$ makes $\mathfrak{Sh}$ the fibered category of [Vis, 3.20] where we take $\mathcal{C} = \mathfrak{Sch}_S$ and $\mathcal{T}$ to be the fppf topology. For an object $(X,\mathcal{F})$ and a morphism $g\colon X' \to X \in \mathfrak{Sch}_S$, the pullback is given by the cartesian morphism $(g, \mathrm{Id}_{\mathcal{F}|_{X'}})\colon (X',\mathcal{F}|_{X'}) \to (X,\mathcal{F})$ as outlined in [Vis, 3.1.3]. Since any cartesian morphism is uniquely isomorphic to one of the form $(g,\mathrm{Id}_{\mathcal{F}|_{X'}})$, we see that a morphism (g,φ) is cartesian if and only if φ is an isomorphism. The fiber over $X \in \mathfrak{Sch}_S$ is the category of sheaves on $\mathfrak{Sch}_X$. In fact, [Vis, 4.11] shows that $\mathfrak{Sh}$ is a stack. Briefly, this is because for any two $\mathcal{F},\mathcal{F}' \in \mathfrak{Sh}(X)$, the homomorphism functor $\mathcal{H}om(\mathcal{F},\mathcal{F}')$ is simply the sheaf of internal homomorphisms between $\mathcal{F}$ and $\mathcal{F}'$ and because $\mathfrak{Sh}$ allows descent since sheaves can be glued along glueing data as in [St, Tag 04TR]. We call $\mathfrak{Sh}$ the stack of sheaves over $\mathfrak{Sch}_S$.

There is a notion of twisting objects, similar to the contracted product discussed above, for objects in a stack. In particular, let $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ be a stack and assume we have and object $x \in \mathfrak{F}(X)$ in the fiber over $X \in \mathfrak{Sch}_S$, a group sheaf $\mathbf{G}\colon \mathfrak{Sch}_X \to \mathfrak{Grp}$, and a homomorphism $\varphi\colon \mathbf{G} \to \mathcal{A}ut(x)$. Then, by [Gir, III.2.3.1], for each $\mathbf{G}$–torsor $\mathcal{P}$ there is an object in $\mathfrak{F}(X)$, denoted by $\mathcal{P} \wedge^{\mathbf{G}} x$ or $\mathcal{P} \wedge^{\varphi} x$ when we wish to emphasize φ, such that

(i) $\mathcal{P} \wedge^{\mathbf{G}} x$ is locally isomorphic to x, and
(ii) by [Gir, III.2.3.2.1] there is an isomorphism of $\mathcal{A}ut(x)$–torsors

$$\mathcal{P} \wedge^{\mathbf{G}} \mathcal{A}ut(x) \xrightarrow{\sim} \mathcal{I}som(x, \mathcal{P} \wedge^{\mathbf{G}} x)$$

where the sheaf on the left is given by the contracted product as in Section 1.7.

We also refer to this construction as the *contracted product*.

Most of the stacks we will be interested in are fibered in groupoids and therefore match the definition of stack from [Ols]. In fact, they are gerbes in the sense of [Gir] or [St]. We note that the notion of gerbe discussed in [Ols, Ch. 12], namely $\boldsymbol{\mu}$–gerbe, is more restrictive.

1.13. **Definition.** [[Gir, III.2.1.1] or [St, Tag 06NZ] (or [CF, 2.2.5.1] in the split case)] A stack $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ is a *gerbe* if

(i) $\mathfrak{F}$ is fibered in groupoids,
(ii) for every $X \in \mathfrak{Sch}_S$ there exists a covering $\{X_i \to X\}_{i\in I}$ such that each $\mathfrak{F}(X_i)$ is non-empty, and
(iii) for every $X \in \mathfrak{Sch}_S$ and objects $x, x' \in \mathfrak{F}(X)$ there exists a cover $\{X_i \to X\}_{i\in I}$ such that there are isomorphisms $x|_{X_i} \cong x'|_{X_i}$ for all $i \in I$.

1.14. **Example.** Consider a group sheaf $\mathbf{G}\colon \mathfrak{Sch}_S \to \mathfrak{Grp}$. Let $\mathfrak{Tors}(\mathbf{G})$ be the fibered category over $\mathfrak{Sch}_S$ whose

(i) objects are pairs $(X, \mathcal{P})$ where $X \in \mathfrak{Sch}_S$ and $\mathcal{P}\colon \mathfrak{Sch}_X \to \mathfrak{Sets}$ is a $\mathbf{G}|_X$–torsor,
(ii) morphisms are pairs $(g, \varphi)\colon (X', \mathcal{P}') \to (X, \mathcal{P})$ where $g\colon X' \to X$ is a morphism of S–schemes and $\varphi\colon \mathcal{P}' \xrightarrow{\sim} \mathcal{P}|_{X'}$ is a morphism (and hence isomorphism) of $\mathbf{G}|_{X'}$–torsors, and whose
(iii) structure functor $p\colon \mathfrak{Tors}(\mathbf{G}) \to \mathfrak{Sch}_S$ sends $(X, \mathcal{P}) \to X$.

This is a gerbe by [Gir, III.1.4.5], however we include a justification in our own notation. The category $\mathfrak{Tors}(\mathbf{G})$ is indeed a fibered category where the pullback of an object $(X, \mathcal{P})$ along a morphism $g\colon X' \to X$ in $\mathfrak{Sch}_S$ is given by the morphism $(g, \mathrm{Id}_{\mathcal{P}|_{X'}})\colon (X', \mathcal{P}|_{X'}) \to (X, \mathcal{P})$. Because all morphisms of torsors are isomorphisms, $\mathfrak{Tors}(\mathbf{G})$ is fibered in groupoids as any morphism over Id_X is of the form (Id_X, φ) for an isomorphism φ. Therefore by Lemma 1.10, all morphisms in $\mathfrak{Tors}(\mathbf{G})$ are cartesian.

For $X \in \mathfrak{Sch}_S$ and two objects $(X, \mathcal{P}_1), (X, \mathcal{P}_2) \in \mathfrak{Tors}(\mathbf{G})(X)$, the functor of homomorphisms $\mathcal{H}om_{\mathfrak{Tors}(\mathbf{G})}((X, \mathcal{P}_1), (X, \mathcal{P}_2))$ is a subfunctor of the homomorphisms $\mathcal{H}om_{\mathfrak{Sh}}((X, \mathcal{P}_1), (X, \mathcal{P}_2))$ in $\mathfrak{Sh}$. Since $\mathbf{G}$–equivariance of morphisms can be checked locally, compatible local torsor isomorphism glue into another torsor isomorphism and so $\mathcal{H}om_{\mathfrak{Tors}(\mathbf{G})}((X, \mathcal{P}_1), (X, \mathcal{P}_2))$ is itself a sheaf. Similarly, gluing torsors along descent data consisting of torsor isomorphism will produce another torsor. Therefore, $\mathfrak{Tors}(\mathbf{G})$ is a stack over $\mathfrak{Sch}_S$. Since torsors are by definition locally isomorphic to $\mathbf{G}$, they are locally isomorphic to each other. Therefore, $\mathfrak{Tors}(\mathbf{G})$ is a gerbe.

There is a clear inclusion functor $\mathfrak{Tors}(\mathbf{G}) \to \mathfrak{Sh}$ which we claim is a morphism of stacks. It is straightforward to see that it respects the structure functors and so we only need to check that cartesian morphism are preserved, i.e., that the image of any morphism is again cartesian in $\mathfrak{Sh}$. A morphism

$(g,\varphi)\colon (X',\mathcal{P}') \to (X,\mathcal{P})$ in $\mathfrak{Tors}(\mathbf{G})$ can be decomposed as

$$\begin{array}{ccc} (X',\mathcal{P}') & & \\ {\scriptstyle (\mathrm{Id}_{X'},\varphi)}\downarrow & \searrow^{(g,\varphi)} & \\ (X',\mathcal{P}|_{X'}) & \xrightarrow{(g,\mathrm{Id}_{\mathcal{P}|_{X'}})} & (X,\mathcal{P}) \end{array}$$

where $(g,\mathrm{Id}_{\mathcal{P}|_{X'}})$ is cartesian in $\mathfrak{Sh}$ and $(\mathrm{Id}_{X'},\varphi)$ is also cartesian in $\mathfrak{Sh}$ by virtue of being an isomorphism. Therefore, their composition (g,φ) is also cartesian in $\mathfrak{Sh}$ as desired. Hence, the inclusion functor $\mathfrak{Tors}(\mathbf{G}) \to \mathfrak{Sh}$ makes the gerbe $\mathfrak{Tors}(\mathbf{G})$ a substack of $\mathfrak{Sh}$.

The following results are analogous to those found in [CF, 2.2.4], which we borrow some notation from. However, the authors there only consider split fibered categories and stacks. See [CF, 2.1.2.1] for their notion of fibered category and [CF, 2.1.3.4] for stack. Because of this, and also because the result corresponding to Example 1.15 is left as an exercise, we include short proofs of these facts in our context.

1.15. **Example.** Let $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ be a stack and let $X \in \mathfrak{Sch}_S$. We call objects $x, x' \in \mathfrak{F}(X)$ *twisted forms* of one another if they are locally isomorphic, that is, if there exists a cover $\{X_i \to X\}_{i\in I}$ and isomorphisms $x|_{X_i} \cong x'|_{X_i}$ in $\mathfrak{F}(X_i)$ for each $i \in I$. Note that this definition is independent of the choice of pullbacks (e.g. [Vis, Remark 3.3]). For an object $s \in \mathfrak{F}(S)$ we denote by $\mathfrak{Forms}(s)$ the subcategory of $\mathfrak{F}$ whose

(i) objects are those $x \in \mathfrak{F}$ (not necessarily in $\mathfrak{F}(S)$) such that x is a twisted form of $s|_{p(x)}$, and whose
(ii) morphisms $y \to x$ are the cartesian morphisms between y and x in $\mathfrak{F}$.

By [Vis, 4.20], the subcategory of $\mathfrak{F}$ consisting of the same objects but only cartesian morphisms, denoted $\mathfrak{F}_{\mathrm{cart}}$, is itself a stack. It is clear that $\mathfrak{Forms}(s)$ is a full subcategory of $\mathfrak{F}_{\mathrm{cart}}$. To see that $\mathfrak{Forms}(s)$ is a substack, we will apply [Vis, 4.19] after checking that the following two conditions hold.

(i) Any arrow in $\mathfrak{F}_{\mathrm{cart}}$ whose target is in $\mathfrak{Forms}(s)$ is also in $\mathfrak{Forms}(s)$.
(ii) If $\{X_i \to X\}_{i\in I}$ is a covering in $\mathfrak{Sch}_S$ and $x \in \mathfrak{F}_{\mathrm{cart}}(X)$ is an object such that the pullbacks $x|_{X_i} \in \mathfrak{Forms}(s)(X_i)$ for all $i \in I$, then $x \in \mathfrak{Forms}(s)(X)$.

To check the first condition, consider an object $x \in \mathfrak{Forms}(s)(X)$ and a morphism $\varphi\colon y \to x$ in $\mathfrak{F}_{\mathrm{cart}}$. Let $g = p(\varphi)\colon Y \to X$ be the underlying map of schemes in $\mathfrak{Sch}_S$. Since all maps in $\mathfrak{F}_{\mathrm{cart}}$ are cartesian, $y \cong x|_Y$ is a pullback of x along g. Let $\{X_i \to X\}_{i\in I}$ be a cover over which x is locally isomorphic to $s|_X$. The local isomorphisms $x|_{X_i} \cong s|_{X_i}$ restrict to isomorphisms $x|_{X_i\times_X Y} \cong s|_{X_i\times_X Y}$, and so $y \cong x|_Y$ is locally isomorphic to $s|_Y$ with respect to the cover $\{X_i\times_X Y \to Y\}_{i\in I}$. Therefore $y \in \mathfrak{Forms}(s)(Y)$ and so the morphism $y \to x$ is in $\mathfrak{Forms}(s)$ as desired.

The second condition is clear since if each $x|_{X_i}$ is locally isomorphic to $s|_{X_i}$, then there exists a refined cover over which x is locally isomorphic to s.

Finally, we show that $\mathfrak{Forms}(s)$ is a gerbe. First, since any arrow in $\mathfrak{Forms}(s)(X)$ is cartesian, it is an isomorphism by [Vis, 3.3], proving condition (i) of Definition 1.13. Next, Definition 1.13(ii) is obvious since $s|_X \in \mathfrak{Forms}(s)(X)$. Finally, by definition, all objects of $\mathfrak{Forms}(s)$ in the same fiber are locally isomorphic, thus also Definition 1.13(iii) holds. This shows that $\mathfrak{Forms}(s)$ is a gerbe.

The following proposition connects isomorphism classes in a stack to cohomology sets of automorphism sheaves. In the context of [CF], part (ii) is [CF, 2.2.4.5] and part (iii) follows from [CF, 2.2.3.6]. Part (ii) is [Gir, III.2.5.1].

1.16. **Proposition.** *Let $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ be a stack and let $s \in \mathfrak{F}(S)$. Consider the gerbe $\mathfrak{Forms}(s)$.*

(i) *For any $X \in \mathfrak{Sch}_S$ and $x \in \mathfrak{Forms}(s)(X)$, the sheaf $\mathcal{I}som(s|_X, x)$ is an $\mathcal{A}ut(s)|_X$–torsor.*

(ii) *There is an equivalence of stacks $\varphi\colon \mathfrak{Forms}(s) \to \mathfrak{Tors}(\mathcal{A}ut(s))$ which acts on objects by*

$$x \mapsto (p(x), \mathcal{I}som(s|_{p(x)}, x))$$

and on morphisms as follows. Let $f : y \to x$ be a morphism in $\mathfrak{Forms}(s)$ and $p(f)\colon Y \to X$ be the underlying morphism in $\mathfrak{Sch}_S$. For a chosen cartesian morphism $f''\colon x|_Y \to x$, there exists a unique isomorphism $f'\colon y \xrightarrow{\sim} x|_Y$ in the fiber $\mathfrak{Forms}(s)(Y)$ such that the composition $y \xrightarrow{f'} x|_Y \xrightarrow{f''} x$ is f. Furthermore, there is a canonical isomorphism

$$g\colon \mathcal{I}som(s|_Y, x|_Y) \xrightarrow{\sim} \mathcal{I}som(s|_X, x)|_Y.$$

Therefore, we define $\varphi(f)$ to be the morphism $(p(f), \phi)$ where ϕ is the isomorphism of $\mathcal{A}ut(s)|_Y$–torsors

$$\begin{aligned} \phi\colon \mathcal{I}som(s|_Y, y) &\xrightarrow{\sim} \mathcal{I}som(s|_X, x)|_Y \\ h &\mapsto g(f' \circ h). \end{aligned}$$

A quasi-inverse is given by the contracted product, namely the functor

$$\begin{aligned} \mathfrak{Tors}(\mathcal{A}ut(s)) &\to \mathfrak{Forms}(s) \\ (X, \mathcal{P}) &\mapsto \mathcal{P} \wedge^{\mathcal{A}ut(s)|_X} s|_X. \end{aligned}$$

(iii) *For any $s' \in \mathfrak{Forms}(s)(S)$, the sheaves $\mathcal{A}ut(s)$ and $\mathcal{A}ut(s')$ are twisted forms of one another as group sheaves. In particular, giving $\mathcal{A}ut(s)$ the left action on itself by inner automorphisms, we have*

$$\mathcal{I}som(s, s') \wedge^{\mathcal{A}ut(s)} \mathcal{A}ut(s) \cong \mathcal{A}ut(s').$$

(iv) *Denote by* $H^1(S, \mathfrak{Forms}(s))$ *the isomorphism classes of the groupoid* $\mathfrak{Forms}(s)(S)$ *and recall that* $H^1(S, \mathcal{A}ut(s))$ *is the set of isomorphism classes of* $\mathcal{A}ut(s)$*–torsors. This is a pointed set and there is a bijective map of pointed sets*

$$\begin{aligned} H^1(S, \mathfrak{Forms}(s)) &\to H^1(S, \mathcal{A}ut(s)) \\ [s'] &\mapsto [\mathcal{I}som(s, s')] \end{aligned}$$

where we choose $[s]$ *as the basepoint of* $H^1(S, \mathfrak{Forms}(s))$.

Proof. (i): The sheaf $\mathcal{I}som(s|_X, x)$ has a natural right action of $\mathcal{A}ut(s)|_X$ given by composition. For any $Y \in \mathfrak{Sch}_X$, the action $\mathcal{I}som(s|_X, x)(Y) \times \mathcal{A}ut(s)(Y) \to \mathcal{I}som(s|_X, x)(Y)$ is simply transitive since any two isomorphisms $\varphi_1, \varphi_2 \in \mathcal{I}som(s|_X, x)(Y)$ differ by $\varphi_1^{-1} \circ \varphi_2 \in \mathcal{A}ut(s)(Y)$. There also exists a cover over which $\mathcal{I}som(s|_X, x)$ is non-empty since x is a twisted form of $s|_X$ by definition. Thus, $\mathcal{I}som(s|_X, x)$ is a $\mathcal{A}ut(s)|_X$–torsor.
(ii): As mentioned above, this is [Gir, III.2.5.1].
(iii): The map of presheaves

$$\begin{aligned} (\mathcal{I}som(s, s') \times \mathcal{A}ut(s))/\sim &\to \mathcal{A}ut(s') \\ (g, \varphi) &\mapsto g \circ \varphi \circ g^{-1}, \end{aligned}$$

where $\sim$ is the equivalence relation defined in Section 1.7, is easily seen to be well defined and it is bijective whenever $\mathcal{I}som(s, s')$ has a section. Therefore, since $\mathcal{I}som(s, s')$ is an $\mathcal{A}ut(s)$–torsor, after sheafification it produces a sheaf isomorphism $\mathcal{I}som(s, s') \wedge^{\mathcal{A}ut(s)} \mathcal{A}ut(s) \xrightarrow{\sim} \mathcal{A}ut(s')$ as desired.
(iv): This follows from (ii). The equivalence of stacks includes an equivalence of categories $\mathfrak{Forms}(s)(S) \to \mathfrak{Tors}(\mathcal{A}ut(s))(S)$ which in turn produces a bijection of pointed sets

$$\begin{aligned} H^1(S, \mathfrak{Forms}(s)) &\xrightarrow{\sim} H^1(S, \mathfrak{Tors}(\mathcal{A}ut(s))) \\ [s'] &\mapsto [\mathcal{I}som(s, s')] \end{aligned}$$

and the latter cohomology set is by definition $H^1(S, \mathcal{A}ut(s))$. □

1.17. **Remark.** If $p\colon \mathfrak{F} \to \mathfrak{Sch}_S$ is itself a gerbe, then $\mathfrak{Forms}(s) = \mathfrak{F}$ by condition 1.13(iii) and we may apply Proposition 1.16 to all of $\mathfrak{F}$. In this case, we will often view the isomorphism $H^1(S, \mathfrak{F}) \xrightarrow{\sim} H^1(S, \mathcal{A}ut(s))$ as an identification and write expressions such as $[s'] \in H^1(S, \mathcal{A}ut(s))$ where $s' \in \mathfrak{F}(S)$. This is a natural shorthand and, for example, is equivalent to considering $H^1(S, \mathbf{GL}_r)$ as the set of isomorphism classes of locally free $\mathcal{O}$–modules of constant rank r.

Our main technique throughout Section 3 will be an application of the following lemma. This lemma is [Gir, III.2.5.3] and it also follows from [CF, 2.2.3.9].

1.18. **Lemma.** *Let $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ be a morphism of gerbes. For $s \in \mathfrak{F}(S)$ there is an associated morphism of group sheaves $\varphi_s\colon \mathcal{A}ut_{\mathfrak{F}}(s) \to \mathcal{A}ut_{\mathfrak{G}}(\varphi(s))$. Then, the map on first cohomology induced by φ_s is the map*

$$\begin{aligned} H^1(S, \mathcal{A}ut_{\mathfrak{F}}(s)) &\to H^1(S, \mathcal{A}ut_{\mathfrak{G}}(\varphi(s))) \\ [s'] &\mapsto [\varphi(s')] \end{aligned}$$

where we view $H^1(S, \mathcal{A}ut_{\mathfrak{F}}(s))$ as the set of isomorphism classes in $\mathfrak{F}(S)$ and $H^1(S, \mathcal{A}ut_{\mathfrak{G}}(\varphi(s)))$ as the isomorphism classes in $\mathfrak{G}(S)$.

We also identify equivalences of gerbes using the following Theorem. This is folklore, but we provide a proof since it is not stated in this exact fashion in [Gir].

1.19. **Theorem** ([Gir])**.** *Let $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ be a morphism of gerbes. Assume there exists $s \in \mathfrak{F}(S)$ such that the induced group sheaf homomorphism $\varphi_s\colon \mathcal{A}ut_{\mathfrak{F}}(s) \to \mathcal{A}ut_{\mathfrak{G}}(\varphi(s))$ is an isomorphism. Then, φ is an equivalence of gerbes.*

Proof. By [Gir, III.2.5.1] there is an equivalence of stacks

$$\begin{aligned} \mathfrak{F} &\to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{F}}(s)) \\ x &\mapsto (p(x), \mathcal{I}som(s|_{p(x)}, x)) \end{aligned}$$

and likewise for $\mathfrak{G} \to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{G}}(\varphi(s)))$. In the case of this second equivalence, we use the quasi-inverse

$$\begin{aligned} \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{G}}(\varphi(s))) &\to \mathfrak{G} \\ (X, \mathcal{P}) &\mapsto \mathcal{P} \wedge^{\mathcal{A}ut_{\mathfrak{G}}(\varphi(s))|_X} \varphi(s)|_X. \end{aligned}$$

where $X \in \mathfrak{Sch}_S$. Additionally, since φ_s is an isomorphism, there is an obvious equivalence of categories

$$\begin{aligned} \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{F}}(s)) &\to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{G}}(\varphi(s))) \\ (X, \mathcal{P}) &\mapsto (X, \mathcal{P} \wedge^{\varphi_s|_X} \mathcal{A}ut_{\mathfrak{G}}(\varphi(s))|_X) \end{aligned}$$

which simply interprets an $\mathcal{A}ut_{\mathfrak{F}}(s)$–torsor $\mathcal{P}$ as an $\mathcal{A}ut_{\mathfrak{G}}(\varphi(s))$–torsor by giving it the left action coming from φ_s^{-1}. Therefore, we have a chain of equivalences

$$\begin{aligned} \mathfrak{F} \to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{F}}(s)) \to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{G}}(\varphi(s))) &\longrightarrow \mathfrak{G} \\ x &\longmapsto \mathcal{I}som(s|_{p(x)}, x) \wedge^{\varphi_s|_{p(x)}} \varphi(s)|_{p(x)}. \end{aligned}$$

Now, by composing the equivalence $\mathfrak{F} \to \mathfrak{Tors}(\mathcal{A}ut_{\mathfrak{F}}(s))$ with its quasi-inverse, we see that there is an isomorphism

$$x \xrightarrow{\sim} \mathcal{I}som(s|_{p(x)}, x) \wedge^{\mathcal{A}ut_{\mathfrak{F}}(s)|_{p(x)}} s|_{p(x)}.$$

Furthermore, by [Gir, III.2.3.11] we have that

$$\varphi(\mathcal{I}som(s|_{p(x)}, x) \wedge^{\mathcal{A}ut_{\mathfrak{F}}(s)|_{p(x)}} s|_{p(x)}) \xrightarrow{\sim} \mathcal{I}som(s|_{p(x)}, x) \wedge^{\varphi_s} \varphi(s)|_{p(x)}.$$

Therefore, φ is canonically isomorphic to the composition of three equivalences above and is hence an equivalence itself, as claimed. □

2. Globalizing Ferrand's Norm Functor

We begin by reviewing Ferrand's construction over rings from [Fer], which generalizes the construction of Knus and Ojanguren [KO] (see [Fer, 5.3]) and also Tignol's construction over a field from [Ti]. In the subsequent sections, we generalize this construction to our setting over a scheme. The norm functor has also been generalized in various other ways, for example to quasi-coherent sheaves on algebraic spaces, by Rydh in his thesis [Ry]. The interested reader can find the details in [RyII].

2.1. **Ferrand's Norm Functor over Rings.** Ferrand's construction begins with the Γ*–algebra*, also called the *divided power algebra*, of a module. We summarize this construction and refer to [B:A2, IV,§5.4] and the related exercises as well as [Ro] for more details.

2.1.1. *The Construction.* Let R be a unital, commutative, associative ring and let M be an R–module. We let $\Gamma_R(M)$ be the unital, commutative, associative R–algebra generated by symbols $\gamma^d(m)$ for $d \in \mathbb{N} = \{0, 1, \ldots\}$ and $m \in M$, subject to the relations

$$\begin{aligned} \gamma^0(m) &= 1_{\Gamma(M)} \\ \gamma^d(rm) &= r^d\gamma^d(m) \\ \gamma^d(m+n) &= \sum_{r=0}^{d} \gamma^r(m)\gamma^{d-r}(n) \\ \gamma^{d_1}(m)\gamma^{d_2}(m) &= \binom{d_1+d_2}{d_1}\gamma^{d_1+d_2}(m) \end{aligned}$$

for all $d, d_1, d_2 \in \mathbb{N}$, $m, n \in M$, and $r \in R$. Here $\binom{d_1+d_2}{d_1} = \frac{(d_1+d_2)!}{d_1!d_2!}$ is the binomial coefficient. We note that [B:A2] uses the notation $\gamma_d(m)$ where we follow [Fer] and use $\gamma^d(m)$. The algebra $\Gamma_R(M)$ is $\mathbb{N}$–graded by total degree of the "exponents", i.e., setting

$$\Gamma_R^d(M) = \mathrm{Span}_R\left(\left\{\gamma^{d_1}(m_1)\gamma^{d_2}(m_2)\ldots\gamma^{d_k}(m_k) \mid \sum_{i=1}^{k} d_i = d, m_i \in M\right\}\right)$$

gives an $\mathbb{N}$–grading $\Gamma_R(M) = \oplus_{d\in\mathbb{N}}\Gamma_R^d(M)$. The assignment $M \mapsto \Gamma_R(M)$ is functorial in M. Given a morphism $f\colon M_1 \to M_2$ of R–modules, we set $\Gamma_R(f)$ to be the map defined on generators as

$$\begin{aligned} \Gamma_R(f)\colon \Gamma_R(M_1) &\to \Gamma_R(M_2) \\ \gamma^d(m) &\mapsto \gamma^d(f(m)). \end{aligned}$$

This is an algebra homomorphism and thus we have a functor $\Gamma_R\colon \mathfrak{Mod}_R \to \mathfrak{Rings}_R$. The morphisms $\Gamma_R(f)$ are also graded morphisms and therefore

via restriction to homogeneous components we also obtain endofunctors $\Gamma^d_R\colon \mathfrak{Mod}_R \to \mathfrak{Mod}_R$.

Given two R–modules M and N, [Fer, 2.4.1] says that there exists a unique R–linear map

$$\mu\colon \Gamma^d_R(M) \otimes_R \Gamma^d_R(N) \to \Gamma^d_R(M \otimes_R N) \tag{2.1.1}$$

with the property that $\gamma^d(m) \otimes \gamma^d(n) \mapsto \gamma^d(m \otimes n)$ for all $m \in M$ and $n \in N$.

Now consider a ring extension $R \to R'$. We may consider R' as an R–module and for $d \in \mathbb{N}$ obtain the R–module $\Gamma^d_R(R')$. Using the map of (2.1.1) and the multiplication of R', $m\colon R' \otimes_R R' \to R'$, we define a multiplication on $\Gamma^d_R(R')$ by

$$\Gamma^d_R(R') \otimes_R \Gamma^d_R(R') \xrightarrow{\mu} \Gamma^d_R(R' \otimes_R R') \xrightarrow{\Gamma^d_R(m)} \Gamma^d_R(R').$$

This makes $\Gamma^d_R(R')$ a unital, commutative, associative R–algebra as it inherits these properties from R'. If we are now also given an R'–module M', we may similarly view it as an R–module and construct $\Gamma^d_R(M')$. This can be equipped with the structure of a $\Gamma^d_R(R')$–module, again using the map (2.1.1) and the map $R' \otimes_R M' \to M'$ defined by $r' \otimes m' \mapsto r'm'$ coming from the R'–module structure of M'. Therefore, the composition

$$\Gamma^d_R(R') \otimes_R \Gamma^d_R(M') \xrightarrow{\mu} \Gamma^d_R(R' \otimes_R M') \to \Gamma^d_R(M')$$

makes $\Gamma^d_R(M')$ a $\Gamma^d_R(R')$–module, as in [Fer, 2.4.6]. This structure has the property that $\gamma^d(r') \cdot \gamma^d(m') = \gamma^d(r'm')$.

Now, assume that the ring extension $R \to R'$ is locally free of finite rank d. We therefore have the determinant map, $\det\colon \mathrm{End}_R(R') \to R$. For $r' \in R'$, the determinant of the left multiplication by r' yields the norm map

$$\mathrm{norm}_{R'/R}\colon R' \to R. \tag{2.1.2}$$

By [Fer, 3.1.2], there exists an R–algebra homomorphism $\pi\colon \Gamma^d_R(R') \to R$ with the property that $\pi(\gamma^d(r')) = \mathrm{norm}_{R'/R}(r')$ for all $r' \in R'$. This is used to define the *norm* of an R'–module M'. Namely, the R–module

$$N_{R'/R}(M') = \Gamma^d_R(M') \otimes_{\Gamma^d_R(R')} R$$

where $\Gamma^d_R(R')$ acts on R via π. Since Γ^d_R is a functor, so is $N_{R'/R}\colon \mathfrak{Mod}_{R'} \to \mathfrak{Mod}_R$. The norm of each M' comes equipped with a canonical (non-linear) function

$$\begin{aligned} \nu_{M'}\colon M' &\to N_{R'/R}(M') \\ m' &\mapsto \gamma^d(m') \otimes 1_R. \end{aligned}$$

This function has the property that for all $m' \in M'$ and $r' \in R'$, we have $\nu_{M'}(r'm') = \mathrm{norm}_{R'/R}(r') \cdot \nu_{M'}(m')$. This can be seen by calculation since

$$\begin{aligned} r'm' \mapsto \gamma^d(r'm') \otimes 1_R &= (\gamma^d(r') \cdot \gamma^d(m')) \otimes 1_R \\ &= \gamma^d(m') \otimes \pi(\gamma^d(r')) \cdot 1_R \end{aligned}$$

$$= \gamma^d(m') \otimes \mathrm{norm}_{R'/R}(r').$$

2.1.2. *Polynomial Laws.* While the above description is the essentials of Ferrand's construction, he instead primarily works with *polynomial laws*. Let $\mathfrak{Rings}_R$ denote the category of R–algebras which are themselves associative, commutative, and unital. For an R–module N, denote by $\mathbf{W}_R(N)\colon \mathfrak{Rings}_R \to \mathfrak{Ab}$ the functor $Q \mapsto N \otimes_R Q$. Note that $\mathbf{W}_R$ is functorial itself. A morphism $\varphi\colon N_1 \to N_2$ of R–modules gives rise to a natural transformation $\mathbf{W}_R(\varphi)\colon \mathbf{W}_R(N_1) \to \mathbf{W}_R(N_2)$ defined over $Q \in \mathfrak{Rings}_R$ by $\mathbf{W}_R(\varphi)(Q) = \varphi \otimes 1\colon N_1 \otimes_R Q \to N_2 \otimes_R Q$. For two R–modules N_1 and N_2, a natural transformation of functors $\mathbf{W}_R(N_1) \to \mathbf{W}_R(N_2)$ is called a *polynomial law*. Of course, $\mathbf{W}_R(\varphi)$ defined above is an example. Such examples are linear, however a general polynomial law need not be. For example, a polynomial law $\boldsymbol{\nu}\colon \mathbf{W}_R(N_1) \to \mathbf{W}_R(N_2)$ is called *homogeneous of degree d* if we have $\boldsymbol{\nu}(qn) = q^d\boldsymbol{\nu}(n)$ for all $Q \in \mathfrak{Rings}_R$, $r \in Q$, and $m \in N_1 \otimes_R Q$. We will generally denote polynomial laws in bold. The canonical, and indeed universal as explained below, example of such a homogeneous of degree d polynomial law is

$$\boldsymbol{\gamma}^d\colon \mathbf{W}_R(N_1) \to \mathbf{W}_R(\Gamma^d_R(N_1)) \tag{2.1.3}$$

which behaves over $Q \in \mathfrak{Rings}_R$ by

$$\sum_{i=1}^{k} n_i \otimes q_i \mapsto \sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(n_1)\dots\gamma^{a_k}(n_k) \otimes q_1^{a_1}\dots q_k^{a_k}.$$

By [Fer, 2.2.4], which itself quotes [Ro, IV.1], if $\boldsymbol{\nu}\colon \mathbf{W}_R(N_1) \to \mathbf{W}_R(N_2)$ is a homogeneous polynomial law of degree d between two R–modules, then there exists a unique R–module homomorphism $\varphi_{\boldsymbol{\nu}}\colon \Gamma^d_R(N_1) \to N_2$ such that $\boldsymbol{\nu} = \mathbf{W}_R(\varphi_{\boldsymbol{\nu}}) \circ \boldsymbol{\gamma}^d$.

If $R \to R'$ is a finite locally free extension, for any $Q \in \mathfrak{Rings}_R$ the extension $Q \to R' \otimes_R Q$ is also finite locally free and hence has a its own norm map. Given a morphism $f\colon Q_1 \to Q_2$ in $\mathfrak{Rings}_R$, the associated norm maps are related by the commutative diagram

$$\begin{array}{ccccc} R' \otimes_R Q_2 & \xrightarrow{\mathrm{norm}_{(R'\otimes_R Q_2)/Q_2}} & Q_2 & \xrightarrow{1\otimes\mathrm{Id}} & R \otimes_R Q_2 \\ \uparrow{\scriptstyle \mathrm{Id}\otimes f} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \mathrm{Id}\otimes f} \\ R' \otimes_R Q_1 & \xrightarrow{\mathrm{norm}_{(R'\otimes_R Q_1)/Q_1}} & Q_1 & \xrightarrow{1\otimes\mathrm{Id}} & R \otimes_R Q_2. \end{array} \tag{2.1.4}$$

Therefore, we have a polynomial law, $\mathbf{norm}\colon \mathbf{W}_R(R') \to \mathbf{W}_R(R)$, given by the norms. If R' is of rank d, then the norm is a homogeneous polynomial law of degree d. The map $\pi\colon \Gamma^d_R(R') \to R$ used above is simply $\varphi_{\mathbf{norm}}$ coming from the universal property of $\Gamma^d_R(R')$.

If M' is an R'–module and N is an R–module, we may consider polynomial laws $\boldsymbol{\nu}\colon \mathbf{W}_R(M') \to \mathbf{W}_R(N)$. If such a $\boldsymbol{\nu}$ has the property that

$$\boldsymbol{\nu}(r'm') = \mathbf{norm}(r')\boldsymbol{\nu}(m')$$

for all $Q \in \mathfrak{Rings}_R$, $r' \in R' \otimes_R Q$, and $m' \in M' \otimes_R Q$, then using the terminology of Ferrand we say that $\boldsymbol{\nu}$ is a *normic polynomial law*. Of course, $\mathbf{norm}\colon \mathbf{W}_R(R') \to \mathbf{W}_R(R)$ is a normic polynomial law since the norm is multiplicative, i.e., we have

$$\mathrm{norm}_{(R'\otimes_R Q)/Q}(ab) = \mathrm{norm}_{(R'\otimes_R Q)/Q}(a)\mathrm{norm}_{(R'\otimes_R Q)/Q}(b)$$

for all $Q \in \mathfrak{Rings}_R$.

2.1.3. *Base Change.* The construction of the Γ–algebra is compatible with ring extensions as follows, demonstrated in [Ro, III.3]. Given an arbitrary ring extension $R \to Q$ and an R–module M, there is a canonical graded isomorphism of Q–algebras

$$\begin{aligned} \varphi_Q\colon \Gamma_R(M) \otimes_R Q &\xrightarrow{\sim} \Gamma_Q(M \otimes_R Q) \\ \gamma^d(m) \otimes q &\mapsto q \cdot \gamma^d(m \otimes 1). \end{aligned} \tag{2.1.5}$$

Since this isomorphism is graded, for each d it restricts to an isomorphism of Q–modules $\varphi^d_Q\colon \Gamma^d_R(M) \otimes_R Q \xrightarrow{\sim} \Gamma^d_R(M \otimes_R Q)$. In the case $R \to R'$ is another ring extension, then $\varphi^d_Q\colon \Gamma^d_R(R') \otimes_R Q \xrightarrow{\sim} \Gamma^d_R(R' \otimes_R Q)$ is an isomorphism of Q–algebras.

Since φ^d_Q is the restriction of a ring homomorphism, for $\sum a_i = d$ and $m_i \in M$ we have

$$\begin{aligned} &\varphi^d_Q(\gamma^{a_1}(m_1)\dots\gamma^{a_k}(m_k) \otimes q) \\ =&\varphi^d_Q\big((\gamma^{a_1}(m_1) \otimes q) \cdot (\gamma^{a_2}(m_2) \otimes 1)\dots(\gamma^{a_k}(m_k) \otimes 1)\big) \\ =&q \cdot \gamma^{a_1}(m_1 \otimes 1)\dots\gamma^{a_k}(m_k \otimes 1). \end{aligned}$$

For our purposes, we will be interested in the following notion of base change. Consider a pushout diagram of commutative R–algebras, or equivalently simply a pushout diagram of rings,

$$D = \begin{array}{ccc} R' & \xrightarrow{f'} & Q' \\ \uparrow & & \uparrow \\ R & \xrightarrow{f} & Q \end{array}$$

where the left vertical arrow is a finite locally free extensions of rank d, which then also holds for the right vertical arrow. Moreover, since this is a pushout diagram, there is a unique isomorphism of R–algebras

$$\psi\colon R' \otimes_R Q \xrightarrow{\sim} Q'. \tag{2.1.6}$$

However, because we have applications to stacks in mind, we avoid identifying Q' and $R' \otimes_R Q$.

2.2. **Lemma.** *Assume we have a pushout diagram of rings as above. Then, there is a commutative diagram*

$$\begin{array}{ccccccc}
\Gamma^d_R(R') & \xrightarrow{\mathrm{Id}\otimes 1} & \Gamma^d_R(R')\otimes_R Q & \xrightarrow{\varphi^d_Q} & \Gamma^d_Q(R'\otimes_R Q) & \xrightarrow{\Gamma^d_Q(\psi)} & \Gamma^d_Q(Q') \\
\downarrow{\scriptstyle \pi_{R'}} & & \downarrow{\scriptstyle \pi_{R'}\otimes \mathrm{Id}} & & \downarrow{\scriptstyle \pi_{R'\otimes_R Q}} & & \downarrow{\scriptstyle \pi_{Q'}} \\
R & \xrightarrow{\mathrm{Id}\otimes 1} & R\otimes_R Q & \xrightarrow{\mathrm{can}} & Q & = & Q
\end{array}$$

$$f$$

where φ^d_Q is as defined in (2.1.5), $\pi_{R'}$ is the unique morphism such that $\mathbf{W}_R(\pi_{R'})\circ \boldsymbol{\gamma}^d = \mathbf{norm}\colon \mathbf{W}_R(R') \to \mathbf{W}_R(R)$, and likewise for $\pi_{R'\otimes_R Q}$ and $\pi_{Q'}$.

Proof. The commutativity of the left square of the diagram is obvious. The commutativity of the middle square of the diagram is more involved, following from the universal properties defining $\pi_{R'}$ and $\pi_{R'\otimes_R Q}$.

We consider the canonical isomorphism $\mathbf{W}_Q(f)\colon \mathbf{W}_R(R)|_Q \xrightarrow{\sim} \mathbf{W}_Q(Q)$ arising from the isomorphisms $f\otimes\mathrm{Id}\colon R\otimes_R P \xrightarrow{\sim} Q\otimes_Q P$ for all $P\in \mathfrak{Rings}_Q$ as well as the isomorphism $\mathbf{W}_Q(\mathrm{Id}\otimes 1_Q)\colon \mathbf{W}_R(R')|_Q \xrightarrow{\sim} \mathbf{W}_Q(R'\otimes_R Q)$ arising from the isomorphisms $R'\otimes_R P \xrightarrow{\sim} (R'\otimes_R Q)\otimes_Q P$. Similarly, there is an isomorphism

$$\mathbf{W}_Q(\varphi^d_Q\circ(\mathrm{Id}\otimes 1_Q))\colon \mathbf{W}_R(\Gamma^d_R(R'))|_Q \xrightarrow{\sim} \mathbf{W}_Q(\Gamma^d_Q(R'\otimes_R Q)).$$

Let $\boldsymbol{\gamma}^d_R\colon \mathbf{W}_R(R') \to \mathbf{W}_R(\Gamma^d_R(R'))$ be the polynomial law of (2.1.3) and likewise let $\boldsymbol{\gamma}^d_Q\colon \mathbf{W}_Q(R'\otimes_R Q) \to \mathbf{W}_Q(\Gamma^d_Q(R'\otimes_R Q))$ be the analogous polynomial law for $R'\otimes_R Q$. We claim that the diagram

$$\begin{array}{ccc}
\mathbf{W}_R(R')|_Q & \xrightarrow{\boldsymbol{\gamma}^d_R|_Q} & \mathbf{W}_R(\Gamma^d_R(R'))|_Q \\
\downarrow{\scriptstyle \mathbf{W}_Q(\mathrm{Id}\otimes 1_Q)} & & \downarrow{\scriptstyle \mathbf{W}_Q(\varphi^d_Q\circ(\mathrm{Id}\otimes 1_Q))} \\
\mathbf{W}_Q(R'\otimes_R Q) & \xrightarrow{\boldsymbol{\gamma}^d_Q} & \mathbf{W}_Q(\Gamma^d_Q(R'\otimes_R Q))
\end{array}$$

commutes. For $P\in\mathfrak{Rings}_Q$, one can trace the image of an element $\sum_{i=1}^k r'_i\otimes p_i \in R'\otimes_R P$ through the diagram, obtaining

$$\begin{array}{ccc}
\displaystyle\sum_{i=1}^k r'_i\otimes p_i & \longmapsto & \displaystyle\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(r'_1)\dots\gamma^{a_k}(r'_k)\otimes p_1^{a_1}\dots p_k^{a_k} \\
\downarrow & & \downarrow \\
\displaystyle\sum_{i=1}^k (r'_i\otimes 1_Q)\otimes p_i & \longmapsto & \displaystyle\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(r'_1\otimes 1_Q)\dots\gamma^{a_k}(r'_k\otimes 1_Q)\otimes p_1^{a_1}\dots p_k^{a_k}
\end{array}$$

which justifies our claim. Next, we let $\mathbf{norm}_{R'}\colon \mathbf{W}_R(R') \to \mathbf{W}_R(R)$ be the normic polynomial law associated to the norm of R' and likewise we let $\mathbf{norm}_{R'\otimes_R Q}\colon \mathbf{W}_Q(R'\otimes_R Q) \to \mathbf{W}_Q(Q)$ be the one associated to $R'\otimes_R Q$. We claim that the diagram

$$\begin{array}{ccc} \mathbf{W}_R(R')|_Q & \xrightarrow{\mathbf{norm}_{R'}|_Q} & \mathbf{W}_R(R)|_Q \\ \downarrow{\scriptstyle \mathbf{W}_Q(\mathrm{Id}\otimes 1_Q)} & & \downarrow{\scriptstyle \mathbf{W}_Q(f)} \\ \mathbf{W}_Q(R'\otimes_R Q) & \xrightarrow{\mathbf{norm}_{R'\otimes_R Q}} & \mathbf{W}_Q(Q) \end{array}$$

commutes. For a ring $P \in \mathfrak{Rings}_Q$ this diagram becomes

$$\begin{array}{ccccc} R'\otimes_R P & \xrightarrow{\mathrm{norm}_{(R'\otimes_R P)/P}} & P & \xrightarrow{1_R\otimes \mathrm{Id}} & R\otimes_R P \\ \downarrow{\scriptstyle (\mathrm{Id}\otimes 1_Q)\otimes\mathrm{Id}} & & \| & & \downarrow{\scriptstyle f\otimes\mathrm{Id}} \\ (R'\otimes_R Q)\otimes QP & \xrightarrow{\mathrm{norm}_{((R'\otimes_R Q)\otimes_Q P)/P}} & P & \xrightarrow{1_Q\otimes\mathrm{Id}} & Q\otimes_Q P. \end{array}$$

The right square clearly commutes. The leftmost arrow in the diagram is an isomorphism of P–modules and the determinant respects such isomorphisms. In particular, the determinant of left multiplication by $x \in R'\otimes_R P$ will be the same as the determinant of left multiplication by $(\mathrm{Id}\otimes 1_Q)(x) \in R'\otimes_R Q)\otimes QP$. Thus, the left square commutes and hence the original diagram commutes as claimed.

Therefore, we have a commutative diagram

$$\begin{array}{ccccc} & & \overset{\mathbf{norm}_{R'}|_Q}{\frown} & & \\ \mathbf{W}_R(R')|_Q & \xrightarrow{\gamma_R^d|_Q} & \mathbf{W}_R(\Gamma_R^d(R'))|_Q & \xrightarrow{\mathbf{W}_R(\pi_{R'})|_Q} & \mathbf{W}_R(R)|_Q \\ \downarrow{\scriptstyle \mathbf{W}_Q(\mathrm{Id}\otimes 1_Q)} & & \downarrow{\scriptstyle \mathbf{W}_Q(\varphi_Q^d\circ(\mathrm{Id}\otimes 1_Q))} & & \downarrow{\scriptstyle \mathbf{W}_Q(f)} \\ \mathbf{W}_Q(R'\otimes_R Q) & \xrightarrow{\gamma_Q^d} & \mathbf{W}_Q(\Gamma_Q^d(R'\otimes_R Q)) & & \mathbf{W}_Q(Q). \\ & & \underset{\mathbf{norm}_{R'\otimes_R Q}}{\smile} & & \end{array}$$

The composition $\mathbf{W}_Q(f)\circ\mathbf{W}_R(\pi_{R'})|_Q\circ\mathbf{W}_Q(\varphi_Q^d\circ(\mathrm{Id}\otimes 1_Q))^{-1}$ appears over $P \in \mathfrak{Rings}_Q$ as the outer edges in the following commutative diagram.

$$\begin{array}{ccccc} \Gamma_R^d(R')\otimes_R P & & \xrightarrow{\pi_{R'}\otimes\mathrm{Id}} & & R\otimes_R P \\ \uparrow{\scriptstyle \mathrm{can}\otimes\mathrm{Id}} & & & & \downarrow{\scriptstyle (\mathrm{Id}\otimes 1_Q)\otimes\mathrm{Id}} \\ (\Gamma_R^d(R')\otimes_R Q)\otimes_Q P & & \xrightarrow{(\pi_{R'}\otimes\mathrm{Id})\otimes\mathrm{Id}} & & (R\otimes_R Q)\otimes_Q P \\ \uparrow{\scriptstyle (\varphi_Q^d)^{-1}\otimes\mathrm{Id}} & & & & \downarrow{\scriptstyle \mathrm{can}\otimes\mathrm{Id}} \\ \Gamma_Q^d(R'\otimes_R Q)\otimes_Q P & & & & Q\otimes_Q P \end{array} \qquad f\otimes\mathrm{Id}$$

By instead using the bottom rectangle, we obtain the equality

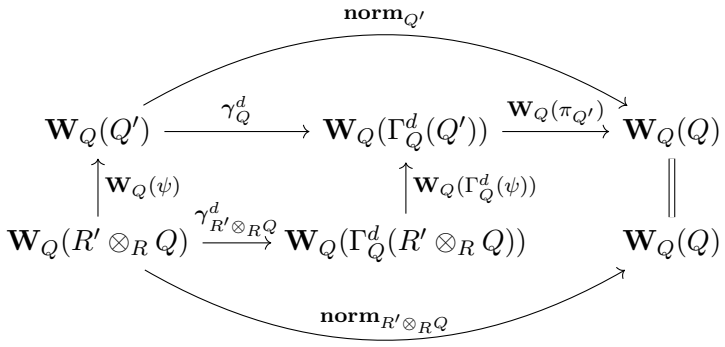
$$\mathbf{W}_Q(f)\circ\mathbf{W}_R(\pi_{R'})|_Q\circ\mathbf{W}_Q(\varphi_Q^d\circ(\mathrm{Id}\otimes 1_Q))^{-1} = \mathbf{W}_Q(\mathrm{can}\circ(\pi_{R'}\otimes\mathrm{Id})\circ(\varphi_Q^d)^{-1}).$$

Finally, the universal property which defines $\pi_{R'\otimes_R Q}$ then enforces that

$$\pi_{R'\otimes_R Q} = \mathrm{can}\circ(\pi_{R'}\otimes\mathrm{Id})\circ(\varphi_Q^d)^{-1}$$

as desired. Thus the middle square commutes.

For the commutativity of the right square we have a similar argument. We consider the commutative diagram



where it is clear the left square commutes and the outer square involving the norms commutes because the determinant is invariant under module isomorphisms as above. Therefore, since $\mathbf{W}_Q(\pi_{Q'})\circ\mathbf{W}_Q(\Gamma_Q^d(\psi)) = \mathbf{W}_Q(\pi_{Q'}\circ\Gamma_Q^d(\psi))$, the universal property of $\pi_{R'\otimes_R Q}$ enforces that

$$\pi_{R'\otimes_R Q} = \pi_{Q'}\circ\Gamma_Q^d(\psi)$$

as claimed. This finishes the proof. □

2.3. **Lemma.** *Consider a pushout diagram of rings*

$$D = \begin{array}{ccc} R' & \xrightarrow{f'} & Q' \\ \uparrow & & \uparrow \\ R & \xrightarrow{f} & Q \end{array}$$

where the vertical arrows are finite locally free extensions of rank d.

(i) *For an R'–module M' there is a canonical isomorphism of Q–modules*

$$\begin{array}{ccc} \theta_{D,M'}\colon N_{R'/R}(M')\otimes_R Q & \xrightarrow{\quad\sim\quad} & N_{Q'/Q}(M'\otimes_{R'} Q') \\ \| & & \| \\ (\Gamma_R^d(M')\otimes_{\Gamma_R^d(R')} R)\otimes_R Q & & \Gamma_Q^d(M'\otimes_{R'} Q')\otimes_{\Gamma_Q^d(Q')} Q \end{array}$$

$$(\gamma^{a_1}(m_1')\dots\gamma^{a_k}(m_k')\otimes r)\otimes q \longmapsto \gamma^{a_1}(m_1'\otimes 1)\dots\gamma^{a_k}(m_k'\otimes 1)\otimes f(r)q$$

for $a_i\in\mathbb{N}$ with $\sum a_i = d$.

(ii) *There is an isomorphism of functors*

$$\theta_D \colon N_{R'/R}(\_) \otimes_R Q \xrightarrow{\sim} N_{Q'/Q}(\_ \otimes_{R'} Q')$$

induced by the isomorphisms of (i).

Proof. (i): Because D is a pushout diagram, we have a unique isomorphism $\psi \colon R' \otimes_R Q \xrightarrow{\sim} Q'$ as in (2.1.6). This isomorphism also gives us an isomorphism

$$\psi_{M'} \colon M' \otimes_R Q \xrightarrow{(\mathrm{Id}\otimes 1)\otimes \mathrm{Id}} M' \otimes_{R'} R' \otimes_R Q \xrightarrow{\mathrm{Id}\otimes\psi} M' \otimes_{R'} Q'$$

which is ψ–equivariant. Now, we consider the following composition of isomorphisms

$$\begin{aligned}(\Gamma^d_R(M') \otimes_{\Gamma^d_R(R')} R) \otimes_R Q \xrightarrow{\mathrm{can}} &(\Gamma^d_R(M') \otimes_R Q) \otimes_{\Gamma^d_R(R')\otimes_R Q} (R \otimes_R Q) \\ \xrightarrow{\varphi^d_Q \otimes \mathrm{can}} &\Gamma^d_Q(M' \otimes_R Q) \otimes_{\Gamma^d_Q(R'\otimes_R Q)} Q \\ \xrightarrow{\Gamma^d_Q(\psi_{M'})\otimes \mathrm{Id}} &\Gamma^d_Q(M' \otimes_{R'} Q') \otimes_{\Gamma^d_Q(Q')} Q\end{aligned}$$

where φ^d_Q is the isomorphism of (2.1.5) and the final two isomorphisms are well-defined due to the results of Lemma 2.2. Tracing an element through this composition yields

$$\begin{aligned}(\gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes r)\otimes q \mapsto &(\gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes 1_Q)\otimes(r\otimes q) \\ \mapsto &\gamma^{a_1}(m'_1\otimes 1_Q)\dots\gamma^{a_k}(m'_k\otimes 1_Q)\otimes f(r)q \\ \mapsto &\gamma^{a_1}(m'_1\otimes 1_{Q'})\dots\gamma^{a_k}(m'_k\otimes 1_{Q'})\otimes f(r)q\end{aligned}$$

which is the claimed formula.
(ii): The fact that θ_D is a natural transformation follows from the functoriality of the Γ–algebras. In particular, for a morphism $\phi \colon M'_1 \to M'_2$ of R'–modules, tracing an element of $N_{R'/R}(M'_1) \otimes_R Q$ through the diagram

$$\begin{array}{ccc} N_{R'/R}(M'_2)\otimes_R Q & \xrightarrow{\theta_D(M'_2)} & N_{Q'/Q}(M'_2\otimes_{R'} Q') \\ \uparrow{\scriptstyle N_{R'/R}(\phi)\otimes\mathrm{Id}} & & \uparrow{\scriptstyle N_{Q'/Q}(\phi\otimes\mathrm{Id})} \\ N_{R'/R}(M'_1)\otimes_R Q & \xrightarrow{\theta_D(M'_1)} & N_{Q'/Q}(M'_1\otimes_{R'} Q') \end{array}$$

yields, using the formula of (i) above,

$$\begin{array}{ccc} \gamma^{a_1}(\phi(m'_1))\dots\gamma^{a_k}(\phi(m'_k))\otimes r\otimes q & \mapsto & \gamma^{a_1}(\phi(m'_1)\otimes 1)\dots\gamma^{a_k}(\phi(m'_k)\otimes 1)\otimes f(r)q \\ \uparrow & & \uparrow \\ \gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes r\otimes q & \longmapsto & \gamma^{a_1}(m'_1\otimes 1)\dots\gamma^{a_k}(m'_k\otimes 1)\otimes f(r)q \end{array}$$

and so we see the diagram commutes. □

2.4. **Remark.** In the case when $Q' = R' \otimes_R Q$, Ferrand states that there is an isomorphism of functors $N_{R'/R}(\_) \otimes_R Q \cong N_{(R' \otimes_R Q)/Q}(\_ \otimes_R Q)$ as his property (N2) in [Fer, §1]. Composing this with the isomorphism $N_{(R' \otimes_R Q)/Q}(\rho)$, where $\rho\colon M' \otimes_R Q \xrightarrow{\sim} M' \otimes_{R'} (R' \otimes_R Q)$ is the canonical isomorphism, yields the isomorphism of Lemma 2.3(ii) above.

2.4.1. *Universal Property.* For an R'–module M', Ferrand assembles the canonical functions $\nu_{M'}\colon M' \to N_{R'/R}(M')$ from Section 2.1.1 into a normic polynomial law as follows. For $Q \in \mathfrak{Rings}_R$, we define the function

$$\boldsymbol{\nu}_{M'}(Q)\colon M' \otimes_R Q \xrightarrow{\nu_{M' \otimes_R Q}} N_{(R' \otimes_R Q)/Q}(M' \otimes_R Q) \xrightarrow{\phi_Q} N_{R'/R}(M') \otimes_R Q$$

where the final map ϕ_Q is the isomorphism $\left((\varphi_Q^d) \otimes \mathrm{can}\right) \circ \mathrm{can}\big)^{-1}$, using the notation of Lemma 2.3. For a morphism $f\colon Q_1 \to Q_2$ in $\mathfrak{Rings}_R$, we claim the diagram

$$\begin{array}{ccccc} M' \otimes_R Q_2 & \xrightarrow{\nu_{M' \otimes_R Q_2}} & N_{(R' \otimes_R Q_2)/Q_2}(M' \otimes_R Q_2) & \xrightarrow{\phi_{Q_2}} & N_{R'/R}(M') \otimes_R Q_2 \\ \uparrow{\scriptstyle \mathrm{Id} \otimes f} & & & & \uparrow{\scriptstyle \mathrm{Id} \otimes f} \\ M' \otimes_R Q_1 & \xrightarrow{\nu_{M' \otimes_R Q_1}} & N_{(R' \otimes_R Q_1)/Q_1}(M' \otimes_R Q_1) & \xrightarrow{\phi_{Q_1}} & N_{R'/R}(M') \otimes_R Q_1 \end{array}$$

commutes. Indeed, tracing an element along the bottom and up we get

$$\begin{aligned} \sum_{i=1}^k m_i' \otimes q_i \mapsto & \gamma^d\Big(\sum_{i=1}^k m_i' \otimes q_i\Big) \otimes 1_{Q_1} \\ &= \sum_{\substack{(a_1,\dots,a_k) \in \mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m_1' \otimes q_1) \dots \gamma^{a_k}(m_k' \otimes q_k) \otimes 1_{Q_1} \\ &= \sum_{\substack{(a_1,\dots,a_k) \in \mathbb{N}^k \\ a_1+\dots+a_k=d}} q_1^{a_1} \dots q_k^{a_k} \gamma^{a_1}(m_1' \otimes 1_{Q_1}) \dots \gamma^{a_k}(m_k' \otimes 1_{Q_1}) \otimes 1_{Q_1} \\ &= \sum_{\substack{(a_1,\dots,a_k) \in \mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m_1' \otimes 1_{Q_1}) \dots \gamma^{a_k}(m_k' \otimes 1_{Q_1}) \otimes q_1^{a_1} \dots q_k^{a_k} \\ &\mapsto \left(\sum_{\substack{(a_1,\dots,a_k) \in \mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m_1') \dots \gamma^{a_k}(m_k') \otimes 1_R \right) \otimes q_1^{a_1} \dots q_k^{a_k} \\ &\mapsto \left(\sum_{\substack{(a_1,\dots,a_k) \in \mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m_1') \dots \gamma^{a_k}(m_k') \otimes 1_R \right) \otimes f(q_1^{a_1} \dots q_k^{a_k}). \end{aligned}$$

Instead, going up and then across the top begins by sending

$$\sum_{i=1}^k m_i' \otimes q_i \mapsto \sum_{i=1}^k m_i' \otimes f(q_i),$$

which then follows similar computations to arrive at

$$\left(\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes 1_R \right) \otimes f(q_1)^{a_1}\dots f(q_k)^{a_k}.$$

The two paths are therefore equal.

This shows that $\boldsymbol{\nu}_{M'}\colon \mathbf{W}_R(M') \to \mathbf{W}_R(N_{R'/R}(M'))$ is a well-defined natural transformation given over $Q \in \mathfrak{Rings}_R$ by $\boldsymbol{\nu}_{M'}(Q)$, justifying our previous notation. It is clear that this is a normic polynomial law since each $\nu_{M'}$ is normic. Ferrand proves that this polynomial law has the following properties.

2.5. **Theorem** (Ferrand)**.** *The functor $N_{R'/R}\colon \mathfrak{Mod}_{R'} \to \mathfrak{Mod}_R$ together with the normic polynomial laws $\boldsymbol{\nu}_{M'}\colon \mathbf{W}_R(M') \to \mathbf{W}_R(N_{R'/R}(M'))$ for every R'–module M' have the following properties.*

(i) *$N_{R'/R}(R') = R$ and $\boldsymbol{\nu}_{R'} = \mathbf{norm}$.*

(ii) *The pair $(N_{R'/R}(M'), \boldsymbol{\nu}_{M'})$ is universal among such pairs. If $(E, \boldsymbol{\nu}')$ is an R–module and normic polynomial law $\boldsymbol{\nu}'\colon \mathbf{W}_R(M') \to \mathbf{W}_R(E)$ pair, then there is a unique morphism of R–modules $\varphi\colon N_{R'/R}(M') \to E$ such that $\boldsymbol{\nu}' = \mathbf{W}_R(\varphi)\circ\boldsymbol{\nu}_{M'}$.*

(iii) *The universal property of $\boldsymbol{\nu}_{M'}$ above induces the image of the norm functor on morphisms. If $\varphi\colon M'_1 \to M'_2$ is an R'–module morphism, then $\boldsymbol{\nu}_{M'_2}\circ\mathbf{W}_R(\varphi)$ is a normic polynomial law and*

$$N_{R'/R}(\varphi)\colon N_{R'/R}(M'_1) \to N_{R'/R}(M'_2)$$

is the unique map making the diagram below commute.

$$\begin{array}{ccc} \mathbf{W}_R(M'_1) & \xrightarrow{\boldsymbol{\nu}_{M'_1}} & \mathbf{W}_R(N_{R'/R}(M'_1)) \\ \downarrow{\scriptstyle \mathbf{W}_R(\varphi)} & & \downarrow{\scriptstyle \mathbf{W}_R(N_{R'/R}(\varphi))} \\ \mathbf{W}_R(M'_2) & \xrightarrow{\boldsymbol{\nu}_{M'_2}} & \mathbf{W}_R(N_{R'/R}(M'_2)) \end{array}$$

To give an example of the form these norms take, and for later reference, we quote another result of Ferrand.

2.6. **Proposition** ([Fer, 3.2.4])**.** *Let $S_1,\dots,S_m$ be finite projective R–algebras of ranks $d_1,\dots,d_m$ respectively. Put $S = S_1\times\dots\times S_m$ and let F be an S–module. Thus, $F = F_1\times\dots\times F_m$ for S_i–modules F_i, $i = 1,\dots,m$. Then, there exists an isomorphism*

$$\phi\colon N_{S/R}(F_1\times\dots\times F_m) \xrightarrow{\sim} N_{S_1/R}(F_1)\otimes_R\dots\otimes_R N_{S_m/R}(F_m)$$

of R–modules such that the normic polynomial $\boldsymbol{\nu}_F$ satisfies

$$(\mathbf{W}_R(\phi)\circ\boldsymbol{\nu}_F)(x_1,\dots,x_m) = \boldsymbol{\nu}_{F_1}(x_1)\otimes\dots\otimes\boldsymbol{\nu}_{F_m}(x_m)$$

for $x_i \in F_i\otimes_R Q$, $Q \in \mathfrak{Rings}_R$.

In particular, if $S = R \times \cdots \times R$ and $E_1, \ldots, E_m$ is a family of R–modules, then we have an isomorphism

$$\phi\colon N_{S/R}(E_1 \times \cdots \times E_m) \xrightarrow{\sim} E_1 \otimes_R \cdots \otimes_R E_m$$

such that the normic polynomial law of $E_1 \times \cdots \times E_m$ is given by

$$(\mathbf{W}_R(\phi) \circ \boldsymbol{\nu}_{E_1 \times \cdots \times E_m})(y_1, \ldots y_m) = y_1 \otimes \cdots \otimes y_m$$

for $y_i \in E_i \otimes_R Q$.

The existence of the isomorphism ϕ in Proposition 2.6 is proven in [Fer, 3.2.4] and the formulas for the normic polynomials can be inferred from the proof of loc. cit.

The universal normic polynomial laws are compatible with the isomorphisms θ_D of Lemma 2.3(ii) in the following way.

2.7. **Lemma.** *Consider a pushout diagram of rings*

$$D = \begin{array}{ccc} R' & \xrightarrow{f'} & Q' \\ \uparrow & & \uparrow \\ R & \xrightarrow{f} & Q \end{array}$$

where the vertical arrows are finite locally free extensions of rank d. Consider the natural isomorphism θ_D of Lemma 2.3(ii). Let $\psi\colon R' \otimes_R Q \xrightarrow{\sim} Q'$ be the unique isomorphism of (2.1.6) and $\psi_{M'}\colon M' \otimes_R Q \xrightarrow{\sim} M' \otimes_{R'} Q'$ the associated isomorphism of Q–modules. For any R'–module M', the universal normic polynomial laws are related via the diagram

$$\begin{array}{ccc} \mathbf{W}_R(M')|_Q & \xrightarrow{\boldsymbol{\nu}_{M'}|_Q} & \mathbf{W}_R(N_{R'/R}(M'))|_Q \\ & & \uparrow{\scriptstyle \mathrm{can}} \\ \uparrow{\scriptstyle \psi'} & & \mathbf{W}_Q(N_{R'/R}(M') \otimes_R Q) \\ & & \downarrow{\scriptstyle \mathbf{W}_Q(\theta_D(M'))} \\ \mathbf{W}_Q(M' \otimes_{R'} Q') & \xrightarrow{\boldsymbol{\nu}_{(M' \otimes_{R'} Q')}} & \mathbf{W}_Q(N_{Q'/Q}(M' \otimes_{R'} Q')). \end{array}$$

Here, the canonical isomorphism is given over $P \in \mathfrak{Rings}_Q$ by the canonical map $(N_{R'/R}(M') \otimes_R Q) \otimes_Q P \xrightarrow{\sim} N_{R'/R}(M') \otimes_R P$ and the isomorphism ψ' is given by

$$(M' \otimes_{R'} Q') \otimes_Q P \xrightarrow{\psi_{M'}^{-1} \otimes \mathrm{Id}} (M' \otimes_R Q) \otimes_Q P \xrightarrow{\mathrm{can}} M' \otimes_R P.$$

Proof. Working over a ring $P \in \mathfrak{Rings}_Q$, the diagram becomes

$$\begin{array}{ccc} M' \otimes_R P & \xrightarrow{\boldsymbol{\nu}_{M'}(P)} & N_{R'/R}(M') \otimes_R P \\ & & \uparrow{\scriptstyle \mathrm{can}} \\ \uparrow{\scriptstyle \psi'(P)} & & N_{R'/R}(M') \otimes_R Q \otimes_Q P \\ & & \downarrow{\scriptstyle \theta_D(M')\otimes \mathrm{Id}} \\ (M' \otimes_{R'} Q') \otimes_Q P & \xrightarrow{\boldsymbol{\nu}_{M'\otimes_{R'}Q'}(P)} & N_{Q'/Q}(M' \otimes_{R'} Q') \otimes_Q P. \end{array}$$

Starting in $M' \otimes_R P$ and tracing an element, we obtain

$$\begin{array}{ccc} \displaystyle\sum_{i=1}^{k} m'_i \otimes p_i & \longmapsto & \left(\displaystyle\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes 1_R \right) \otimes p_1^{a_1}\dots p_k^{a_k} \\ \uparrow & & \updownarrow \\ & & \left(\displaystyle\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m'_1)\dots\gamma^{a_k}(m'_k)\otimes 1_R \right) \otimes 1_Q \otimes p_1^{a_1}\dots p_k^{a_k} \\ & & \updownarrow \\ \displaystyle\sum_{i=1}^{k} (m'_i \otimes 1) \otimes p_i & \mapsto & \left(\displaystyle\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k \\ a_1+\dots+a_k=d}} \gamma^{a_1}(m'_1\otimes 1)\dots\gamma^{a_k}(m'_k\otimes 1)\otimes 1_Q \right) \otimes p_1^{a_1}\dots p_k^{a_k} \end{array}$$

which verifies that the diagram commutes as claimed. □

2.8. **The Norm of Modules.** We globalize Ferrand's norm to our setting over a scheme by applying the constructions of Appendix C to produce a norm morphism of stacks.

2.8.1. *The Construction.* We will apply Proposition C.10 in the following context. First, as in Appendix C.8, define the stack of quasi-coherent modules over S, denoted $p\colon \mathfrak{QCoh} \to \mathfrak{Sch}_S$, to have the following.

(i) Its objects are pairs $(X, \mathcal{F})$ with $X \in \mathfrak{Sch}_S$ and $\mathcal{F}$ a quasi-coherent $\mathcal{O}|_X$–module on $\mathfrak{Sch}_X$.
(ii) Its morphisms are pairs $(g, \varphi)\colon (X', \mathcal{F}') \to (X, \mathcal{F})$ where $g\colon X' \to X$ is a morphism of S–schemes and $\varphi\colon \mathcal{F}' \to g^*(\mathcal{F})$ is a morphism of $\mathcal{O}|_{X'}$–modules. Composition is given by $(g,\varphi)\circ(h,\psi) = (g\circ h, h^*(\varphi)\circ \psi)$.
(iii) Its structure functor is given by $(X, \mathcal{F}) \mapsto X$ and $(g, \varphi) \mapsto g$.

For a scheme $X \in \mathfrak{Sch}_S$, the fiber $\mathfrak{QCoh}(X)$ over X is the category of quasi-coherent $\mathcal{O}|_X$–modules.

Next, define the stack of quasi-coherent modules over a finite locally free extension of rank d, denoted $p\colon \mathfrak{QCoh}^d_{\mathrm{fff}} \to \mathfrak{Sch}_S$, to have

(i) objects which are pairs $(h\colon T' \to T, \mathcal{M})$ where $h\colon T' \to T$ is a finite locally free morphism of constant rank d in $\mathfrak{Sch}_S$ and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}|_{T'}$–module,

(ii) morphisms which are triples

$$(f, g, \varphi)\colon (j\colon X' \to X, \mathcal{N}) \to (h\colon T' \to T, \mathcal{M})$$

where f and g are morphisms in $\mathfrak{Sch}_S$ such that

$$\begin{array}{ccc} X' & \xrightarrow{g} & T' \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & T \end{array}$$

is a fiber product diagram and $\varphi\colon \mathcal{N} \xrightarrow{\sim} g^*(\mathcal{M})$ is an isomorphism of $\mathcal{O}|_{X'}$–modules, and

(iii) structure functor given by $(h\colon T' \to T, \mathcal{M}) \mapsto T$ and $(f, g, \varphi) \mapsto f$.

In the notation of Appendix C.8, we write $\mathfrak{QCoh}^d_{\mathrm{fff}} = \mathfrak{QCoh}_{\mathfrak{I}}$ where $\mathfrak{I}$ is the stack of finite locally free morphisms of constant rank d in $\mathfrak{Sch}_S$ (viewed as a full substack of the stack of affine morphisms $\mathfrak{AffMor}$, also defined in C.8). For each object $h\colon U' \to U$ in $\mathfrak{I}$, i.e., each finite locally free morphism of constant rank d, where U and U' are affine schemes, we set

$$\mathcal{F}_h = N_{\mathcal{O}(U')/\mathcal{O}(U)}\colon \mathfrak{Mod}_{\mathcal{O}(U')} \to \mathfrak{Mod}_{\mathcal{O}(U)}$$

to be Ferrand's norm functor. For each fiber product diagram in $\mathfrak{Sch}_S$ of the form on the left below with associated pushout diagram of rings on the right below

$$D = \begin{array}{ccc} V' & \xrightarrow{f'} & U' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ V & \xrightarrow{f} & U, \end{array} \qquad \begin{array}{ccc} \mathcal{O}(U') & \longrightarrow & \mathcal{O}(V') \\ \uparrow & & \uparrow \\ \mathcal{O}(U) & \longrightarrow & \mathcal{O}(V). \end{array}$$

where $h, h' \in \mathfrak{I}$ and where U, U', V, V' are all affine schemes, we use the isomorphism of functors

$$\theta_D\colon \mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}_{h'}(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')).$$

from Lemma 2.3(ii) corresponding to the pushout diagram of rings. We now verify that these isomorphisms satisfy the required assumptions.

2.9. **Lemma.** *For the stack $\mathfrak{I}$ of finite locally free morphisms of rank d in $\mathfrak{Sch}_S$, the functors $\mathcal{F}_h$ and natural isomorphisms θ_D chosen above satisfy assumptions (C.8.c) and (C.8.d).*

Proof. To verify assumption (C.8.c) holds, consider a fiber product diagram of schemes and associated pushout diagram of rings of the following form.

$$D = \begin{array}{ccc} U' & = & U' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ U & = & U, \end{array} \qquad \begin{array}{ccc} \mathcal{O}(U') & = & \mathcal{O}(U') \\ \uparrow & & \uparrow \\ \mathcal{O}(U) & = & \mathcal{O}(U). \end{array}$$

For any $\mathcal{O}(U')$–module M', we trace an element through the diagram

$$\begin{array}{ccc} N_{\mathcal{O}(U')/\mathcal{O}(U)}(M') \otimes_{\mathcal{O}(U)} \mathcal{O}(U) & & \\ \downarrow{\scriptstyle \theta_D(M')} & \searrow^{\text{can}} & \\ N_{\mathcal{O}(U')/\mathcal{O}(U)}(M' \otimes_{\mathcal{O}(U')} \mathcal{O}(U')) & \xrightarrow[N_{\mathcal{O}(U')/\mathcal{O}(U)}(\text{can})]{} & N_{\mathcal{O}(U')/\mathcal{O}(U)}(M') \end{array}$$

to obtain

$$\begin{array}{ccc} (\gamma^{a_1}(m'_1) \dots \gamma^{a_k}(m'_k) \otimes u_1) \otimes u_2 & & \\ \downarrow & \searrow & \\ \gamma^{a_1}(m'_1 \otimes 1) \dots \gamma^{a_k}(m'_k \otimes 1) \otimes u_1 u_2 & \longmapsto & \gamma^{a_1}(m'_1) \dots \gamma^{a_k}(m'_k) \otimes u_1 u_2 \end{array}$$

where $\sum a_i = d$ and $m'_i \in M'$. This shows that the diagram

$$\begin{array}{ccc} \mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) & & \\ \downarrow{\scriptstyle \theta_D} & \searrow^{\text{can}} & \\ \mathcal{F}_h(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(U')) & \xrightarrow{\mathcal{F}_h(\text{can})} & \mathcal{F}_h(\_) \end{array}$$

commutes as required.

To verify assumption (C.8.d) holds, consider fiber product diagrams of affine schemes

$$D_f = \begin{array}{ccc} V' & \xrightarrow{f'} & U' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ V & \xrightarrow{f} & U \end{array}, \quad D_g = \begin{array}{ccc} W' & \xrightarrow{g'} & V' \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} \\ W & \xrightarrow{g} & V \end{array}, \text{ and } D_{f\circ g} = \begin{array}{ccc} W' & \xrightarrow{f'\circ g'} & U' \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h} \\ W & \xrightarrow{f\circ g} & U \end{array}$$

with $h, h', h'' \in \mathfrak{I}$ and the associated pushout diagrams of rings

$$\begin{array}{ccc} \mathcal{O}(U') & \xrightarrow{f'^*} & \mathcal{O}(V') \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h'^*} \\ \mathcal{O}(U) & \xrightarrow{f^*} & \mathcal{O}(V), \end{array} \quad \begin{array}{ccc} \mathcal{O}(V') & \xrightarrow{g'^*} & \mathcal{O}(W') \\ \uparrow{\scriptstyle h'^*} & & \uparrow{\scriptstyle h''^*} \\ \mathcal{O}(V) & \xrightarrow{g^*} & \mathcal{O}(W), \end{array} \text{ and } \begin{array}{ccc} \mathcal{O}(U') & \xrightarrow{(f'\circ g')^*} & \mathcal{O}(W') \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h''^*} \\ \mathcal{O}(U) & \xrightarrow{(f\circ g)^*} & \mathcal{O}(W). \end{array}$$

For an $\mathcal{O}(U')$–module M', we claim that the diagram

$$\begin{array}{ccc}
 & & N_{\mathcal{O}(U')/\mathcal{O}(U)}(M') \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \\
 & \overset{\text{can}}{\nearrow} & \\
\big(N_{\mathcal{O}(U')/\mathcal{O}(U)}(M') \otimes_{\mathcal{O}(U)} \mathcal{O}(V)\big) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & & \\
\big\downarrow{\scriptstyle \theta_{D_f}(M')\otimes \mathrm{Id}} & & \\
N_{\mathcal{O}(V')/\mathcal{O}(V)}\big(M' \otimes_{\mathcal{O}(U')} \mathcal{O}(V')\big) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & & \big\downarrow{\scriptstyle \theta_{D_{f\circ g}}(M')} \\
\big\downarrow{\scriptstyle \theta_{D_g}(M'\otimes_{\mathcal{O}(U')}\mathcal{O}(V'))} & & \\
N_{\mathcal{O}(W')/\mathcal{O}(W)}\big(\big(M' \otimes_{\mathcal{O}(U')} \mathcal{O}(V')\big) \otimes_{\mathcal{O}(V')} \mathcal{O}(W')\big) & & \\
 & \underset{N_{\mathcal{O}(W')/\mathcal{O}(W)}(\text{can})}{\searrow} & \\
 & & N_{\mathcal{O}(W')/\mathcal{O}(W)}(M' \otimes_{\mathcal{O}(U')} \mathcal{O}(W'))
\end{array}$$

commutes. Indeed, this can be seen by tracing an element through the diagram. For $\overline{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ with $\sum a_i = d$ and $\overline{m'} = (m'_1, \dots, m'_k) \in (M')^k$, we use the notation $\gamma^{\overline{a}}(\overline{m'}) = \gamma^{a_1}(m'_1) \dots \gamma^{a_k}(m'_k)$. We have

$$\begin{array}{ccc}
(\gamma^{\overline{a}}(\overline{m'}) \otimes u) \otimes v \otimes w & \longmapsto & (\gamma^{\overline{a}}(\overline{m'}) \otimes u) \otimes g^*(v)w \\
\big\downarrow & & \\
(\gamma^{\overline{a}}(\overline{m' \otimes 1_{\mathcal{O}(V')}}) \otimes f^*(u)v) \otimes w & & \\
\big\downarrow & & \\
\gamma^{\overline{a}}(\overline{m' \otimes 1_{\mathcal{O}(V')} \otimes 1_{\mathcal{O}(W')}}) \otimes g^*(f^*(u)v)w & & \\
 & \searrow & \\
 & & \gamma^{\overline{a}}(\overline{m' \otimes 1_{\mathcal{O}(W')}}) \otimes (f\circ g)^*(u)g^*(v)w \\
 & & = \gamma^{\overline{a}}(\overline{m' \otimes 1_{\mathcal{O}(W')}}) \otimes g^*(f^*(u)v)w
\end{array}$$

for all $u \in \mathcal{O}(U)$, $v \in \mathcal{O}(V)$, and $w \in \mathcal{O}(W)$, which shows that the diagram

$$\begin{array}{ccc}
\big(\mathcal{F}_h(_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)\big) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & \xrightarrow{\text{can}} & \mathcal{F}_h(_) \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \\
\big\downarrow{\scriptstyle \theta_{D_f}\otimes \mathrm{Id}} & & \\
\mathcal{F}_{h'}\big(_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')\big) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & & \big\downarrow{\scriptstyle \theta_{D_{f\circ g}}} \\
\big\downarrow{\scriptstyle \theta_{D_g}} & & \\
\mathcal{F}_{h''}\big(\big(_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')\big) \otimes_{\mathcal{O}(V')} \mathcal{O}(W')\big) & \xrightarrow{\mathcal{F}_{h''}(\text{can})} & \mathcal{F}_{h''}(_ \otimes_{\mathcal{O}(U')} \mathcal{O}(W'))
\end{array}$$

commutes as required. This finishes the proof. □

Lemma 2.9 allows us to apply Proposition C.10 to our choice of functors and natural isomorphisms, yielding a stack morphism

$$N\colon \mathfrak{QCoh}^d_{\mathrm{fff}} \to \mathfrak{QCoh} \tag{2.9.1}$$

which we call the *norm morphism*. Since Proposition C.10 uses the construction of Lemma C.7, for each $h\colon T' \to T$ in $\mathfrak{I}$ we have a norm functor $N_{T'/T}\colon \mathfrak{QCoh}(T') \to \mathfrak{QCoh}(T)$ between categories of quasi-coherent modules. In particular, for a finite locally free cover $T \to S$ of degree d of our fixed base scheme S, we have a norm functor

$$N_{T/S}\colon \mathfrak{QCoh}(T) \to \mathfrak{QCoh}(S). \tag{2.9.2}$$

By construction, this functor has the property that for $\mathcal{M} \in \mathfrak{QCoh}(T)$ and $U \in \mathfrak{Aff}_S$, we have $N_{T/S}(\mathcal{M})(U) = N_{\mathcal{O}(T\times_S U)/\mathcal{O}(U)}(\mathcal{M}(T \times_S U))$. Furthermore, the restriction along a morphism $V \to U$ in $\mathfrak{Aff}_S$ has the following nice expression,

$$\begin{aligned} N_{\mathcal{O}(T\times_S U)/\mathcal{O}(U)}(\mathcal{M}(T \times_S U)) &\to N_{\mathcal{O}(T\times_S V)/\mathcal{O}(V)}(\mathcal{M}(T \times_S V)) \\ \gamma^{\overline{a}}(\overline{m}) \otimes u &\mapsto \gamma^{\overline{a}}(\overline{m|_{T\times_S V}}) \otimes u|_V \end{aligned} \tag{2.9.3}$$

for $\overline{m} = (m_1, \dots, m_k) \in \mathcal{M}(T \times_S U)^k$ and $u \in \mathcal{O}(U)$.

2.9.1. *The Universal Normic Polynomial Law.* Let $T \to S$ be a finite locally free cover of degree d and consider the norm functor $N_{T/S}\colon \mathfrak{QCoh}(T) \to \mathfrak{QCoh}(S)$ of (2.9.2). Theorem 2.5(ii) is preserved in a sense for this globalized norm functor. Since $f\colon T \to S$ is finite locally free, there is a canonical norm

$$\mathbf{norm}\colon f_*(\mathcal{O}|_T) \to \mathcal{O}$$

defined in [St, Tag 0BD2] or [EGA, II 6.5.1] for sheaves on schemes, but which generalizes immediately to sheaves on $\mathfrak{Sch}_S$. It is the globalized version of the norm of a finite locally free ring extension as in (2.1.2). For a quasi-coherent $\mathcal{O}|_T$–module $\mathcal{M}$, we define a *normic polynomial law* to be a natural transformation $\boldsymbol{\nu}\colon f_*(\mathcal{M}) \to \mathcal{N}$, where $\mathcal{N}$ is a quasi-coherent $\mathcal{O}$–module, such that

$$\boldsymbol{\nu}(tm) = \mathbf{norm}(t)\boldsymbol{\nu}(m)$$

for all appropriate sections $t \in f_*(\mathcal{O}|_T)$ and $m \in f_*(\mathcal{M})$. For a fixed $\mathcal{O}|_T$–module $\mathcal{M}$, we can form the category of normic polynomial laws, denoted $\mathfrak{NPL}_{T/S}(\mathcal{M})$, whose

(i) objects are pairs $(\mathcal{N}, \boldsymbol{\nu})$ where $\mathcal{N}$ is a quasi-coherent $\mathcal{O}$–module and $\boldsymbol{\nu}\colon f_*(\mathcal{M}) \to \mathcal{N}$ is a normic polynomial law, and whose
(ii) morphisms $(\mathcal{N}, \boldsymbol{\nu}) \to (\mathcal{N}', \boldsymbol{\nu}')$ are $\mathcal{O}$–module maps $\varphi\colon \mathcal{N} \to \mathcal{N}'$ such that $\boldsymbol{\nu}' = \varphi \circ \boldsymbol{\nu}$.

2.10. **Proposition.** *Let $f\colon T \to S$ be a finite locally free morphism and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}|_T$–module. Then, there is a normic polynomial law $\boldsymbol{\nu}_{\mathcal{M}}\colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$, given over $U \in \mathfrak{Aff}_S$ by the function*

$$\nu_{\mathcal{M}(T\times_S U)}\colon \mathcal{M}(T \times_S U) \to N_{\mathcal{O}(T\times_S U)/\mathcal{O}(U)}(\mathcal{M}(T \times_S U))$$

of Section 2.4.1, such that $(N_{T/S}(\mathcal{M}), \boldsymbol{\nu}_{\mathcal{M}})$ *is an initial object in the category* $\mathfrak{NPL}_{T/S}(\mathcal{M})$.

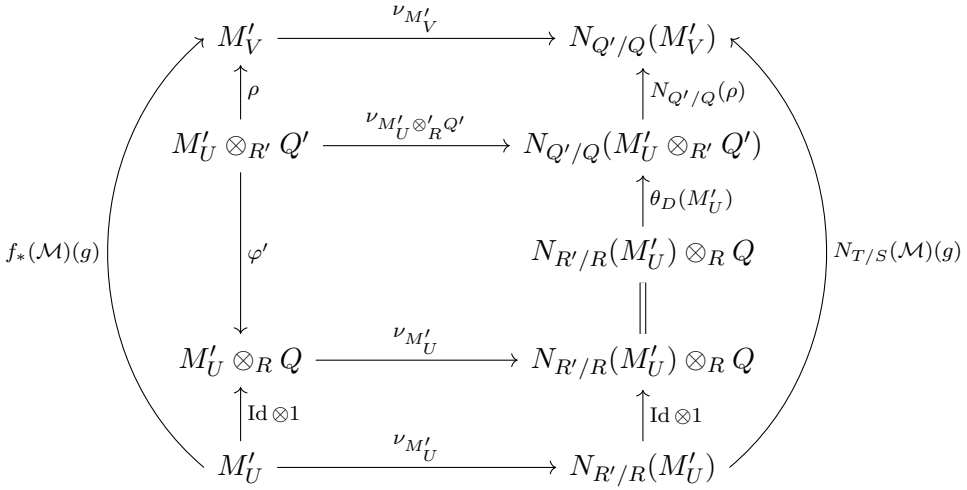
Proof. We first check that the proposed functions over each $U \in \mathfrak{Aff}_S$ assemble into a natural transformation of sheaves on $\mathfrak{Aff}_S$. Let $g\colon V \to U$ be a morphism in $\mathfrak{Aff}_S$. We then have a fiber product diagram

$$D = \begin{array}{ccc} T \times_S V & \xrightarrow{g'} & T \times_S U \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & U \end{array}$$

in $\mathfrak{Sch}_S$. To shorten notation, we set

$$\begin{aligned} R &= \mathcal{O}(U) & R' &= \mathcal{O}(T \times_S U) \\ Q &= \mathcal{O}(V) & Q' &= \mathcal{O}(T \times_S V) \\ M'_U &= \mathcal{M}(T \times_S U) & M'_V &= \mathcal{M}(T \times_S V). \end{aligned}$$

Because this is a fiber product diagram, there is a canonical isomorphism $\varphi'\colon M'_U \otimes_{R'} Q' \xrightarrow{\sim} M'_U \otimes_R Q$ and since $\mathcal{M}$ is quasi-coherent, there is a canonical isomorphism $\rho\colon M'_U \otimes_{R'} Q' \xrightarrow{\sim} M'_V$. We have a diagram



where the top square commutes by Theorem 2.5(iii), the middle square commutes by Lemma 2.7, and the bottom square commutes because Ferrand's universal normic polynomial law $\boldsymbol{\nu}_{M'_U}\colon \mathbf{W}_R(M'_U) \to \mathbf{W}_R(N_{R'/R}(M'_U))$ is a natural transformation. The two faces involving curved arrows commute by definition. Hence, we have a polynomial law

$$\boldsymbol{\nu}_{\mathcal{M}}\colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$$

which is clearly normic since each $\nu_{M'_U}$ is so.

To justify that this polynomial law has the claimed universal property, let $\boldsymbol{\nu}\colon f_*(\mathcal{M}) \to \mathcal{N}$ be another normic polynomial law into a quasi-coherent $\mathcal{O}$–module. Fix an affine scheme $U \in \mathfrak{Aff}_S$ and a morphism $g\colon W \to V$ in $\mathfrak{Aff}_U$.

Using the same notation as above, as well as $\mathcal{O}(W) = P$, $\mathcal{O}(T \times_S W) = P'$, and $\mathcal{M}(T \times_S W) = M'_W$, we have a diagram

$$\begin{array}{ccccccccc}
M'_U \otimes_R P & \xleftarrow{\sim} & M'_U \otimes_{R'} P' & \xrightarrow{\sim} & M'_W & \xrightarrow{\boldsymbol{\nu}(W)} & \mathcal{N}(W) & \xleftarrow{\sim} & \mathcal{N}(U) \otimes_R P \\
\uparrow{\scriptstyle \mathrm{Id} \otimes g^*} & & \uparrow{\scriptstyle \mathrm{Id} \otimes g'^*} & & \uparrow{\scriptstyle f_*(\mathcal{M})(g)} & & \uparrow{\scriptstyle \mathcal{N}(g)} & & \uparrow{\scriptstyle \mathrm{Id} \otimes g} \\
M'_U \otimes_R Q & \xleftarrow{\sim} & M'_U \otimes_{R'} Q' & \xrightarrow{\sim} & M'_V & \xrightarrow{\boldsymbol{\nu}(V)} & \mathcal{N}(V) & \xleftarrow{\sim} & \mathcal{N}(U) \otimes_R Q
\end{array}$$

and hence we may take the long horizontal compositions as $\boldsymbol{\nu}_U(V)$ and $\boldsymbol{\nu}_U(W)$ respectively to define a normic polynomial law $\boldsymbol{\nu}_U \colon \mathbf{W}_R(M'_U) \to \mathbf{W}_R(\mathcal{N}(U))$. Therefore, by Theorem 2.5(ii), there exists a unique R–linear map $\phi_U \colon N_{T/S}(\mathcal{M})(U) = N_{R'/R}(M'_U) \to \mathcal{N}(U)$ such that $\mathbf{W}_R(\phi_U) \circ \boldsymbol{\nu}_{M'_U} = \boldsymbol{\nu}_U$. For the affine scheme V, we similarly obtain a Q–linear homomorphism $\phi_V \colon N_{T/S}(\mathcal{M})(V) \to \mathcal{N}(V)$. Due to the isomorphisms $M'_V \cong M'_U \otimes_R Q$ and $\mathcal{N}(V) \cong \mathcal{N}(U) \otimes_R Q$ as well as the uniqueness of ϕ_U, we will have a diagram

$$\begin{array}{ccc}
N_{Q'/Q}(M'_V) & \xrightarrow{\phi_V} & \mathcal{N}(V) \\
\uparrow{\scriptstyle \wr} & & \uparrow{\scriptstyle \wr} \\
N_{R'/R}(M'_U) \otimes_R Q & \xrightarrow{\phi_U \otimes 1} & \mathcal{N}(U) \otimes_R Q \\
\uparrow{\scriptstyle \mathrm{Id} \otimes 1} & & \uparrow{\scriptstyle \mathrm{Id} \otimes 1} \\
N_{R'/R}(M'_U) & \xrightarrow{\phi_U} & \mathcal{N}(U)
\end{array}$$

which shows that the various ϕ_U assemble into a map $\phi \colon N_{T/S}(\mathcal{M}) \to \mathcal{N}$ which is $\mathcal{O}$–linear and satisfies $\phi \circ \boldsymbol{\nu}_{\mathcal{M}} = \boldsymbol{\nu}$. If $\phi_2 \colon N_{T/S}(\mathcal{M}) \to \mathcal{N}$ is any other such map, then in a similar manner to above we can extract a natural transformation

$$\phi_{2,U} \colon \mathbf{W}_R(M'_U) \to \mathbf{W}_R(N_{R'/R}(M'_U))$$

satisfying $\mathbf{W}_R(\phi_{2,U}) \circ \boldsymbol{\nu}_{M'_U} = \boldsymbol{\nu}_U$. Then, uniqueness of ϕ_U requires that $\phi_{2,U} = \phi_U$ and since this holds for all $U \in \mathfrak{Aff}_S$, we must have $\phi_2 = \phi$ globally. Thus, ϕ is the unique such morphism as desired. □

For an $\mathcal{O}|_T$–module $\mathcal{M}$, we will refer to $\boldsymbol{\nu}_{\mathcal{M}} \colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$ as the universal polynomial law associated to $\mathcal{M}$. The analogue of Theorem 2.5(iii) also holds.

2.11. **Corollary.** *The universal property of Proposition 2.10 above induces the image of the norm functor on morphisms. If $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of quasi-coherent $\mathcal{O}|_T$–modules, then $\boldsymbol{\nu}_{\mathcal{M}_2} \circ f_*(\varphi)$ is a normic polynomial law and $N_{T/S}(\varphi)$ is the unique $\mathcal{O}$–module map making the diagram*

$$\begin{array}{ccc} f_*(\mathcal{M}_1) & \xrightarrow{\boldsymbol{\nu}_{\mathcal{M}_1}} & N_{T/S}(\mathcal{M}_1) \\ \downarrow{\scriptstyle f_*(\varphi)} & & \downarrow{\scriptstyle N_{T/S}(\varphi)} \\ f_*(\mathcal{M}_2) & \xrightarrow{\boldsymbol{\nu}_{\mathcal{M}_2}} & N_{T/S}(\mathcal{M}_2) \end{array}$$

commute.

Proof. It is clear from the explicit definitions of $\boldsymbol{\nu}_{\mathcal{M}_i}$ as well as $N_{T/S}(\varphi)$ that the above diagram commutes. Therefore, the uniqueness claim follows from Proposition 2.10. □

The next corollary shows that the universal normic polynomial law is stable under base change.

2.12. **Corollary.** *Consider a fiber product diagram in $\mathfrak{Sch}_S$*

$$D = \begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

where $f\colon T \to S$ is finite locally free of degree d and hence so is $f'\colon T' \to S'$. Then, for any $\mathcal{M} \in \mathfrak{QCoh}(T)$, the diagram

$$\begin{array}{ccc} f'_*(\mathcal{M}|_{T'}) & \xrightarrow{\boldsymbol{\nu}_{(\mathcal{M}|_{T'})}} & N_{T'/S'}(\mathcal{M}|_{T'}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi_D} \\ f_*(\mathcal{M})|_{S'} & \xrightarrow{(\boldsymbol{\nu}_{\mathcal{M}})|_{S'}} & N_{T/S}(\mathcal{M})|_{S'} \end{array}$$

commutes. Here, ψ is the canonical isomorphism coming from the isomorphisms $T \times_S X \xrightarrow{\sim} T' \times_{S'} X$ for any $X \in \mathfrak{Sch}_{S'}$ and ϕ_D is the isomorphism of Lemma C.9.

Proof. We may check over affine schemes $U \in \mathfrak{Aff}_{S'}$, where the diagram becomes

$$\begin{array}{ccc} \mathcal{M}(T' \times_{S'} U) & \xrightarrow{\nu_{\mathcal{M}(T' \times_{S'} U)}} & N_{\mathcal{O}(T' \times_{S'} U)/\mathcal{O}(U)}(\mathcal{M}(T' \times_{S'} U)) \\ \downarrow{\scriptstyle \psi(U)} & & \downarrow{\scriptstyle \phi_D} \\ \mathcal{M}(T \times_S U) & \xrightarrow{\nu_{\mathcal{M}(T \times_S U)}} & N_{\mathcal{O}(T \times_S U)/\mathcal{O}(U)}(\mathcal{M}(T \times_S U)). \end{array}$$

Tracing an element through yields

$$\begin{array}{ccc} m' & \longmapsto & \gamma^d(m') \otimes 1 \\ \downarrow & & \downarrow \\ m'|_{T\times_S U} & \longmapsto & \gamma^d(m'|_{T\times_S U}) \otimes 1 \end{array}$$

where ϕ_D is the composition

$$\gamma^d(m') \otimes 1 \mapsto (\gamma^d(m') \otimes 1) \otimes 1 \mapsto \gamma^d(m' \otimes 1) \otimes 1 \mapsto \gamma^d(m'|_{T\times_S U}) \otimes 1.$$

This verifies that the diagram commutes. □

We will need the following result for the next section.

2.13. **Lemma.** *Assume that $T \to S$ is finite étale of constant degree d. If $\mathcal{M}$ is a locally free $\mathcal{O}|_T$–module of constant rank r, then $N_{T/S}(\mathcal{M})$ is a locally free $\mathcal{O}$–module of constant rank r^d.*

Proof. By Lemma 1.3 we may examine the restriction of $N_{T/S}(\mathcal{M})$ to $\mathfrak{Aff}_S$. There, for each $U \in \mathfrak{Aff}_S$ the $\mathcal{O}(T \times_S U)$–module $\mathcal{M}(T \times_S U)$ is projective of rank r. Therefore, by [Fer, 4.1.3], $N_{T/S}(\mathcal{M})(U)$ is a projective $\mathcal{O}(U)$–module of rank r^d, which implies the stated claim globally. □

2.14. **Example.** Consider the split étale cover $f\colon S^{\sqcup d} \to S$. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}|_{S^{\sqcup d}}$–module. Any $S^{\sqcup d}$–scheme is of the form $T_1 \sqcup \ldots \sqcup T_d$ for $T_i \in \mathfrak{Sch}_S$ and the module $\mathcal{E}$ is given by the formula

$$\mathcal{E}(T_1 \sqcup \ldots \sqcup T_d) = \mathcal{E}_1(T_1) \times \ldots \times \mathcal{E}_d(T_d).$$

where $\mathcal{E}_1, \ldots, \mathcal{E}_d$ are quasi-coherent $\mathcal{O}$–modules. Then, we have the following.

(i) The norm of $\mathcal{E}$ is $N_{S^{\sqcup d}/S}(\mathcal{E}) = \mathcal{E}_1 \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{E}_d$.
(ii) We have $f_*(\mathcal{E}) = \mathcal{E}_1 \times \ldots \times \mathcal{E}_d$ and the universal normic polynomial law is

$$\begin{aligned} \boldsymbol{\nu}\colon \mathcal{E}_1 \times \ldots \times \mathcal{E}_d &\mapsto \mathcal{E}_1 \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{E}_d \\ (e_1, \ldots, e_d) &\mapsto e_1 \otimes \ldots \otimes e_d. \end{aligned}$$

Proof. Both claims follow from Proposition 2.6 after localizing with respect to an affine cover. □

2.15. **The Norm of Algebras.** The functor $N_{T/S}$ of (2.9.2) restricts to the category of quasi-coherent $\mathcal{O}|_T$–algebras. This is shown in Lemma 2.17, which is a globalization of parts of [Fer, 3.2.5]. Alternatively, when $T \to S$ is étale, it is a globalization of [KO, 4.5]. In turn, this means that the morphism of stacks N of (2.9.1) restricts to the stack of quasi-coherent algebras. However, we begin with two technical lemmas about norms of modules which will be needed.

2.16. **Lemma.** *Let $f\colon T \to S$ be a finite locally free morphism and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}|_T$–module. Consider its universal normic polynomial law $(N_{T/S}(\mathcal{M}), \boldsymbol{\nu}_{\mathcal{M}})$ from Proposition 2.10. Then, $N_{T/S}(\mathcal{M})$ is generated as an $\mathcal{O}$–module by the image of $\boldsymbol{\nu}_{\mathcal{M}}$. Precisely, we mean that $N_{T/S}(\mathcal{M})$ is the sheaf associated to the presheaf*

$$U \mapsto \mathrm{Span}_{\mathcal{O}(U)}(\{\boldsymbol{\nu}_{\mathcal{M}}(m) \mid m \in f_*(\mathcal{M})(U)\})$$

for $U \in \mathfrak{Sch}_S$.

Proof. This result follows from the analogous statement over rings that is used throughout [Fer], which in turn follows from [Fer, 2.3.1]. In particular, for any affine scheme $U \in \mathfrak{Aff}_S$ over which $T \times_S U \to U$ is of constant degree d, there exists a cover (depending on d) $\{V \to U\}$ such that V is an affine scheme, $\mathcal{O}(U) \to \mathcal{O}(V)$ is a finite free ring extension, and $N_{T/S}(\mathcal{M})(V)$ is generated as an $\mathcal{O}(V)$–module by the image under $\boldsymbol{\nu}_{\mathcal{M}}$ of $f_*(\mathcal{M})(V)$. Therefore, any scheme in $\mathfrak{Sch}_S$ has an affine cover on which the presheaf above and $N_{T/S}(\mathcal{M})$ agree, so the statement follows. □

Now we can argue that the norm functor preserves algebras.

2.17. **Lemma.** *Let $T \to S$ be a finite locally free morphism of schemes of degree d. Let $\mathcal{B}$ be an $\mathcal{O}|_T$–algebra and let $\boldsymbol{\nu}_{\mathcal{B}}\colon f_*(\mathcal{B}) \to N_{T/S}(\mathcal{B})$ be the universal normic polynomial law of Proposition 2.10.*

(i) *There is a unique morphism of $\mathcal{O}$–modules*

$$\Phi\colon N_{T/S}(\mathcal{B}) \otimes_{\mathcal{O}} N_{T/S}(\mathcal{B}) \to N_{T/S}(\mathcal{B} \otimes_{\mathcal{O}|_T} \mathcal{B})$$

such that $\Phi(\boldsymbol{\nu}_{\mathcal{B}}(b_1) \otimes \boldsymbol{\nu}_{\mathcal{B}}(b_2)) = \boldsymbol{\nu}'(b_1 \otimes b_2)$ where $\boldsymbol{\nu}'$ is the universal normic polynomial associated to $\mathcal{B} \otimes_{\mathcal{O}|_T} \mathcal{B}$. In particular, since $\mathcal{B}$ has an algebra structure morphism $\mu\colon \mathcal{B} \otimes_{\mathcal{O}|_T} \mathcal{B} \to \mathcal{B}$, the composition

$$N_{T/S}(\mathcal{B}) \otimes_{\mathcal{O}} N_{T/S}(\mathcal{B}) \xrightarrow{\Phi} N_{T/S}(\mathcal{B} \otimes_{\mathcal{O}|_T} \mathcal{B}) \xrightarrow{N_{T/S}(\mu)} N_{T/S}(\mathcal{B})$$

gives $N_{T/S}(\mathcal{B})$ a natural algebra structure. It is associative or unital or commutative if $\mathcal{B}$ is so. Further, the universal normic polynomial law $\boldsymbol{\nu}_{\mathcal{B}}$ is multiplicative with respect to this natural structure. If $\mathcal{B}$ is unital, then $\nu_{\mathcal{B}}$ preserves the unit as well.

(ii) *The norm preserves algebra homomorphisms. If $\varphi\colon \mathcal{B}_1 \to \mathcal{B}_2$ is an $\mathcal{O}|_T$–algebra homomorphism, then $N_{T/S}(\varphi)\colon N_{T/S}(\mathcal{B}_1) \to N_{T/S}(\mathcal{B}_2)$ is an $\mathcal{O}$–algebra homomorphism with respect to the natural algebra structures from (i).*

(iii) *If $T \to S$ is finite étale, then Φ is an isomorphism.*

Proof. (i): The property $\Phi(\boldsymbol{\nu}_{\mathcal{B}}(b_1) \otimes \boldsymbol{\nu}_{\mathcal{B}}(b_2)) = \boldsymbol{\nu}'(b_1 \otimes b_2)$ is sufficient to define a unique $\mathcal{O}$–module morphism by Lemma 2.16. We leave the verification that the resulting algebra structure preserves being associative or

commutative to the reader. The multiplicativity of $\boldsymbol{\nu}_{\mathcal{B}}$ follows from the calculation

$$\begin{aligned}\boldsymbol{\nu}_{\mathcal{B}}(b_1 b_2) &= \boldsymbol{\nu}_{\mathcal{B}} \circ f_*(\mu)(b_1 \otimes b_2) = N_{T/S}(\mu) \circ \boldsymbol{\nu}'(b_1 \otimes b_2) \\ &= N_{T/S}(\mu) \circ \Phi(\boldsymbol{\nu}_{\mathcal{B}}(b_1) \otimes \boldsymbol{\nu}_{\mathcal{B}}(b_2)) = \boldsymbol{\nu}_{\mathcal{B}}(b_1)\boldsymbol{\nu}_{\mathcal{B}}(b_2)\end{aligned}$$

for b_1, b_2 sections in $f_*(\mathcal{B})$.

Now, assume $\mathcal{B}$ is unital. To see that $N_{T/S}(\mathcal{B})$ is also unital and that $\nu_{\mathcal{B}}$ preserves this unit, we argue that $\nu_{\mathcal{B}}(1_{\mathcal{B}})$ is the identity in $N_{T/S}(\mathcal{B})$. Let $x \in N_{T/S}(\mathcal{B})(U)$ be a section over some $U \in \mathfrak{Sch}_S$. By Lemma 2.16, there is a cover $\{U_i \to U\}_{i\in I}$ over which $x|_{U_i} = \sum a_j \nu_{\mathcal{B}}(b_j)$ for sections $a_j \in \mathcal{O}(U_i)$ and $b_j \in f_*(\mathcal{B})(U_i)$. Then, the product $\nu_{\mathcal{B}}(1_{\mathcal{B}})|_U \cdot x$ is locally of the form

$$\begin{aligned}\nu_{\mathcal{B}}(1_{\mathcal{B}})|_{U_i} \cdot x|_{U_i} &= \nu_{\mathcal{B}}(1_{\mathcal{B}})|_{U_i} \cdot \sum a_j \nu_{\mathcal{B}}(b_j) \\ &= \sum a_j \nu_{\mathcal{B}}(1_{\mathcal{B}}|_{U_i} \cdot b_j) \\ &= \sum a_j \nu_{\mathcal{B}}(b_j) \\ &= x|_{U_i}\end{aligned}$$

where we use the multiplicativity of $\nu_{\mathcal{B}}$ established above. This implies that $\nu_{\mathcal{B}}(1_{\mathcal{B}})|_U \cdot x = x$ and therefore $\nu_{\mathcal{B}}(1_{\mathcal{B}})$ is the identity in $N_{T/S}(\mathcal{B})$ as claimed.
(ii): This can be verified via direct computation on the generators given by Lemma 2.16.
(iii): This follows since, under the new assumptions, the map is an isomorphism over affine schemes by [Fer, 3.2.5 (c)]. □

We now turn our attention to Azumaya algebras and assume that $T \to S$ is finite étale. The following is inspired by [Fer, 3.2.5], where Ferrand claims that for arbitrary R'–modules M_1' and M_2' there exists an R–linear map between $N_{R'/R}(\operatorname{Hom}_{R'}(M_1', M_2'))$ and $\operatorname{Hom}_R(N_{R'/R}(M_1'), N_{R'/R}(M_2'))$ arising via the universal property from a normic polynomial law

$$\mathbf{W}_R(\operatorname{Hom}_{R'}(M_1', M_2')) \to \mathbf{W}_R(\operatorname{Hom}_R(N_{R'/R}(M_1'), N_{R'/R}(M_2'))).$$

We have not been able to verify that Ferrand's construction of this normic polynomial law works in the stated generality unless M_1' is finitely generated projective. In any case, we make an equivalent assumption in Lemma 2.18 below in order to ensure that our $\mathcal{H}om$ modules are quasi-coherent. A similar assumption is made by Knus-Ojanguren in [KO, Prop. 4.4], where the authors prove Lemma 2.18 over rings. We recall that by [Fer, 5.3], in the setting of [KO], Ferrand's norm functor and the one constructed by Knus-Ojanguren are isomorphic.

2.18. **Lemma.** *Assume that $T \to S$ is finite étale. Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{QCoh}(T)$ and assume that $\mathcal{M}_1$ is finite locally free. There is a normic polynomial law*

$$\boldsymbol{\eta}\colon f_*(\mathcal{H}om_{\mathcal{O}|_T}(\mathcal{M}_1, \mathcal{M}_2)) \to \mathcal{H}om_{\mathcal{O}}(N_{T/S}(\mathcal{M}_1), N_{T/S}(\mathcal{M}_2))$$

defined over $\mathfrak{Aff}_S$ as follows. For $U \in \mathfrak{Aff}_S$ we consider the fiber product diagram

$$D = \begin{array}{ccc} T \times_S U & \longrightarrow & T \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S \end{array}$$

and then for

$$\varphi \in f_*(\mathcal{H}om_{\mathcal{O}|_T}(\mathcal{M}_1, \mathcal{M}_2))(U) = \mathrm{Hom}_{\mathcal{O}|_{T\times_S U}}(\mathcal{M}_1|_{T\times_S U}, \mathcal{M}_2|_{T\times_S U})$$

we set $\boldsymbol{\eta}(\varphi)$ to be the composition

$$\begin{array}{ccccc} N_{T/S}(\mathcal{M}_1)|_U & \xleftarrow{\phi_D(\mathcal{M}_1)} & N_{(T\times_S U)/U}(\mathcal{M}_1|_{T\times_S U}) & \xrightarrow{N_{(T\times_S U)/U}(\varphi)} & N_{(T\times_S U)/U}(\mathcal{M}_2|_{T\times_S U}) \\ & \searrow^{\boldsymbol{\eta}(\varphi)} & & & \downarrow{\scriptstyle \phi_D(\mathcal{M}_2)} \\ & & & & N_{T/S}(\mathcal{M}_2)|_U \end{array}$$

where ϕ_D is the isomorphism of functors of Lemma C.9.

Proof. Assuming $\mathcal{M}_1$ is finite locally free ensures that $\mathcal{H}om_{\mathcal{O}|_T}(\mathcal{M}_1, \mathcal{M}_2)$ is quasi-coherent by Lemma C.2. It also ensures that $N_{T/S}(\mathcal{M}_1)$ is finite locally free by Lemma 2.13 since we are assuming $T \to S$ is étale. Therefore, we know $\mathcal{H}om_{\mathcal{O}}(N_{T/S}(\mathcal{M}_1), N_{T/S}(\mathcal{M}_2))$ is quasi-coherent as well by another application of Lemma C.2.

We verify that $\boldsymbol{\eta}$ defines a natural transformation. Let $g\colon V \to U$ be a morphism in $\mathfrak{Aff}_S$ and let $W \in \mathfrak{Aff}_V$. We will show that $\boldsymbol{\eta}(\varphi)|_V(W) = \boldsymbol{\eta}(\varphi|_{T\times_S V})(W)$ for a morphism $\varphi \in f_*(\mathcal{H}om_{\mathcal{O}|_T}(\mathcal{M}_1, \mathcal{M}_2))(U)$. Since W is arbitrary, we will conclude that $\boldsymbol{\eta}(\varphi)|_V = \boldsymbol{\eta}(\varphi|_{T\times_S V})$ as is required.

We have the following commutative diagram

$$\begin{array}{ccccc} (T \times_S V) \times_V W & \xrightarrow{g''} & (T \times_S U) \times_U W & \xrightarrow{g_0''} & T \times_S W \\ \downarrow \searrow^{h'_V} & & \downarrow \searrow^{h'_U} & & \downarrow \searrow^{h'_S} \\ T \times_S V & \xrightarrow{g'} & T \times_S U & \xrightarrow{g'_0} & T \\ \downarrow & & \downarrow & & \downarrow \\ W & = & W & = & W \\ \downarrow{\scriptstyle h_V} \searrow & & \downarrow{\scriptstyle h_U} \searrow & & \downarrow{\scriptstyle h_S} \searrow \\ V & \xrightarrow{g} & U & \xrightarrow{g_0} & S \end{array}$$

where the vertical faces are fiber product diagrams. This means that g'' and g_0'' are isomorphisms. We set

$$D_U = \begin{array}{ccc} T \times_S U & \xrightarrow{g'_0} & T \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g_0} & S \end{array} \quad \text{and} \quad D_V = \begin{array}{ccc} T \times_S V & \xrightarrow{g'_0 \circ g'} & T \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g_0 \circ g} & S. \end{array}$$

The morphism $\boldsymbol{\eta}(\varphi)|_V(W)$ is then the composition

$$\begin{array}{c} N_{\mathcal{O}(T\times_S W)/\mathcal{O}(W)}(\mathcal{M}_1(T\times_S W)) \\ \uparrow{\scriptstyle \phi_{D_U}(\mathcal{M}_1)(W)} \\ N_{\mathcal{O}((T\times_S U)\times_U W)/\mathcal{O}(W)}(\mathcal{M}_1((T\times_S U)\times_U W)) \\ \downarrow{\scriptstyle \Gamma^d_{\mathcal{O}(W)}(\varphi((T\times_S U)\times_U W))\otimes \mathrm{Id}} \\ N_{\mathcal{O}((T\times_S U)\times_U W)/\mathcal{O}(W)}(\mathcal{M}_2((T\times_S U)\times_U W)) \\ \downarrow{\scriptstyle \phi_{D_U}(\mathcal{M}_2)(W)} \\ N_{\mathcal{O}(T\times_S W)/\mathcal{O}(W)}(\mathcal{M}_2(T\times_S W)). \end{array}$$

Since g_0' is an isomorphism, the restriction map $\mathcal{M}_1(g_0')\colon \mathcal{M}_1(T\times_S W)\to \mathcal{M}_1((T\times_S U)\times_U W)$ is also an isomorphism. Therefore, the isomorphism $\phi_{D_U}(\mathcal{M}_1)(W)$ takes the form

$$\gamma^{\overline{a}}(\overline{m''})\otimes w \mapsto \gamma^{\overline{a}}(\overline{\mathcal{M}_1(g_0')^{-1}(m'')})\otimes w.$$

for $\overline{a}=(a_1,\dots,a_k)$ with $\sum a_i = d$ and $\overline{m''}=(m_1'',\dots,m_k'')\in \mathcal{M}_1((T\times_S U)\times_U W)^k$. So, denoting $\varphi((T\times_S U)\times_U W)=\varphi''$, the morphism $\boldsymbol{\eta}(\varphi)|_V(W)$ behaves as

$$\begin{aligned} \gamma^{\overline{a}}(\overline{m'})\otimes w &\mapsto \gamma^{\overline{a}}(\overline{\mathcal{M}_1(g_0')(m')})\otimes w \\ &\mapsto \gamma^{\overline{a}}(\overline{(\varphi''\circ \mathcal{M}_1(g_0'))(m')})\otimes w \\ &\mapsto \gamma^{\overline{a}}(\overline{(\mathcal{M}_2(g_0')^{-1}\circ\varphi''\circ \mathcal{M}_1(g_0'))(m')})\otimes w \end{aligned}$$

for $\overline{m'}\in \mathcal{M}_1(T\times_S W)^k$. However, since φ is a natural transformation, the diagram

$$\begin{array}{ccc} \mathcal{M}_1(T\times_S W) & \xrightarrow{\varphi(T\times_S W)} & \mathcal{M}_2(T\times_S W) \\ \downarrow{\scriptstyle \mathcal{M}_1(g_0')} & & \downarrow{\scriptstyle \mathcal{M}_2(g_0')} \\ \mathcal{M}_1((T\times_S U)\times_U W) & \xrightarrow{\varphi''} & \mathcal{M}_2((T\times_S U)\times_U W) \end{array}$$

commutes and therefore we have that

$$\boldsymbol{\eta}(\varphi)|_V(W)(\gamma^{\overline{a}}(\overline{m'})\otimes w) = \gamma^{\overline{a}}(\overline{\varphi(T\times_S W)(m')})\otimes w.$$

Next, we consider $\boldsymbol{\eta}(\varphi|_{T\times_S V})(W)$. This is the composition

$$\begin{array}{c} N_{\mathcal{O}(T\times_S W)/\mathcal{O}(W)}(\mathcal{M}_1(T\times_S W)) \\ \uparrow{\scriptstyle \phi_{D_V}(\mathcal{M}_1)(W)} \\ N_{\mathcal{O}((T\times_S V)\times_V W)/\mathcal{O}(W)}(\mathcal{M}_1((T\times_S V)\times_V W)) \\ \downarrow{\scriptstyle \Gamma^d_{\mathcal{O}(W)}(\varphi((T\times_S V)\times_V W))\otimes \mathrm{Id}} \\ N_{\mathcal{O}((T\times_S V)\times_V W)/\mathcal{O}(W)}(\mathcal{M}_2((T\times_S V)\times_V W)) \\ \downarrow{\scriptstyle \phi_{D_V}(\mathcal{M}_2)(W)} \\ N_{\mathcal{O}(T\times_S W)/\mathcal{O}(W)}(\mathcal{M}_2(T\times_S W)). \end{array}$$

A symmetric argument, simply swapping U for V, then yields that

$$\boldsymbol{\eta}(\varphi|_{T\times_S V})(W)(\gamma^{\overline{a}}(\overline{m'})\otimes w) = \gamma^{\overline{a}}(\overline{\varphi(T\times_S W)(m')})\otimes w$$

and hence $\boldsymbol{\eta}(\varphi)|_V(W) = \boldsymbol{\eta}(\varphi|_{T\times_S V})(W)$ as desired. Thus $\boldsymbol{\eta}$ is a well-defined natural transformation.

Finally, it follows from the explicit descriptions of $\boldsymbol{\eta}(\varphi)(W)$ above, which also hold when $W \in \mathfrak{Aff}_U$, that $\boldsymbol{\eta}$ is normic. For $t \in \mathcal{O}(T\times_S W)$, the morphism $\boldsymbol{\eta}(t\varphi)$ will have the following formula over W.

$$\begin{aligned} \boldsymbol{\eta}(t\varphi)(\gamma^{\overline{a}}(\overline{m'})\otimes w) =& \gamma^{\overline{a}}(\overline{t\varphi(T\times_S W)(m')})\otimes w \\ =& \gamma^d(t)\cdot\gamma^{\overline{a}}(\overline{\varphi(T\times_S W)(m')})\otimes w \\ =& \gamma^{\overline{a}}(\overline{\varphi(T\times_S W)(m')})\otimes \mathrm{norm}_{\mathcal{O}(T\times_S W)/\mathcal{O}(W)}(t)\cdot w \\ =& \mathbf{norm}(t)\cdot\boldsymbol{\eta}(\varphi)(\gamma^{\overline{a}}(\overline{m'})\otimes w) \end{aligned}$$

Thus, we conclude $\boldsymbol{\eta}(t\varphi) = \mathbf{norm}(t)\cdot\boldsymbol{\eta}(\varphi)$, which finishes the proof. □

2.19. **Lemma.** *Let $T \to S$ be a finite étale morphism of schemes of degree d. Consider a neutral Azumaya $\mathcal{O}|_T$–algebra $\mathcal{B} = \mathcal{E}nd_{\mathcal{O}|_T}(\mathcal{Q})$ for a finitely locally free $\mathcal{O}|_T$–module $\mathcal{Q}$. Let $\boldsymbol{\nu}_{\mathcal{B}}\colon f_*(\mathcal{B}) \to N_{T/S}(\mathcal{B})$ be the universal normic polynomial law of Proposition 2.10. Then, there is a unique isomorphism of $\mathcal{O}$–algebras Ψ making the following diagram commute*

$$\begin{array}{ccc} f_*(\mathcal{B}) & \xrightarrow{\boldsymbol{\nu}_{\mathcal{B}}} & N_{T/S}(\mathcal{B}) \\ & {\scriptstyle \boldsymbol{\eta}}\searrow & \downarrow{\scriptstyle \Psi} \\ & & \mathcal{E}nd_{\mathcal{O}}(N_{T/S}(\mathcal{Q})) \end{array}$$

where $\boldsymbol{\eta}$ is the normic polynomial law of Lemma 2.18.

Proof. Since $\boldsymbol{\eta}$ is a normic polynomial law, a unique such $\mathcal{O}$–module map Ψ exists by Proposition 2.10. Additionally, since $\boldsymbol{\eta}$ is induced by the norm functor, multiplication in endomorphism algebras is by composition, and

functors respect composition, we see that $\boldsymbol{\eta}$ is multiplicative. Using this along with the fact that $\boldsymbol{\nu}_{\mathcal{B}}$ is multiplicative by Lemma 2.17(i), we compute

$$\begin{aligned}\Psi(\boldsymbol{\nu}_{\mathcal{B}}(b_1)\boldsymbol{\nu}_{\mathcal{B}}(b_2)) &= \Psi \circ N_{T/S}(\mu) \circ \Phi(\boldsymbol{\nu}_{\mathcal{B}}(b_1) \otimes \boldsymbol{\nu}_{\mathcal{B}}(b_2)) \\ &= \Psi \circ N_{T/S}(\mu) \circ \boldsymbol{\nu}'(b_1 \otimes b_2) \\ &= \Psi \circ \boldsymbol{\nu}_{\mathcal{B}} \circ f_*(\mu)(b_1 \otimes b_2) = \Psi \circ \boldsymbol{\nu}_{\mathcal{B}}(b_1 b_2) \\ &= \boldsymbol{\eta}(b_1 b_2) = \boldsymbol{\eta}(b_1)\boldsymbol{\eta}(b_2) \\ &= \Psi(\boldsymbol{\nu}_{\mathcal{B}}(b_1))\Psi(\boldsymbol{\nu}_{\mathcal{B}}(b_2))\end{aligned}$$

and so Ψ is multiplicative on the image of $\boldsymbol{\nu}_{\mathcal{B}}$. Therefore, by Lemma 2.16 and the linearity of Ψ, this means Ψ is multiplicative in general and hence is an algebra homomorphism. The fact that Ψ is an isomorphism follows since it is an isomorphism over affine schemes by [Fer, 3.2.5 (c)]. □

The following is the Azumaya algebra analogue of Lemma 2.13.

2.20. **Lemma.** *Assume that $T \to S$ is finite étale of constant degree d. If $\mathcal{A}$ is an Azumaya $\mathcal{O}|_T$–algebra of constant degree r, then $N_{T/S}(\mathcal{A})$ is an Azumaya $\mathcal{O}$–algebra of constant degree r^d.*

Proof. This follows from Lemma 2.17(i) and Lemma 2.19. There will be a cover $\{U_i \to S\}_{i\in I}$ over which we have $\mathcal{A}|_{T\times_S U_i} \cong \mathcal{E}nd_{\mathcal{O}|_{T\times_S U_i}}(\mathcal{Q}_i)$ for a locally free $\mathcal{O}|_{T\times_S U_i}$–module $\mathcal{Q}_i$ of constant rank r, and so we have the isomorphism of Lemma 2.19

$$N_{T/S}(\mathcal{A})|_{U_i} = N_{T\times_S U_i/U_i}(\mathcal{A}|_{T\times_S U_i}) \xrightarrow{\Psi} \mathcal{E}nd_{\mathcal{O}|_{U_i}}(N_{T\times_S U_i/U_i}(\mathcal{Q}_i))$$

where $N_{T\times_S U_i/U_i}(\mathcal{Q}_i)$ is a locally free $\mathcal{O}|_{U_i}$–module of constant rank r^d by Lemma 2.13. Therefore, $N_{T/S}(\mathcal{A})$ is an Azumaya $\mathcal{O}$–algebra of degree r^d as claimed. □

2.21. **Remark.** Since Lemma 2.17(i) and (ii) show that the norm respects quasi-coherent algebras, it is immediate that the morphism $N\colon \mathfrak{QCoh}^d_{\mathrm{fff}} \to \mathfrak{QCoh}$ of (2.9.1) restricts to a morphism

$$N_{\mathrm{alg}}\colon \mathfrak{QAlg}^d_{\mathrm{fff}} \to \mathfrak{QAlg} \tag{2.21.1}$$

between stacks of quasi-coherent algebras. In detail, we define $\mathfrak{QAlg}^d_{\mathrm{fff}}$ to be the substack of $\mathfrak{QCoh}^d_{\mathrm{fff}}$ which has

(i) objects $(T' \to T, \mathcal{B}) \in \mathfrak{QCoh}^d_{\mathrm{fff}}$ where $\mathcal{B}$ is a quasi-coherent $\mathcal{O}|_{T'}$–algebra, and
(ii) morphisms $(f, g, \varphi)\colon (X' \to X, \mathcal{B}_1) \to (T' \to T, \mathcal{B}_2)$ of $\mathfrak{QCoh}^d_{\mathrm{fff}}$ where $\varphi\colon \mathcal{B}_1 \to g^*(\mathcal{B}_2)$ is an $\mathcal{O}|_{X'}$–algebra isomorphism.

Similarly, the unadorned $\mathfrak{QAlg}$ is the substack of $\mathfrak{QCoh}$ which has

(i) objects $(X, \mathcal{B}) \in \mathfrak{QCoh}$ where $\mathcal{B}$ is a quasi-coherent $\mathcal{O}|_X$–algebra, and
(ii) morphisms $(g, \varphi)\colon (X', \mathcal{B}') \to (X, \mathcal{B})$ of $\mathfrak{QCoh}$ where $\varphi\colon \mathcal{B}' \to g^*(\mathcal{B})$ is an $\mathcal{O}|_{X'}$–algebra morphism.

3. Cohomological Description

In this section we give a cohomological description of the norm functor over finite étale covers of degree d by analyzing restrictions of the morphisms N and N_{alg} of (2.9.1) and (2.21.1) to various substacks of the stacks $\mathfrak{QCoh}^d_{\mathrm{fff}}$ of Section 2.8.1 and $\mathfrak{QAlg}^d_{\mathrm{fff}}$ of Remark 2.21 respectively. In particular, we will consider the following stacks.

(i) Let $\mathfrak{Mod}_r$ be the substack of $\mathfrak{QCoh}$ whose objects are those $(X, \mathcal{M}) \in \mathfrak{QCoh}$ where $\mathcal{M}$ is a finite locally free $\mathcal{O}|_X$–module of constant rank r, and whose morphisms are the cartesian morphisms from $\mathfrak{QCoh}$. This is equivalent to the split stack $\mathfrak{Vec}_r$ considered in [CF, 2.4.1.8].

(ii) Let $\mathfrak{Azu}_r$ be the full substack of $\mathfrak{QAlg}$ whose objects are those $(X, \mathcal{A}) \in \mathfrak{QAlg}$ where $\mathcal{A}$ is an Azumaya $\mathcal{O}|_X$–algebra of constant degree r, and whose morphisms are the cartesian morphisms of $\mathfrak{QAlg}$. This is equivalent to the split stack of [CF, 2.5.3.10].

(iii) Let $\mathfrak{Mod}_r^{d-\text{ét}}$ be the full substack of $\mathfrak{QCoh}^d_{\mathrm{fff}}$ whose objects are those $(T' \to T, \mathcal{M}) \in \mathfrak{QCoh}^d_{\mathrm{fff}}$ where $T' \to T$ is an étale cover (of degree d) and $\mathcal{M}$ is a locally free $\mathcal{O}|_{T'}$–module of constant rank r, and whose morphisms are the cartesian morphisms of $\mathfrak{QCoh}^d_{\mathrm{fff}}$.

(iv) Let $\mathfrak{Azu}_r^{d-\text{ét}}$ be the full substack of $\mathfrak{QAlg}^d_{\mathrm{fff}}$ whose objects are those $(T' \to T, \mathcal{A}) \in \mathfrak{QAlg}^d_{\mathrm{fff}}$ where $T' \to T$ is an étale cover (of degree d) and $\mathcal{A}$ is an Azumaya $\mathcal{O}|_{T'}$–algebra of constant degree r, and whose morphisms are the cartesian morphisms of $\mathfrak{QAlg}^d_{\mathrm{fff}}$.

Since all four of the above stacks only contain cartesian morphisms, they are fibered in groupoids by Lemma 1.10. In fact, all four stacks are gerbes, which we will justify for the first two in 3.1, for $\mathfrak{Mod}_r^{d-\text{ét}}$ before Lemma 3.3, and for $\mathfrak{Azu}_r^{d-\text{ét}}$ after Lemma 3.3.

The results of Lemma 2.20 imply that the norm morphism $N\colon \mathfrak{QCoh}^d_{\mathrm{fff}} \to \mathfrak{QCoh}$ restricts to two morphisms of stacks,

$$N_{\mathfrak{Mod}}\colon \mathfrak{Mod}_r^{d-\text{ét}} \to \mathfrak{Mod}_{r^d} \text{ and } N_{\mathfrak{Azu}}\colon \mathfrak{Azu}_r^{d-\text{ét}} \to \mathfrak{Azu}_{r^d}. \tag{3.0.1}$$

Since all four of these stacks are gerbes, we will obtain a cohomological description of $N_{\mathfrak{Mod}}$ and $N_{\mathfrak{Azu}}$ by applying Lemma 1.18. To do so, we first identify some of the automorphism sheaves of objects in these stacks.

3.1. **Automorphism Sheaves.** To begin, if we consider the object $(S, \mathcal{O}^r) \in \mathfrak{Mod}_r$, then it is clear that its automorphism sheaf is $\mathcal{A}ut(S, \mathcal{O}^r) = \mathbf{GL}_r$. Any locally free module of rank r is by definition locally isomorphic to $\mathcal{O}^r$, the stack $\mathfrak{Mod}_r$ is fibered in groupoids, and the fibers $\mathfrak{Mod}_r(U)$ for $U \in \mathfrak{Sch}_S$ are nonempty since they contain the free module, so we know $\mathfrak{Mod}_r$ is a gerbe. We call $(S, \mathcal{O}^r)$ the split object in $\mathfrak{Mod}_r(S)$.

If we consider the object $(S, \mathrm{M}_r(\mathcal{O})) \in \mathfrak{Azu}_r$, then we have $\mathcal{A}ut(S, \mathrm{M}_r(\mathcal{O})) = \mathbf{PGL}_r$. Any Azumaya algebra of degree r is locally isomorphic to $\mathrm{M}_r(\mathcal{O})$ and so $\mathfrak{Azu}_r$ is a gerbe as well. We call $(S, \mathrm{M}_r(\mathcal{O}))$ the split object of $\mathfrak{Azu}_r(S)$.

Next, we define some semi-direct products of groups which will appear later as automorphism sheaves. Let $f\colon T \to S$ be a degree d étale cover of our base scheme.

First, we define the group $f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T)$. Let $X \in \mathfrak{Sch}_S$ and let $g\colon T \times_S X \xrightarrow{\sim} T \times_S X$ be an isomorphism of X–schemes, i.e., $g \in \mathcal{A}ut_S(T)(X)$. We then have a pullback functor g^* from the category of $\mathcal{O}|_{T\times_S X}$–modules to itself and $g^*(\mathcal{O}|_{T\times_S X}) = \mathcal{O}|_{T\times_S X}$. This also means that $g^*(\mathcal{O}|^r_{T\times_S X}) = \mathcal{O}^r_{T\times_S X}$. Therefore, for a section $\varphi \in f_*(\mathbf{GL}_r|_T)(X) = \mathrm{Aut}_{\mathcal{O}_{T\times_S X}}(\mathcal{O}|^r_{T\times_S X})$ we also have that $g^*(\varphi)$ is an automorphism of $\mathcal{O}|^r_{T\times_S X}$. We use this to define the semidirect product structure on $f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T)$ on by

$$\varphi \cdot g = g \cdot g^*(\varphi)$$

for appropriate sections.

When $T = S^{\sqcup d}$, this group becomes $(\mathbf{GL}_r)^d \rtimes \mathbb{S}_d$ where $\mathbb{S}_d$ acts on $(\mathbf{GL}_r)^d$ be permuting the factors as follows. To keep track of position, write $S^{\sqcup d} = S_1 \sqcup \ldots \sqcup S_d$ where each $S_j = S$. Let $X \in \mathfrak{Sch}_S$ be any scheme. Since X is possibly disconnected, let $X = \bigsqcup_{i\in I} X_i$ be its decomposition into connected components. Then $S^{\sqcup d} \times_S X = \bigsqcup_{i\in I}(X_{1,i} \sqcup \ldots \sqcup X_{d,i})$ where each $X_{j,i} = X_i$. For $\sigma = (\sigma_i)_{i\in I} \in \mathbb{S}_d(X) = \prod_{i\in I} \mathbb{S}_d(\mathbb{Z})$ (here $\mathbb{S}_d(\mathbb{Z})$ is simply the abstract group of permutations since $\mathrm{Spec}(\mathbb{Z})$ is connected), we view it as the scheme isomorphism which sends component $X_{j,i} \to X_{\sigma_i(j),i}$ via Id_{X_i}.

Now, let $\mathcal{M}$ be an $\mathcal{O}|_{X^{\sqcup d}}$–module. A $X^{\sqcup d}$ scheme is of the form $\sqcup_{i\in I}(Y_{1,i} \sqcup \ldots \sqcup Y_{d,i})$ where each $Y_{j,i}$ are arbitrary X–schemes and the structure morphism sends $Y_{j,i} \to X_{j,i}$. The module $\mathcal{M}$ will then evaluate as

$$\mathcal{M}(\sqcup_{i\in I}(Y_{1,i} \sqcup \ldots \sqcup Y_{d,i})) = \prod_{i\in I}(\mathcal{M}_{1,i}(Y_{1,i}) \times \ldots \times \mathcal{M}_{d,i}(Y_{d,i}))$$

for $\mathcal{O}|_X$–modules $\mathcal{M}_{j,i}$. We express this as $\mathcal{M} = (\mathcal{M}_{1,i}, \ldots, \mathcal{M}_{d,i})_{i\in I}$. The pullback module $\sigma^*(\mathcal{M})$ will evaluate the $X^{\sqcup d}$-scheme $\sqcup_{i\in I}(Y_{1,i} \sqcup \ldots \sqcup Y_{d,i}$ as if the structure morphism sends $Y_{j,i} \to X_{\sigma_i(j),i}$. Therefore,

$$\sigma^*(\mathcal{M})(\sqcup_{i\in I}(Y_{1,i} \sqcup \ldots \sqcup Y_{d,i})) = \prod_{i\in I}(\mathcal{M}_{\sigma_i(1),i}(Y_{1,i}) \times \ldots \times \mathcal{M}_{\sigma_i(d),i}(Y_{d,i})),$$

i.e., $\sigma^*(\mathcal{M}) = (\mathcal{M}_{\sigma_i(1),i}, \ldots, \mathcal{M}_{\sigma_i(d),i})_{i\in I}$. So, for an automorphism $\varphi = (\varphi_{1,i}, \ldots, \varphi_{d,i})_{i\in I}$ of $\mathcal{M}$, we have

$$\sigma^*(\varphi) = (\varphi_{\sigma_i(1),i}, \ldots, \varphi_{\sigma_i(d),i})_{i\in I}.$$

In particular, this applies when all $\mathcal{M}_{j,i} = \mathcal{O}|^r_{X_i}$ to describe $\sigma^*(\varphi)$ for $\varphi \in (\mathbf{GL}_r)^d(X)$. As an example, if X is connected, $d = 3$, and $\sigma = (1\ 2\ 3)$ is a cycle, then

$$\sigma^*(\varphi_1, \varphi_2, \varphi_3) = (\varphi_2, \varphi_3, \varphi_1).$$

Likewise, we define the group $f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T)$. With g still as above, we have $g^*(\mathcal{E}nd_{\mathcal{O}|_{T\times_S X}}(\mathcal{O}|^r_{T\times_S X})) = \mathcal{E}nd_{\mathcal{O}|_{T\times_S X}}(\mathcal{O}|^r_{T\times_S X})$ and so we also have a semidirect product structure on $f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T)$ using

the same formula, just when φ is an algebra automorphism. Once again, when $T = S^{\sqcup d}$, this group becomes $(\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d$.

3.2. **Lemma.** *Consider objects of the form* $(f\colon T \to S, \mathcal{O}|_T^r) \in \mathfrak{Mod}_r^{d-\text{ét}}(S)$.

(i) *We have that*

$$\mathcal{A}ut(T \to S, \mathcal{O}|_T^r) \cong f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T).$$

(ii) *In particular,*

$$\mathcal{A}ut(S^{\sqcup d} \to S, \mathcal{O}_{S^{\sqcup d}}^r) \cong (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d.$$

Proof. (i): Let $X \in \mathfrak{Sch}_S$. A section in $\mathcal{A}ut(T \to S, \mathcal{O}|_T^r)(X)$ is a triple of the form $(\mathrm{Id}_X, g, \varphi)$ where $g \in \mathcal{A}ut_S(T)(X)$ and $\varphi\colon \mathcal{O}|_{T\times_S X}^r \xrightarrow{\sim} g^*(\mathcal{O}|_{T\times_S X}^r) = \mathcal{O}|_{T\times_S X}^r$, and so $\varphi \in f_*(\mathbf{GL}_r|_T)(X)$. Unsurprisingly, the map of sheaves

$$\begin{aligned} \mathcal{A}ut(T \to S, \mathcal{O}|_T^r) &\to f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T) \\ (\mathrm{Id}_S, g, \varphi) &\mapsto g \cdot \varphi \end{aligned}$$

will give our desired isomorphism. It is clearly bijective and it is a group isomorphism since we have

$$\begin{aligned} &(\mathrm{Id}_S, h, \psi)(\mathrm{Id}_S, g, \varphi) = (\mathrm{Id}_S, hg, g^*(\psi)\varphi) \\ \mapsto &h \cdot g \cdot g^*(\psi) \cdot \varphi = h \cdot \psi \cdot g \cdot \varphi = (h \cdot \psi)(g \cdot \varphi) \end{aligned}$$

(ii): This is a specific instance of Lemma B.3. Of course, (ii) also follows immediately from (i). □

Every degree d étale cover is locally isomorphic to $S^{\sqcup d} \to S$ and likewise every rank r finite locally free module over such a cover is locally isomorphic to $\mathcal{O}|_{S^{\sqcup d}}^r$. Therefore, $\mathfrak{Mod}_r^{d-\text{ét}}$ is a gerbe. By choosing $(S^{\sqcup d} \to S, \mathcal{O}_{S^{\sqcup d}}^r)$ as the split object, we view the groupoid $\mathfrak{Mod}_r^{d-\text{ét}}(S)$ as the groupoid of twisted forms of $(S^{\sqcup d}, \mathcal{O}_{S^{\sqcup d}}^r)$. By Lemma 3.2(ii) this groupoid is equivalent to the category of $(\mathbf{GL}_r^d) \rtimes \mathbb{S}_d$–torsors. The isomorphism classes in $\mathfrak{Mod}_r^{d-\text{ét}}(S)$ are classified by $H^1(S, (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d)$. In the notation of Appendix B, $\mathfrak{Mod}_r^{d-\text{ét}}$ is equivalent to the stack $\mathfrak{F}(\mathbf{GL}_r)^{d-\text{ét}}$.

3.3. **Lemma.** *Consider objects of the form* $(f\colon T \to S, \mathrm{M}_r(\mathcal{O}|_T)) \in \mathfrak{Azu}_r^{d-\text{ét}}(S)$.

(i) *We have that*

$$\mathcal{A}ut(T \to S, \mathrm{M}_r(\mathcal{O}|_T)) \cong f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T).$$

(ii) *In particular,*

$$\mathcal{A}ut(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}})) \cong (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d.$$

Proof. (i): Because $g^*(\mathcal{E}nd_{\mathcal{O}|_{T\times_S Y}}(\mathcal{O}|_{T\times_S Y}^r)) = \mathcal{E}nd_{\mathcal{O}|_{T\times_S Y}}(\mathcal{O}|_{T\times_S Y}^r)$, this proof is the same as the proof of Lemma 3.2(i) except $\psi, \varphi \in f_*(\mathbf{PGL}_r|_T)$ will instead be algebra automorphisms.
(ii): This also is a specific case of Lemma B.3 or follows from (i). □

Via a similar argument as above, any object of $\mathfrak{Azu}_r^{d-\text{ét}}$ is locally isomorphic to $(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))$. Since $(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))$ is a global object, every fiber is non-empty. Hence, $\mathfrak{Azu}_r^{d-\text{ét}}$ is a gerbe. We choose $(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))$ as the split object. We view $\mathfrak{Azu}_r^{d-\text{ét}}(S)$ as the groupoid of its twisted forms, see Proposition 1.16(ii). By Lemma 3.3(ii), this category is equivalent to the category of $(\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d$–torsors. The isomorphism classes in $\mathfrak{Azu}_r^{d-\text{ét}}(S)$ are classified by $H^1(S, (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d)$. This stack is equivalent to the stack $\mathfrak{F}(\mathbf{PGL}_r)^{d-\text{ét}}$ of Appendix B.

3.4. **Cohomology Maps.** Since we know from Example 2.14 that the morphism $N_{\mathfrak{Mod}}$ maps $(S^{\sqcup d} \to S, \mathcal{O}^r_{S^{\sqcup d}}) \in \mathfrak{Mod}_r^{d-\text{ét}}$ to $(\mathcal{O}^r)^{\otimes d} \cong \mathcal{O}^{(r^d)} \in \mathfrak{Mod}_{r^d}$, functoriality yields an associated group homomorphism between the automorphism groups

$$N_{\mathfrak{Mod},(S^{\sqcup d} \to S, \mathcal{O}|^r_{S^{\sqcup d}})} \colon (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{GL}_{r^d}$$

and we seek to describe the resulting map $\widetilde{N_{\mathfrak{Mod}}} \colon H^1(S, (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d) \to H^1(S, \mathbf{GL}_{r^d})$ on isomorphism classes.

First, we consider the Segre homomorphism

$$\begin{aligned} \mathrm{Seg} \colon \mathbf{GL}_r \times_S \cdots \times_S \mathbf{GL}_r &\to \mathbf{GL}_{r^d} \\ (A_1, \ldots, A_d) &\mapsto A_1 \otimes \cdots \otimes A_d \end{aligned} \tag{3.4.1}$$

which we extend slightly. We view the isomorphism $\mathcal{O}^{(r^d)} \cong (\mathcal{O}^r)^{\otimes d}$ as an identification. Let $X \in \mathfrak{Sch}_S$ by any scheme. Since X may be disconnected, let $X = \bigsqcup_{i \in I} X_i$ be its decomposition into connected components. The $\mathcal{O}(X)$–module $\mathcal{O}^{(r^d)}(X)$ is spanned by elements of the form $x_1 \otimes \ldots \otimes x_d$ with each $x_j = (x_{j,i})_{i \in I} \in \mathcal{O}^r(X) = \prod_{i \in I} \mathcal{O}^r(X_i)$.

For each $\sigma = (\sigma_i)_{i \in I} \in \mathbb{S}_d(X) = \prod_{i \in I} \mathbb{S}_d(\mathbb{Z})$, we obtain a linear transformation of $\mathcal{O}^{(r^d)}(X)$ by sending

$$(x_{1,i})_{i \in I} \otimes \ldots \otimes (x_{d,i})_{i \in I} \mapsto (x_{\sigma_i^{-1}(1),i})_{i \in I} \otimes \ldots \otimes (x_{\sigma_i^{-1}(d),i})_{i \in I}.$$

Here, $x_{k,i}$ is ending up in the $\sigma_i(k), i$–position. For example, if X is connected, $d = 3$, and $\sigma = (1\ 2\ 3)$ is a cycle, then

$$x_1 \otimes x_2 \otimes x_3 \mapsto x_3 \otimes x_1 \otimes x_2.$$

This yields an injective group homomorphism $j(X) \colon \mathbb{S}_d(X) \hookrightarrow \mathbf{GL}_{r^d}(X)$ and together for all $X \in \mathfrak{Sch}_S$ these yield an injective morphism of group sheaves $j \colon \mathbb{S}_d \hookrightarrow \mathbf{GL}_{r^d}$. For clarity in the following computation we assume X is connected, however the computation in the general case is the same but with added indices as above. For $A_i \in \mathbf{GL}_r(X)$ and $\sigma \in \mathbb{S}_d(X)$, we have

$$\begin{aligned} &(\mathrm{Seg}(A_1, \ldots, A_d) \circ j(\sigma))(x_1 \otimes \ldots \otimes x_d) \\ =&(A_1 \otimes \ldots \otimes A_d)(x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(d)}) \\ =&A_1(x_{\sigma^{-1}(1)}) \otimes \ldots \otimes A_d(x_{\sigma^{-1}(d)}) \\ =&j(\sigma)(A_{\sigma(1)}(x_1) \otimes \ldots \otimes A_{\sigma(d)}(x_d)) \end{aligned}$$

$$=(j(\sigma)\circ(A_{\sigma(1)}\otimes\ldots\otimes A_{\sigma(d)}))(x_1\otimes\ldots\otimes x_d)$$
$$=(j(\sigma)\circ \mathrm{Seg}(A_{\sigma(1)},\ldots,A_{\sigma(d)}))(x_1\otimes\ldots\otimes x_d)$$

which shows that $\mathrm{Seg}(A_1,\ldots,A_d)\circ j(\sigma) = j(\sigma)\circ\mathrm{Seg}(A_{\sigma(1)},\ldots,A_{\sigma(d)})$. Therefore, combining Seg with j we get a well defined group homomorphism

$$\mathrm{Seg}'\colon (\mathbf{GL}_r)^d\rtimes\mathbb{S}_d\to\mathbf{GL}_{r^d}. \tag{3.4.2}$$

3.5. **Theorem.** *Let $N_{\mathfrak{Mod}}$ be the morphism of (3.0.1). The group homomorphism*

$$N_{\mathfrak{Mod},(S^{\sqcup d}\to S,\mathcal{O}|^r_{S^{\sqcup d}})}\colon (\mathbf{GL}_r)^d\rtimes\mathbb{S}_d\to\mathbf{GL}_{r^d}$$

is the homomorphism Seg′.

Proof. Let $f\colon S^{\sqcup d}\to S$ be the canonical projection. By Example 2.14, $N_{\mathfrak{Mod}}(S^{\sqcup d}\to S,(\mathcal{O}|_{S^{\sqcup d}})^r)=\mathcal{O}^{(r^d)}$ and the universal normic polynomial law is

$$\boldsymbol{\nu}\colon f_*((\mathcal{O}|_{S^{\sqcup d}})^r)=(\mathcal{O}^r)^d\to(\mathcal{O}^r)^{\otimes d}=\mathcal{O}^{(r^d)}$$
$$(x_1,\ldots,x_d)\mapsto x_1\otimes\ldots\otimes x_d.$$

Furthermore, $\boldsymbol{\nu}$ is stable under base change by Corollary 2.12. Therefore, $\boldsymbol{\nu}|_X$ also has the same universal property as $\boldsymbol{\nu}$.

Now, let $\varphi=(A_1,\ldots,A_d)\sigma\in((\mathbf{GL}_r)^d\rtimes\mathbb{S}_d)(X)$ be a section over some $X\in\mathfrak{Sch}_S$. Denote by $f'\colon X^{\sqcup d}\to X$ the standard cover which is the pullback of f. Here as well we write as if X is connected, but indices may be added for the general case. The automorphism φ acts on $(\mathcal{O}|^r_X)^d$ by

$$(x_1,\ldots,x_d)\mapsto(A_1x_{\sigma^{-1}(1)},\ldots,A_dx_{\sigma^{-1}(d)}).$$

The composition $\boldsymbol{\nu}|_X\circ\varphi\colon f_*((\mathcal{O}|_{X^{\sqcup d}})^r)\to\mathcal{O}|_X^{(r^d)}$ is also a normic law and it is described by

$$(x_1,\ldots,x_d)\mapsto A_1x_{\sigma^{-1}(1)}\otimes\ldots\otimes A_dx_{\sigma^{-1}(d)}.$$

It is therefore clear that the map $\mathrm{Seg}'(\varphi)$ makes the diagram below commute

$$\begin{array}{ccc} f'_*((\mathcal{O}|_{X^{\sqcup d}})^r) & \xrightarrow{\boldsymbol{\nu}} & \mathcal{O}|_X^{(r^d)} \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\mathrm{Seg}'(\varphi)} \\ f'_*((\mathcal{O}|_{X^{\sqcup d}})^r) & \xrightarrow{\boldsymbol{\nu}} & \mathcal{O}|_X^{(r^d)} \end{array}$$

and therefore, by Corollary 2.11, it is the unique such $\mathcal{O}|_X$–module isomorphism which does so. This means $N_{X^{\sqcup d}/X}(\varphi)=\mathrm{Seg}'(\varphi)$. Since we have that $N_{\mathfrak{Mod},(S^{\sqcup d}\to S,\mathcal{O}|^r_{S^{\sqcup d}})}(X)=N_{X^{\sqcup d}/X}$ on morphisms by definition, we conclude that $N_{\mathfrak{Mod},(S^{\sqcup d}\to S,\mathcal{O}|^r_{S^{\sqcup d}})}=\mathrm{Seg}'$ as natural transformations, as desired. □

3.6. **Corollary.** *The map on cohomology induced by the Segre homomorphism* Seg′ *of (3.4.2) is*

$$\widetilde{\mathrm{Seg}'}\colon H^1(S,(\mathbf{GL}_r)^d\rtimes\mathbb{S}_d)\to H^1(S,\mathbf{GL}_{r^d})$$

$$[(T \to S, \mathcal{M})] \mapsto [N_{T/S}(\mathcal{M})].$$

Proof. Since $\mathfrak{Mod}_r^{d-\text{ét}}$ and $\mathfrak{Mod}_{r^d}$ are gerbes and we know by Theorem 3.5 that $N_{\mathfrak{Mod},(S^{\sqcup d}\to S,\mathcal{O}|^r_{S^{\sqcup d}})} = \mathrm{Seg}'$, this follows by applying Lemma 1.18. □

Under the Segre homomorphism, the center of each $\mathbf{GL}_r$ maps into the center of $\mathbf{GL}_{r^d}$. The center of $\mathbf{GL}_r$ is also the kernel of the canonical projection $\mathbf{GL}_r \to \mathbf{PGL}_r$ and so there exists a group homomorphism PSeg' which makes the diagram

$$\begin{array}{ccc} (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d & \xrightarrow{\mathrm{Seg}'} & \mathbf{GL}_{r^d} \\ \downarrow & & \downarrow \\ (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d & \xrightarrow{\mathrm{PSeg}'} & \mathbf{PGL}_{r^d}. \end{array} \tag{3.6.1}$$

commute. By viewing an algebra isomorphism in $\mathbf{PGL}_r$ as simply a module isomorphism of a locally free $\mathcal{O}$–module of rank r^2, we get a canonical inclusion $\mathbf{PGL}_r \hookrightarrow \mathbf{GL}_{r^2}$ which fits into the commutative diagram

$$\begin{array}{ccc} (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d & \hookrightarrow & (\mathbf{GL}_{r^2})^d \rtimes \mathbb{S}_d \\ \downarrow{\scriptstyle \mathrm{PSeg}'_r} & & \downarrow{\scriptstyle \mathrm{Seg}'_{r^2}} \\ \mathbf{PGL}_{r^d} & \hookrightarrow & \mathbf{GL}_{r^{2d}}. \end{array} \tag{3.6.2}$$

where we add subscripts to PSeg' and Seg' to track the ranks.

3.7. **Corollary.** *Let $N_{\mathfrak{Azu}}$ be the morphism of (3.0.1). We know the automorphism group of $(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}})) \in \mathfrak{Azu}_r^{d-\text{ét}}$ is $(\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d$ by Lemma 3.3(ii) and we have that $N_{\mathfrak{Azu}}(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}})) \cong \mathrm{M}_{(r^d)}(\mathcal{O})$. Therefore, we get a group homomorphism*

$$N_{\mathfrak{Azu},(S^{\sqcup d}\to S,\mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))} \colon (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d \to \mathbf{PGL}_{r^d}.$$

This homomorphism is PSeg'.

Proof. By Lemma 2.19 and Example 2.14, we have that

$$N_{\mathfrak{Azu}}((S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}})) \cong \mathrm{M}_{(r^d)}(\mathcal{O}).$$

Let $\varphi \in (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d$. Let φ' denote this isomorphism viewed as a morphism in $\mathfrak{Mod}_{r^2}^{d-\text{ét}}$. We then have

$$N_{\mathfrak{Azu},(S^{\sqcup d}\to S,\mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))}(\varphi) = N_{\mathfrak{Mod},(S^{\sqcup d}\to S,\mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))}(\varphi') = \mathrm{Seg}'(\varphi').$$

where the second equality is given by Theorem 3.5. However, due to diagram (3.6.2), this is simply $\mathrm{PSeg}'(\varphi)$ viewed as a module morphism. Thus, $N_{\mathfrak{Azu},(S^{\sqcup d}\to S,\mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}}))} = \mathrm{PSeg}'$ as claimed. □

3.8. **Corollary.** *The map on cohomology induced by the morphism* PSeg' *of* (3.6.1) *is*

$$\widetilde{\mathrm{PSeg}'} \colon H^1(S, (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d) \to H^1(S, \mathbf{PGL}_{r^d})$$

$$[(T \to S, \mathcal{A})] \mapsto [N_{T/S}(\mathcal{A})].$$

Proof. This follows from Lemma 1.18 because of the result of Corollary 3.7. □

The gerbes $\mathfrak{Azu}_r^{d-\text{ét}}$ and $\mathfrak{Azu}_{r^d}$ fit into the commutative diagram of stack morphisms

$$\begin{array}{ccc} \mathfrak{Azu}_r^{d-\text{ét}} & \longrightarrow & \mathfrak{Mod}_{r^2}^{d-\text{ét}} \\ \downarrow{\scriptstyle N_{\mathfrak{Azu}}} & & \downarrow{\scriptstyle N_{\mathfrak{Mod}}} \\ \mathfrak{Azu}_{r^d} & \longrightarrow & \mathfrak{Mod}_{r^{2d}} \end{array}$$

where the horizontal maps are the canonical inclusions (equivalently, forgetful functors). By Corollary 3.6 and Corollary 3.8, both the above diagram and (3.6.2) induce the same diagram on cohomology, namely

$$\begin{array}{ccc} H^1(S, (\mathbf{PGL}_r)^d \rtimes \mathbb{S}_d) & \longrightarrow & H^1(S, (\mathbf{GL}_{r^2})^d \rtimes \mathbb{S}_d) \\ \downarrow{\scriptstyle \widetilde{\text{PSeg}'}} & & \downarrow{\scriptstyle \widetilde{\text{Seg}'}} \\ H^1(S, \mathbf{PGL}_{r^d}) & \longrightarrow & H^1(S, \mathbf{GL}_{r^{2d}}) \end{array}$$

where the horizontal maps send isomorphism classes of algebras to their isomorphism class simply as modules.

3.9. **The Norm and the Brauer Group.** In this section we fix a degree d étale cover $f\colon T \to S$ of our base scheme and we describe how the functor $N_{T/S}$ acts on the Brauer classes of Azumaya algebras. We work with Brauer-Grothendieck groups as in [CTS], which are second cohomology groups. For example, these are denoted $\text{Br}(S) = H^2_{\text{fppf}}(S, \mathbb{G}_m)$ and $\text{Br}(T) = H^2_{\text{fppf}}(T, \mathbb{G}_m|_T)$. This is in contrast to [CF, 3.6.1.1] where the notation "$\text{Br}(S)$" is used for the Brauer-Azumaya group consisting of classes of Azumaya algebras up to Brauer equivalence. These two notions are not isomorphic in general, but they are isomorphic over fields or more broadly in the case covered by Gabber's Theorem, see [CTS, 4.2.1].

We will show in Proposition 3.14(i) that the norm functor is compatible with the *trace map* $H^2_{\text{fppf}}(T, \mathbb{G}_m|_T) \to H^2_{\text{fppf}}(S, \mathbb{G}_m)$ of [SGA4, IX.5.1.3]. The work in [SGA4] uses étale cohomology, but by [CF, 2.2.5.15] or [M, III.3.9], this agrees with flat cohomology since $\mathbb{G}_m$ is smooth. The trace map is defined as follows. First, as noted in [SGA4, IX.5.1], since $f\colon T \to S$ is finite étale, there is an isomorphism $H^2_{\text{fppf}}(T, \mathbb{G}_m|_T) \cong H^2_{\text{fppf}}(S, f_*(\mathbb{G}_m|_T))$. Then, the product map $\mu\colon \mathbb{G}_m^d \to \mathbb{G}_m$ can have its domain twisted by the $\mathbb{S}_d$–torsor $\mathcal{I}som(S^{\sqcup d}, T)$ as in Lemma A.1, which yields the trace map $\text{tr}\colon f_*(\mathbb{G}_m|_T) \to \mathbb{G}_m$ of [SGA4, IX.5.1.2]. This trace in turn induces the desired trace map between cohomology.

Further, for any group sheaf over S, we have a restriction map, $\text{res}\colon \mathbf{G} \to f_*(\mathbf{G}|_T)$, which is the diagonal embedding $\mathbf{G} \hookrightarrow \mathbf{G}^d$ twisted by $\mathcal{I}som(S^{\sqcup d}, T)$.

Alternatively, for $X \in \mathfrak{Sch}_S$ there are the restriction maps

$$\mathbf{G}(X) \to \mathbf{G}(T \times_S X) = f_*(\mathbf{G}|_T)(X)$$
$$\varphi \mapsto \varphi|_{T\times_S X}$$

which are part of the definition of the sheaf. These homomorphisms assemble into the restriction map $\mathrm{res}\colon \mathbf{G} \to f_*(\mathbf{G}|_T)$. Since $\mathbb{G}_m$ is abelian, we have by [SGA4, IX.5.1.4] that the composition

$$\mathbb{G}_m \xrightarrow{\mathrm{res}} f_*(\mathbb{G}_m|_T) \xrightarrow{\mathrm{tr}} \mathbb{G}_m$$

is the "multiplication" by d map, i.e., $x \mapsto x^d$ since $\mathbb{G}_m$ is written multiplicatively. In turn, the composition on cohomology

$$H^2(S, \mathbb{G}_m) \xrightarrow{\mathrm{res}} H^2(S, f_*(\mathbb{G}_m|_T)) \xrightarrow{\mathrm{tr}} H^2(S, \mathbb{G}_m) \tag{3.9.1}$$

is also multiplication by d.

We now define two new stacks and compute some of their automorphism sheaves. First, let $T\text{-}\mathfrak{Mod}_r$ be the stack with

(i) objects $(X, \mathcal{M})$ where $X \in \mathfrak{Sch}_S$ and $\mathcal{M}$ is a locally free $\mathcal{O}|_{T\times_S X}$–module of constant rank r,
(ii) morphisms $(g, \varphi)\colon (Y, \mathcal{M}_1) \to (X, \mathcal{M}_2)$ where $g\colon Y \to X$ is an S–scheme morphism and $\varphi\colon \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2|_{T\times_S Y}$ is a $\mathcal{O}|_{T\times_S Y}$–module isomorphism, where $\mathcal{M}_2$ is restricted along the map $T \times_S Y \to T \times_S X$ which is the pullback of g, and
(iii) structure functor $(X, \mathcal{M}) \mapsto X$ and $(g, \varphi) \mapsto g$.

It is clear that $T\text{-}\mathfrak{Mod}_r$ is fibered in groupoids and since two locally free modules of the same rank are locally isomorphic, it is also a gerbe. The fiber $T\text{-}\mathfrak{Mod}_r(S)$ is the groupoid of locally free $\mathcal{O}|_T$–modules of constant rank r. We designate $(S, \mathcal{O}|_T^r) \in T\text{-}\mathfrak{Mod}_r(S)$ as the split object.

3.10. **Lemma.** *Let $f\colon T \to S$ be a degree d étale cover and let $T\text{-}\mathfrak{Mod}_r$ be defined as above. Consider an object $(S, \mathcal{M}) \in T\text{-}\mathfrak{Mod}_r(S)$.*

(i) *We have that*

$$\mathcal{A}ut(S, \mathcal{M}) \cong f_*(\mathbf{GL}(\mathcal{M})).$$

(ii) *In particular,*

$$\mathcal{A}ut(S, \mathcal{O}|_T^r) \cong f_*(\mathbf{GL}_r|_T).$$

Proof. (i): For a scheme $X \in \mathfrak{Sch}_S$, a section $(g, \varphi) \in \mathcal{A}ut(S, \mathcal{M})(X)$ is an automorphism of $(X, \mathcal{M}|_{T\times_S X})$ in $T\text{-}\mathfrak{Mod}_r(X)$. Since it is a morphism in the fiber $T\text{-}\mathfrak{Mod}_r(X)$, it must have $g = \mathrm{Id}_X$ and therefore $\varphi\colon \mathcal{M}|_{T\times_S X} \xrightarrow{\sim} \mathcal{M}_{T\times_S X}$ is an automorphism in $\mathbf{GL}(\mathcal{M})(T \times_S X) = f_*(\mathbf{GL}(\mathcal{M}))(X)$. It is clear this yields an isomorphism of groups $\mathcal{A}ut(S, \mathcal{M})(X) \cong f_*(\mathbf{GL}(\mathcal{M}))(X)$ and that these assemble into an automorphism of sheaves as claimed.
(ii): This is immediate from (i) since $\mathbf{GL}(\mathcal{O}|_T^r) = \mathbf{GL}_r|_T$. □

Second, we define an analogous stack for Azumaya algebras. Let $T\text{-}\mathfrak{Azu}_r$ be the substack of $T\text{-}\mathfrak{Mod}_{r^2}$ with

(i) objects $(X, \mathcal{A})$ where $X \in \mathfrak{Sch}_S$ and $\mathcal{A}$ is an Azumaya $\mathcal{O}|_{T\times_S X}$–algebra of constant degree r,
(ii) morphisms $(g, \varphi)\colon (Y, \mathcal{A}_1) \to (X, \mathcal{A}_2)$ where $g\colon Y \to X$ is an S–scheme morphism and $\varphi\colon \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2|_{T\times_S Y}$ is an $\mathcal{O}|_{T\times_S Y}$–algebra isomorphism.

Since all such Azumaya algebras are locally isomorphic, T-$\mathfrak{Azu}_r$ is a gerbe as well. The fiber T-$\mathfrak{Azu}_r(S)$ is the groupoid of degree r Azumaya $\mathcal{O}|_T$–algebras.

3.11. **Lemma.** *Let $f\colon T \to S$ be a degree d étale cover and let T-$\mathfrak{Azu}_r$ be defined as above. Consider an object $(S, \mathcal{A}) \in T$-$\mathfrak{Azu}_r(S)$.*

(i) *We have that*

$$\mathcal{A}ut(S, \mathcal{A}) \cong f_*(\mathbf{PGL}(\mathcal{A})).$$

(ii) *In particular,*

$$\mathcal{A}ut(S, \mathrm{M}_r(\mathcal{O}|_T)) \cong f_*(\mathbf{PGL}_r|_T).$$

Proof. The proof of Lemma 3.10 may be replicated here, replacing module automorphisms with algebra automorphisms and replacing **GL** with **PGL**. □

Now, we define various stack morphism from which we will later extract a commutative diagram of group sheaves.

$$\begin{aligned}
\mathrm{res}\colon \mathfrak{Mod}_r &\to T\text{-}\mathfrak{Mod}_r & \mathrm{res}\colon \mathfrak{Azu}_r &\to T\text{-}\mathfrak{Azu}_r\\
(X, \mathcal{M}) &\mapsto (X, \mathcal{M}|_{T\times_S X}) & (X, \mathcal{A}) &\mapsto (X, \mathcal{A}|_{T\times_S X})\\
(g, \varphi) &\mapsto (g, \varphi|_{T\times_S Y}) & (g, \varphi) &\mapsto (g, \varphi|_{T\times_S Y})\\
\mathrm{inc}\colon T\text{-}\mathfrak{Mod}_r &\to \mathfrak{Mod}_r^{d-\text{ét}} & \mathrm{inc}\colon T\text{-}\mathfrak{Azu}_r &\to \mathfrak{Azu}_r^{d-\text{ét}}\\
(X, \mathcal{M}) &\mapsto (T \times_S X \to X, \mathcal{M}) & (X, \mathcal{A}) &\to (T \times_S X \to X, \mathcal{A})\\
(g, \varphi) &\mapsto (g, g', \varphi) & (g, \varphi) &\mapsto (g, g', \varphi)\\
\mathcal{E}nd\colon \mathfrak{Mod}_r &\to \mathfrak{Azu}_r & T\text{-}\mathcal{E}nd\colon T\text{-}\mathfrak{Mod}_r &\to T\text{-}\mathfrak{Azu}_r\\
(X, \mathcal{M}) &\mapsto (X, \mathcal{E}nd_{\mathcal{O}|_X}(\mathcal{M})) & (X, \mathcal{M}) &\mapsto (X, \mathcal{E}nd_{\mathcal{O}|_{T\times_S X}}(\mathcal{M}))\\
(g, \varphi) &\mapsto (g, \mathcal{E}nd(\varphi)) & (g, \varphi) &\mapsto (g, \mathcal{E}nd(\varphi))
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}nd^{d-\text{ét}}\colon \mathfrak{Mod}_r^{d-\text{ét}} &\to \mathfrak{Azu}_r^{d-\text{ét}}\\
(X' \to X, \mathcal{M}) &\mapsto (X' \to X, \mathcal{E}nd_{\mathcal{O}|_Y}(\mathcal{M}))\\
(f', g, \varphi) &\mapsto (f', g, \mathcal{E}nd(\varphi)).
\end{aligned}$$

where $g'\colon T\times_S Y \to T\times_S X$ denotes the pullback of $g\colon Y \to X$. We abuse notation by reusing res and inc for two different restriction and inclusion maps, however the second instance is the same map but on a substack. For the maps $\mathrm{res}\colon \mathfrak{Mod}_r \to T$-$\mathfrak{Mod}_r$, if $(g, \varphi)\colon (Y, \mathcal{M}_1) \to (X, \mathcal{M}_2)$ is a morphism,

then $\varphi\colon \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2|_Y$ is an isomorphism. Since $(\mathcal{M}_2|_{T\times_S X})|_{T\times_S Y} = (\mathcal{M}_2|_Y)|_{T\times_S Y}$, the restricted morphism is

$$\varphi|_{T\times_S Y}\colon \mathcal{M}_1|_{T\times_S Y} \xrightarrow{\sim} (\mathcal{M}_2|_{T\times_S X})|_{T\times_S Y}$$

as required and similarly for $\mathrm{res}\colon \mathfrak{Azu}_r \to T\text{-}\mathfrak{Azu}_r$.

If $\varphi\colon \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2|_Y$ is an $\mathcal{O}|_Y$–module isomorphism (or similarly, over $\mathcal{O}|_{T\times_S Y}$ or $\mathcal{O}|_{Y'}$ as would be the case for $T\text{-}\mathcal{E}nd$ or $\mathcal{E}nd^{d-\text{ét}}$), then $\mathcal{E}nd(\varphi)$ is the algebra automorphism

$$\mathcal{E}nd(\varphi)\colon \mathcal{E}nd_{\mathcal{O}|_Y}(\mathcal{M}_1) \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}|_Y}(\mathcal{M}_2|_Y) = \mathcal{E}nd_{\mathcal{O}|_X}(\mathcal{M}_2)|_Y$$
$$\alpha \mapsto \varphi\circ\alpha\circ\varphi^{-1}.$$

These morphisms fit into a commutative diagram

$$\begin{array}{ccc}
\mathfrak{Mod}_r & \xrightarrow{\mathcal{E}nd} & \mathfrak{Azu}_r \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
T\text{-}\mathfrak{Mod}_r & \xrightarrow{T\text{-}\mathcal{E}nd} & T\text{-}\mathfrak{Azu}_r \\
\downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\
\mathfrak{Mod}_r^{d-\text{ét}} & \xrightarrow{\mathcal{E}nd^{d-\text{ét}}} & \mathfrak{Azu}_r^{d-\text{ét}}.
\end{array} \tag{3.11.1}$$

3.12. **Lemma.** *Tracing the image of $(S,\mathcal{O}^r)\in\mathfrak{Mod}_r$ through the diagram (3.11.1) we obtain*

$$\begin{array}{ccc}
(S,\mathcal{O}^r) & \longmapsto & (S,\mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^r)) \\
\downarrow & & \downarrow \\
(S,\mathcal{O}|_T^r) & \longmapsto & (S,\mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^r)|_T) = (S,\mathcal{E}nd_{\mathcal{O}|_T}(\mathcal{O}|_T^r)) \\
\downarrow & & \downarrow \\
(T\to S,\mathcal{O}|_T^r) & \longmapsto & (T\to S,\mathcal{E}nd_{\mathcal{O}|_T}(\mathcal{O}|_T^r)).
\end{array}$$

The corresponding induced homomorphisms between automorphism sheaves are given by the following diagram.

$$\begin{array}{ccc}
\mathbf{GL}_r & \xrightarrow{\pi} & \mathbf{PGL}_r \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
f_*(\mathbf{GL}_r|_T) & \xrightarrow{\pi'} & f_*(\mathbf{PGL}_r|_T) \\
\downarrow & & \downarrow \\
f_*(\mathbf{GL}_r|_T)\rtimes \mathcal{A}ut_S(T) & \xrightarrow{\pi'\times\mathrm{Id}} & f_*(\mathbf{PGL}_r|_T)\rtimes \mathcal{A}ut_S(T)
\end{array}$$

where π and π' are the canonical projections and the hooked arrows indicate the inclusions.

Proof. The objects appearing in the first diagram have the corresponding automorphism sheaves in the second diagram either by definition or by Lemmas 3.2(i), 3.3(i), 3.10(ii), or 3.11(ii).

Since the restriction maps send a morphism φ over $X \in \mathfrak{Sch}_S$ to $\varphi|_{T\times_S X}$, it is clear they induce the restriction map between groups.

The claim that the horizontal maps are the canonical projections is clear since, by definition, $\mathcal{E}nd$, T-$\mathcal{E}nd$, and $\mathcal{E}nd^{d-\text{ét}}$ send a module automorphism to its corresponding inner automorphism of the endomorphism algebra and additionally $\mathcal{E}nd^{d-\text{ét}}$ acts as the identity on the scheme part of morphisms, i.e., it preserves f' and g in (f', g, φ).

The fact that the hooked arrows are the inclusion is immediate since the inclusion maps sends morphisms of the form (Id_X, φ) to $(\mathrm{Id}_X, \mathrm{Id}_{T\times_S X}, \varphi)$. □

Next, we extend the commutative diagram (3.11.1) by appending the functors $N_{\mathfrak{Mod}}$ and $N_{\mathfrak{Azu}}$ of (3.0.1) on the bottom. This produces

$$\begin{array}{ccc}
\mathfrak{Mod}_r & \xrightarrow{\mathcal{E}nd} & \mathfrak{Azu}_r \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
T\text{-}\mathfrak{Mod}_r & \xrightarrow{T\text{-}\mathcal{E}nd} & T\text{-}\mathfrak{Azu}_r \\
\downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\
\mathfrak{Mod}_r^{d-\text{ét}} & \xrightarrow{\mathcal{E}nd^{d-\text{ét}}} & \mathfrak{Azu}_r^{d-\text{ét}} \\
\downarrow{\scriptstyle N_{\mathfrak{Mod}}} & & \downarrow{\scriptstyle N_{\mathfrak{Azu}}} \\
\mathfrak{Mod}_{r^d} & \xrightarrow{\mathcal{E}nd} & \mathfrak{Azu}_{r^d}.
\end{array} \tag{3.12.1}$$

where the bottom square only commutes up to canonical isomorphism. In particular, for each $(X' \to X, \mathcal{M}) \in \mathfrak{Mod}_r^{d-\text{ét}}$ we have

$$\begin{aligned}
(N_{\mathfrak{Azu}} \circ \mathcal{E}nd^{d-\text{ét}})(X' \to X, \mathcal{M}) &= (X, N_{X'/X}(\mathcal{E}nd_{\mathcal{O}|_{X'}}(\mathcal{M}))) \\
(\mathcal{E}nd \circ N_{\mathfrak{Mod}})(X' \to X, \mathcal{M}) &= (X, \mathcal{E}nd_{\mathcal{O}|_X}(N_{X'/X}(\mathcal{M})))
\end{aligned}$$

and by Lemma 2.19 there is a canonical isomorphism of $\mathcal{O}|_X$–algebras

$$\Psi_{(X'\to X,\mathcal{M})} \colon N_{X'/X}(\mathcal{E}nd_{\mathcal{O}|_{X'}}(\mathcal{M})) \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}|_X}(N_{X'/X}(\mathcal{M})).$$

Tracing the object $(T \to S, \mathcal{O}|_T^r)$ through the bottom square (and through its canonical isomorphism) produces

$$\begin{array}{ccc}
(T \to S, \mathcal{O}|_T^r) & \longmapsto & (T \to S, \mathcal{E}nd_{\mathcal{O}|_T}(\mathcal{O}|_T^r)) \\
\downarrow & & \downarrow \\
(S, N_{T/S}(\mathcal{O}|_T^r)) & \longmapsto & (S, \mathcal{E}nd_{\mathcal{O}}(N_{T/S}(\mathcal{O}|_T^r)))
\end{array}$$

with corresponding group sheaf homomorphisms

$$\begin{array}{ccc} f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T) & \xrightarrow{\pi' \times \mathrm{Id}} & f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \mathbf{GL}(N_{T/S}(\mathcal{O}|_T^r)) & \xrightarrow{\pi''} & \mathbf{PGL}(N_{T/S}(\mathcal{O}|_T^r)) \end{array} \tag{3.12.2}$$

where π'' is the canonical projection. In fact, ϕ and ϕ' are twists of the modified Segre embeddings Seg' and PSeg' respectively. This can be seen by taking a sufficiently fine cover which splits T and then applying Theorem 3.5 or Corollary 3.7 respectively.

Combining the diagrams of Lemma 3.12 and (3.12.2) and extending the rows into their canonical short exact sequences, we obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}_r & \xrightarrow{\pi} & \mathbf{PGL}_r & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \\ 1 & \to & f_*(\mathbb{G}_m|_T) & \longrightarrow & f_*(\mathbf{GL}_r|_T) & \xrightarrow{\pi'} & f_*(\mathbf{PGL}_r|_T) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & f_*(\mathbb{G}_m|_T) & \to & f_*(\mathbf{GL}_r|_T) \rtimes \mathcal{A}ut_S(T) & \xrightarrow{\pi' \times \mathrm{Id}} & f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T) & \to & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}(N_{T/S}(\mathcal{O}|_T^r)) & \xrightarrow{\pi''} & \mathbf{PGL}(N_{T/S}(\mathcal{O}|_T^r)) & \longrightarrow & 1. \end{array}$$

3.13. **Lemma.** *The dashed morphism in the diagram above is the trace homomorphism,* $\mathrm{tr}\colon f_*(\mathbb{G}_m|_T) \to \mathbb{G}_m$.

Proof. If we instead consider the following diagram involving the Segre homomorphism,

$$\begin{array}{ccc} \mathbb{G}_m^d & \longrightarrow & (\mathbf{GL}_r)^d \rtimes \mathbb{S}_d \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{Seg}'} \\ \mathbb{G}_m & \longrightarrow & \mathbf{GL}_{(r^d)} \end{array}$$

where the trace morphism coincides with the multiplication map, it is clear this commutes since

$$\mathrm{Seg}(c_1 I, \ldots, c_d I) = c_1 I \otimes \ldots \otimes c_d I = (c_1 \ldots c_d) I.$$

This describes the dashed morphism we are interested in locally and by points (i) and (ii) after [SGA4, IX.5.1.3], this characterizes the trace map. Therefore, the dashed morphism is $\mathrm{tr}\colon f_*(\mathbb{G}_m|_T) \to \mathbb{G}_m$ as claimed. □

At this point, the third row of the large diagram above is no longer needed and we consider the compressed diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}_r & \xrightarrow{\pi} & \mathbf{PGL}_r & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \\
1 & \to & f_*(\mathbb{G}_m|_T) & \longrightarrow & f_*(\mathbf{GL}_r|_T) & \xrightarrow{\pi'} & f_*(\mathbf{PGL}_r|_T) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}(N_{T/S}(\mathcal{O}|_T^r)) & \xrightarrow{\pi''} & \mathbf{PGL}(N_{T/S}(\mathcal{O}|_T^r)) & \to & 1
\end{array}$$

where $\rho = \phi \circ \mathrm{inc}$ and $\rho' = \phi' \circ \mathrm{inc}$. Finally, we may take the associated diagram of long exact cohomology sequences to obtain the following result.

3.14. **Proposition.** *Let $T \to S$ be a degree d étale cover. Let $\mathcal{B}$ be an Azumaya $\mathcal{O}|_T$–algebra of constant degree and $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra of constant degree.*

(i) $[N_{T/S}(\mathcal{B})] = \mathrm{tr}([\mathcal{B}]) \in \mathrm{Br}(S)$.
(ii) $[N_{T/S}(\mathcal{A}|_T)] = d[\mathcal{A}] \in \mathrm{Br}(S)$.

Proof. The diagram of long exact cohomology contains the following,

$$\begin{array}{ccc}
H^1(S, \mathbf{PGL}_r) & \longrightarrow & H^2_{\mathrm{fppf}}(S, \mathbb{G}_m) = \mathrm{Br}(S) \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
H^1(S, f_*(\mathbf{PGL}_r|_T)) & \longrightarrow & H^2_{\mathrm{fppf}}(S, f_*(\mathbb{G}_m|_T)) = \mathrm{Br}(T) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\
H^1(S, \mathbf{PGL}(N_{T/S}(\mathcal{O}|_T^r))) & \longrightarrow & H^2_{\mathrm{fppf}}(S, \mathbb{G}_m) = \mathrm{Br}(S)
\end{array}$$

where the horizontal maps are the natural boundary morphisms taking an isomorphism class of an algebra to its Brauer class.

By Lemma 1.18 and Lemma 3.12, the maps on first cohomology induced by the group homomorphisms

$$\mathbf{PGL}_r \xrightarrow{\mathrm{res}} f_*(\mathbf{PGL}_r|_T) \hookrightarrow f_*(\mathbf{PGL}_r|_T) \rtimes \mathcal{A}ut_S(T) \xrightarrow{\phi'} \mathbf{PGL}(N_{T/S}(\mathcal{O}|_T^r))$$

are the same as the maps induced by the functors

$$\mathfrak{Azu}_r \xrightarrow{\mathrm{res}} T\text{-}\mathfrak{Azu}_r \xrightarrow{\mathrm{inc}} \mathfrak{Azu}_r^{d-\text{ét}} \xrightarrow{N_{\mathfrak{Azu}}} \mathfrak{Azu}_{r^d}.$$

Therefore, tracing the image of the isomorphism class $[\mathcal{B}] \in H^1(T, \mathbf{PGL}_r|_T)$ $= H^1(S, f_*(\mathbf{PGL}_r|_T))$, we obtain

$$\begin{array}{ccc}
[\mathcal{B}] & \longmapsto & [\mathcal{B}] \in \mathrm{Br}(T) \\
\downarrow & & \downarrow \\
[N_{T/S}(\mathcal{B})] & \longmapsto & [N_{T/S}(\mathcal{B})] = \mathrm{tr}([\mathcal{B}]) \in \mathrm{Br}(S)
\end{array}$$

justifying claim (i). Similarily, for $[\mathcal{A}] \in H^1(S, \mathbf{PGL}_r)$, we can chase it through the diagram as follows

$$\begin{array}{ccc}
[\mathcal{A}] & \longmapsto & [\mathcal{A}] \in \mathrm{Br}(S) \\
\downarrow & & \downarrow \\
[\mathcal{A}|_T] & \longmapsto & [\mathcal{A}|_T] \in \mathrm{Br}(T) \\
\downarrow & & \downarrow \\
[N_{T/S}(\mathcal{A}|_T)] & \longmapsto & [N_{T/S}(\mathcal{A}|_T)] = d[\mathcal{A}] \in \mathrm{Br}(S)
\end{array}$$

where the factor of d appears by (3.9.1). This justifies claim (ii). □

4. An Equivalence $A_1^2 \equiv D_2$

In this section we show that the norm functor provides an equivalence between the following two stacks. First, $\mathfrak{A}_1^2 = \mathfrak{Azu}_2^{2-\text{ét}}$ is the gerbe of quaternion algebras over degree 2 étale extensions, see (iv) at the start of Section 3. Second, $\mathfrak{D}_2$ is the stack whose

(i) objects are pairs $(X, (\mathcal{A}, \sigma, f))$ where $X \in \mathfrak{Sch}_S$ and $(\mathcal{A}, \sigma, f)$ is a quadratic triple with $\mathcal{A}$ a degree 4 Azumaya $\mathcal{O}|_X$–algebra,
(ii) morphisms are pairs $(g, \varphi)\colon (Y, (\mathcal{A}_1, \sigma_1, f_1)) \to (X, (\mathcal{A}_2, \sigma_2, f_2))$ where $g\colon Y \to X$ is an S–scheme morphism and the map $\varphi\colon (\mathcal{A}_1, \sigma_1, f_1) \to g^*(\mathcal{A}_2, \sigma_2, f_2)$ is an isomorphism of quadratic triples over Y, and
(iii) structure functor sends $(X, (\mathcal{A}, \sigma, f)) \mapsto X$ and $(g, \varphi) \mapsto g$.

The stack $\mathfrak{D}_2$ is also a gerbe since we know by [GNR, 4.6] that all quadratic triples are isomorphic étale locally and thus also fppf locally.

4.1. **A Quadratic Triple over $\mathbb{Z}$.** As preparation, we begin by constructing a quadratic triple over the integers from a tensor product of symplectic involutions. Let $n_1, \dots, n_d$ be an even number of positive integers (so d is even), and let n be the integer such that $2n = (2n_1) \dots (2n_d)$. We then have an isomorphism of $\mathbb{Z}$–algebras

$$\mathrm{M}_{2n_1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathrm{M}_{2n_d}(\mathbb{Z}) \xrightarrow{\sim} \mathrm{M}_{2n}(\mathbb{Z})$$

given by the tensor product of matrices. On each $\mathrm{M}_{2n_i}(\mathbb{Z})$, we consider the standard symplectic involution σ_{n_i} defined by

$$\sigma_{n_i}(B) = J_{n_i}^{-1} B^T J_{n_i} = -J_{n_i} B^T J_{n_i} \text{ with } J_{n_i} = \begin{bmatrix} 0 & -I_{n_i} \\ I_{n_i} & 0 \end{bmatrix}.$$

The involution σ_{n_i} is adjoint to the skew-symmetric bilinear form ψ_{n_i} on $\mathbb{Z}^{2n_i}$ defined by $\psi_{n_i}(v, v') = v^T J_{n_i} v'$, considering v and v' as columns. The tensor product $\sigma = \sigma_{n_1} \otimes \cdots \otimes \sigma_{n_d}$ of these involutions is then an orthogonal involution on $\mathrm{M}_{2n}(\mathbb{Z})$. Precisely, it is adjoint to the regular symmetric bilinear form b on $\mathbb{Z}^{2n} = \mathbb{Z}^{2n_1} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^{2n_d}$ defined by

$$b(v_1 \otimes \cdots \otimes v_d, v_1' \otimes \cdots \otimes v_d') = \psi_{n_1}(v_1, v_1') \dots \psi_{n_d}(v_d, v_d').$$

Since each ψ_{n_i} is symplectic, if $v = v_1 \otimes \cdots \otimes v_d$ is a pure tensor, then $b(v,v) = 0$. Writing a general vector $w = w_1 + \cdots + w_k$ as a sum of pure tensors, this means that

$$b(w,w) = \sum_{i=1}^{k} b(w_i,w_i) + \sum_{\substack{i,j=1\\ i<j}}^{k} (b(w_i,w_j) + b(w_j,w_i)) = \sum_{\substack{i,j=1\\ i<j}}^{k} 2b(w_i,w_j),$$

i.e., $b(w,w) \in 2\mathbb{Z}$. Therefore, we may define a quadratic $\mathbb{Z}$–form $q(x) = \frac{1}{2}b(x,x)$ whose polar will be b. The form q is regular since b is and therefore it has an adjoint involution $\sigma_q = \sigma$. It follows from [GNR, 4.4(i)] or [CF, 2.7.0.31] that σ_q is part of a quadratic pair (σ_q, f_q). Since $\frac{1}{2} \in \mathbb{Q}^\times$, after extension to $\mathbb{Q}$ we must have that

$$f_q(s) = \frac{1}{2}\operatorname{Tr}(s) \tag{4.1.1}$$

and hence this also holds for each $s \in \operatorname{Sym}(\mathrm{M}_{2n}(\mathbb{Z}), \sigma_q)$. Thus, the quadratic pair (σ_q, f_q) is unique and therefore q is unique also.

4.2. **Remark.** The involution σ_q is isomorphic over $\mathbb{Z}$ to the split involution η_0 of [GNR, 4.5 (b)] and so for uniqueness reasons as in [GNR, 4.3(b)], the isomorphism is also one of quadratic pairs, i.e., $(\sigma_q, f_q) \cong (\eta_0, f_0)$. Further, when $d = 2$ and $\sigma_q = \sigma_{n_1} \otimes \sigma_{n_2}$, there is a quadratic pair $(\sigma_q, f_\otimes)$ on $\mathrm{M}_4(\mathbb{Z})$ arising from the construction in [GNR, 5.6]. Uniqueness also implies that $f_q = f_\otimes$.

4.3. **Restricting the Segre Homomorphism.** We consider the orthogonal groups reviewed in Section 1.6 with respect to the quadratic form and quadratic triple defined in Section 4.1 above. In particular, for this subsection we will consider them over the base scheme $\operatorname{Spec}(\mathbb{Z})$. The geometric fibers of $\mathbf{O}_q^+$ are split semisimple groups of type D_n by [KMRT, 25.12] because $n \geq 2$, so $\mathbf{O}_q^+$ is a semisimple $\mathbb{Z}$-group scheme of type D_n. Since $\mathbf{O}_q^+|_{\mathbb{Q}}$ is a split semisimple group, $\mathbf{O}_q^+$ is a Chevalley $\mathbb{Z}$–group scheme in view of the uniqueness of integral models, as in [Con2, 1.4].

We also consider the symplectic groups $\mathbf{Sp}_{2n_i} = \mathbf{Sp}_{\mathrm{M}_{2n_i}(\mathbb{Z}),\sigma_{n_i}}$ and $\mathbf{PSp}_{2n_i} = \mathit{Aut}(\mathrm{M}_{2n_i}(\mathbb{Z}), \sigma_{n_i})$ associated to the symplectic involutions defined in Section 4.1. The group $\mathbf{Sp}_{2n_i}$ is isomorphic to the symplectic group of the alternating form ψ_{n_i} and so by [KMRT, 25.11], the geometric fibers of $\mathbf{Sp}_{2n_i}$ are split semisimple and simply connected groups of type C_{n_i}. Again, $\mathbf{Sp}_{2n_i}|_{\mathbb{Q}}$ is a split semisimple group, so $\mathbf{Sp}_{2n_i}$ is a Chevalley $\mathbb{Z}$–group scheme.

4.4. **Lemma.** *Consider the Segre homomorphism* (3.4.1) *and its extension* Seg' *of* (3.4.2) *as well as the orthogonal and symplectic groups as reviewed above.*

(i) *The mapping* $\mathrm{Seg}\colon \mathbf{GL}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{GL}_{2n_d} \to \mathbf{GL}_{2n}$ *induces a closed immersion of* $\mathbb{Z}$*–group schemes*

$$h\colon \left(\mathbf{Sp}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{Sp}_{2n_d}\right)/(\boldsymbol{\mu}_2)^{d,0} \to \mathbf{O}_q^+$$

where $(\boldsymbol{\mu}_2)^{d,0} = \ker\big((\boldsymbol{\mu}_2)^d \xrightarrow{\Pi} \boldsymbol{\mu}_2\big)$ *and* Π *is the product map.*

(ii) *If* $n_1 = \cdots = n_d = m$ *(so that* $2n = (2m)^d$*), then* h *extends to a closed immersion of* $\mathbb{Z}$*–group schemes*

$$\widetilde{h}\colon \Big(\big(\mathbf{Sp}_{2m}\big)^d/(\boldsymbol{\mu}_2)^{d,0}\Big) \rtimes_{\mathbb{Z}} \mathbb{S}_d \to \mathbf{O}_q$$

where the permutation groups acts as in (3.4.2)*. Furthermore, recalling the Dickson homomorphism from* (1.6.1)*, the composition map*

$$\mathbb{S}_d \to \mathbf{O}_q \xrightarrow{\text{Dickson}} \mathbb{Z}/2\mathbb{Z}$$

is the signature homomorphism if m *is odd and is trivial if* m *is even.*

(iii) *If* $m = 1$ *and* $d = 2$*, then* $\widetilde{h}$ *is an isomorphism.*

Proof. (i): The Segre mapping, $\mathrm{Seg}\colon \mathbf{GL}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{GL}_{2n_d} \to \mathbf{GL}_{2n}$, induces a homomorphism of $\mathbb{Z}$–group schemes

$$h'\colon \mathbf{Sp}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{Sp}_{2n_d} \to \mathbf{O}_q.$$

Since the symplectic groups have connected geometric fibers and $\mathbf{O}_q^+$ is the identity component of $\mathbf{O}_q$, the map h' factors through $\mathbf{O}_q^+$. The kernel of h' is the intersection of $\mathbf{Sp}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{Sp}_{2n_d}$ with the kernel of the Segre mapping, which is $\ker(\mathbb{G}_m^d \xrightarrow{\Pi} \mathbb{G}_m)$. It follows that $\ker(h') = (\boldsymbol{\mu}_2)^{d,0}$. According to [SGA3, VIII.5], we can quotient out by the diagonalizable $\mathbb{Z}$–group $(\boldsymbol{\mu}_2)^{d,0}$ and get a monomorphism

$$h\colon \Big(\mathbf{Sp}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{Sp}_{2n_d}\Big)/(\boldsymbol{\mu}_2)^{d,0} \to \mathbf{O}_q^+.$$

This is a closed immersion according to [Con1, 5.3.5].

(ii): This follows from the fact that the construction of (σ_q, f_q) is equivariant with respect to the action of the symmetric group $\mathbb{S}_d$. It remains to deal with the composition map $\mathbb{S}_d \to \mathbf{O}_q \to \mathbb{Z}/2\mathbb{Z}$. It is enough to check it over the $\mathbb{Q}$-points and, in this case, the Dickson map $\mathbf{O}_q(\mathbb{Q}) \to (\mathbb{Z}/2\mathbb{Z})(\mathbb{Q}) \cong \boldsymbol{\mu}_2(\mathbb{Q})$ is nothing but the determinant by [Knu, IV.5.1.2]. To prove our claim, it is then enough to compute the image of the transposition (1 2) by the morphism $j\colon \mathbb{S}_d \to \mathbb{S}_{(2m)^d}$. Without loss of generality, we can assume that $d = 2$ so that $2n = (2m)^2$ and

$$j((1\ 2)) = \prod_{1 \le i < j \le 2m} ((i,j)\ (j,i)),$$

which is a product of $m(2m-1)$ transpositions. It follows that we have $\det\Big(\mathrm{Seg}'((1\ 2))\Big) = 1$ if and only m is even, which justifies the claim.

(iii): Since $(\mathbf{Sp}_2)^2/\boldsymbol{\mu}_2$ is smooth according to [SGA3, VI_B.9.2], it is both flat and locally of finite presentation. Since $\mathbf{O}_q^+$ is also smooth, the fiberwise isomorphism criterion of [EGA, IV_4.17.9.5] allows us to reduce to the case of an algebraically closed field k. The map $(\mathbf{Sp}_{2,k})^2/\boldsymbol{\mu}_2 \to (\mathbf{O}_q^+)_k$ is a closed

embedding between two smooth connected algebraic groups of the same dimension (i.e., 6), which is an isomorphism according to [GW, cor. 5.8]. Using that the composition $\mathbb{Z}/2\mathbb{Z} \to \mathbf{O}_{q,k} \to \mathbb{Z}/2\mathbb{Z}$ is an isomorphism, we conclude that $\widetilde{h}_k$ is an isomorphism. □

4.5. **Remark.** Another way to see that the map $\widetilde{h}$ in Lemma 4.4(iii) is an isomorphism is to use the map $f\colon \mathbf{SL}_2 \times_{\mathbb{Z}} \mathbf{SL}_2 \to \mathbf{O}^+_{\det}$ defined by $f(B_1, B_2)(A) = B_1 A B_2^{-1}$. Here $\det\colon \mathrm{M}_2(\mathbb{Z}) \to \mathbb{Z}$ is a quadratic form where we view $\mathrm{M}_2(\mathbb{Z})$ simply as a rank 4 $\mathbb{Z}$–module. See [Con1, C.6.3].

To get maps of adjoint groups, we can quotient out the maps of Lemma 4.4 by the center $\boldsymbol{\mu}_2$ of each $\mathbf{Sp}_{2n_i}$ on the left and the center $\boldsymbol{\mu}_2$ of $\mathbf{O}^+_q$ on the right. This yields a closed immersion

$$\underline{h}\colon \mathbf{PSp}_{2n_1} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbf{PSp}_{2n_d} \to \mathbf{PGO}^+_q. \tag{4.5.1}$$

In the second case, we get a closed immersion

$$\underline{h}'\colon (\mathbf{PSp}_{2m})^d \rtimes_{\mathbb{Z}} \mathbb{S}_d \to \mathbf{PGO}_q, \tag{4.5.2}$$

which is the restriction of the map PSeg' of (3.6.1). In particular, if $m = 1$ and $d = 2$, the group homomorphism $\underline{h}'$ is an isomorphism since the map $\tilde{h}$ of Lemma 4.4(iii) is an isomorphism in this case.

4.6. **Twisting Quadratic Triples.** We now use the morphism $\underline{h}$ of (4.5.1) to define a morphism of stacks. Let the numbers d and $n_1, \ldots, n_d$ be as in the beginning of Section 4.1. The first stack will be denoted $\mathfrak{C}_{(n_1,\ldots,n_d)}$ and will have

(i) objects $(X, (\mathcal{A}_1, \sigma_1), \ldots, (\mathcal{A}_d, \sigma_d))$ where $X \in \mathfrak{Sch}_S$ and $(\mathcal{A}_i, \sigma_i)$ is an Azumaya $\mathcal{O}|_X$–algebra of degree $2n_i$ with symplectic involution,

(ii) morphisms

$$(g, \varphi_1, \ldots, \varphi_d)\colon (Y, (\mathcal{A}_{1,i}, \sigma_{1,i})_{i=1}^d) \to (X, (\mathcal{A}_{2,i}, \sigma_{2,i})_{i=1}^d)$$

where $g\colon Y \to X$ is an S–scheme morphism and $\varphi_i\colon (\mathcal{A}_{1,i}, \sigma_{1,i}) \xrightarrow{\sim} (\mathcal{A}_{2,i}, \sigma_{2,i})|_Y$ is an isomorphism of Azumaya $\mathcal{O}|_Y$–algebras with involution, and

(iii) structure functor which sends $(X, (\mathcal{A}_1, \sigma_1), \ldots, (\mathcal{A}_d, \sigma_d)) \mapsto X$ and $(g, \varphi_1, \ldots, \varphi_d) \mapsto g$.

It is clear $\mathfrak{C}_{(n_1,\ldots,n_d)}$ is a gerbe. We take the split object to be $\mathcal{M} = (S, (\mathrm{M}_{2n_1}(\mathcal{O}), \sigma_{n_1}), \ldots, (\mathrm{M}_{2n_d}(\mathcal{O}), \sigma_{n_d}))$ and so $\mathcal{A}ut(\mathcal{M}) = \mathbf{PSp}_{2n_1} \times \ldots \times \mathbf{PSp}_{2n_d}$. Hence, $\mathfrak{C}_{(n_1,\ldots,n_d)}$ is equivalent to the stack of $(\mathbf{PSp}_{2n_1} \times \ldots \mathbf{PSp}_{2n_d})$–torsors by Lemma 1.16(ii) and Remark 1.17.

The second stack is $\mathfrak{D}_n$, for $n \geq 2$, which consists of

(i) objects $(X, (\mathcal{A}, \sigma, f))$ where $X \in \mathfrak{Sch}_S$ and $(\mathcal{A}, \sigma, f)$ is an Azumaya $\mathcal{O}|_X$–algebra of degree $2n$ with quadratic pair,

(ii) morphisms (g, φ) where $g\colon Y \to X$ is an S–scheme morphism and $\varphi\colon (\mathcal{A}_1, \sigma_1, f_1) \to (\mathcal{A}_2, \sigma_2, f_2)|_Y$ is an isomorphism of quadratic triples over $\mathcal{O}|_Y$, and

(iii) structure functor $(X,(\mathcal{A},\sigma,f))\mapsto X$ and $(g,\varphi)\mapsto g$.

This is also a gerbe. We take $(S,(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q))$ to be the split object where $(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)$ is the quadratic triple defined in (4.1.1) base changed to S. Then, $\mathcal{A}ut(S,(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q))=\mathbf{PGO}_q$ and so $\mathfrak{D}_n$ is equivalent to the stack of $\mathbf{PGO}_q$–torsors by Proposition 1.16(ii) and Remark 1.17. In [CF, 2.7.0.30], the stack $\mathfrak{D}_n$ is denoted as $\mathcal{P}aires\mathcal{Q}uad_{2n}$.

We use the group homomorphism

$$\psi\colon \mathbf{PSp}_{2n_1}\times\ldots\times\mathbf{PSp}_{2n_d}\xrightarrow{\overline{h}}\mathbf{PGO}_q^+\hookrightarrow\mathbf{PGO}_q \tag{4.6.1}$$

to give a left action of $(\mathbf{PSp}_{2n_1}\times\ldots\times\mathbf{PSp}_{2n_d})|_X$ to $(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)|_X$ for all $X\in\mathfrak{Sch}_S$. This allows us to define a stack morphism as follows. For an object $C=(X,(\mathcal{A}_1,\sigma_1),\ldots,(\mathcal{A}_d,\sigma_d))\in\mathfrak{C}_{(n_1,\ldots,n_d)}$, the sheaf $\mathcal{E}=\mathcal{I}som(\mathcal{M}|_X,C)$ is an $\mathcal{A}ut(\mathcal{M}|_X)=\mathbf{PSp}_{2n_1}|_X\times\ldots\times\mathbf{PSp}_{2n_d}|_X$–torsor by Proposition 1.16(i). Since ψ maps $\mathcal{A}ut(\mathcal{M}|_X)$ into $\mathbf{PGO}_q|_X$, the contracted product $\mathcal{E}\wedge^{\mathcal{A}ut(\mathcal{M}|_X)}(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)|_X$ will be another quadratic triple over $\mathcal{O}|_X$. Then, defining

$$\begin{aligned}\Psi\colon \mathfrak{C}_{(n_1,\ldots,n_d)}&\to\mathfrak{D}_n\\ (X,(\mathcal{A}_1,\sigma_1),\ldots,(\mathcal{A}_d,\sigma_d))&\mapsto(X,\mathcal{E}\wedge^{\mathcal{A}ut(\mathcal{M}|_X)}(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)|_X)\end{aligned} \tag{4.6.2}$$

and naturally on morphisms yields a functor which is a morphism of stacks because $\mathfrak{D}_n$ is a gerbe.

4.7. **Lemma.** *We use numbers d and $n_1,\ldots,n_d$ as in the beginning of Section 4.1 and consider the morphism Ψ of (4.6.2) defined above. Let $(\mathcal{A}_i,\sigma_{\mathcal{A}_i})$ for $i=1,\ldots,d$ be Azumaya $\mathcal{O}|_X$–algebras of respective degree $2n_i$ equipped with symplectic involutions.*

(i) *The induced group sheaf homomorphism*

$$\Psi_{\mathcal{M}}\colon \mathbf{PSp}_{2n_1}\times\ldots\times\mathbf{PSp}_{2n_d}\to\mathbf{PGO}_q$$

is the map ψ of (4.6.1).

(ii) *The image $\Psi(X,(\mathcal{A}_1,\sigma_1),\ldots,(\mathcal{A}_d,\sigma_d))$ is a quadratic triple of the form $(\mathcal{A}_1\otimes_{\mathcal{O}|_X}\cdots\otimes_{\mathcal{O}|_X}\mathcal{A}_d,\sigma_{\mathcal{A}_1}\otimes\cdots\otimes\sigma_{\mathcal{A}_d},f)$.*

(iii) *If $d=2$, this quadratic triple this agrees with the one constructed in* [GNR, 5.6].

Proof. (i): When we consider $\Psi(S,\mathcal{M})$, the relevant torsor $\mathcal{E}$ is the trivial torsor $\mathcal{I}som(\mathcal{M},\mathcal{M})=\mathcal{A}ut(\mathcal{M})$. Therefore,

$$\Psi(S,\mathcal{M})=\mathcal{A}ut(\mathcal{M})\wedge^{\mathcal{A}ut(\mathcal{M})}(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)\cong(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)$$

and so $\Psi_{\mathcal{M}}$ maps into $\mathcal{A}ut(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)=\mathbf{PGO}_q$ as claimed. Given an automorphism $\varphi\in\mathcal{A}ut(\mathcal{M})$, its image under Ψ is the map which acts on the presheaf underlying $\mathcal{A}ut(\mathcal{M})\wedge^{\mathcal{A}ut(\mathcal{M})}(\mathrm{M}_{2n}(\mathcal{O}),\sigma_q,f_q)$ by

$$(\rho,B)\mapsto(\varphi\circ\rho,B).$$

for $\rho \in \mathcal{A}ut(\mathcal{M})$ and $B \in (\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)$. However, using the equivalence relation defining the contracted product, this is equivalently the map

$$\big(\mathrm{Id}, \psi(\rho)(B)\big) \mapsto \big(\mathrm{Id}, \psi(\varphi)(\psi(\rho)(B))\big),$$

i.e., it is the map $\psi(\varphi) \in \mathcal{A}ut(\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)$. This justifies the claim.
(ii): By definition, $\Psi(X, (\mathcal{A}_1, \sigma_1), \dots, (\mathcal{A}_d, \sigma_d))$ is a quadratic triple which we denote $(\mathcal{A}, \sigma, f)$. For functoriality reasons, since ψ is a restriction of Seg, $(\mathcal{A}, \sigma)$ must be the tensor product of the twists of each $(\mathrm{M}_{2n_i}(\mathcal{O}|_X), \sigma_i)$ by the $\mathbf{PSp}_{2n_i}|_X$-torsors $\mathcal{I}som\big((\mathrm{M}_{2n_i}(\mathcal{O}|_X), \sigma_{n_i}), (\mathcal{A}_i, \sigma_{\mathcal{A}_i})\big)$, so

$$(\mathcal{A}, \sigma, f) = (\mathcal{A}_1 \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{A}_d, \sigma_{\mathcal{A}_1} \otimes \cdots \otimes \sigma_{\mathcal{A}_d}, f).$$

(iii): In the proof, we denote by $(\sigma_{n_1} \otimes \sigma_{n_2}, f_\otimes)$ the canonical quadratic pair on $\mathcal{A}_1 \otimes_{\mathcal{O}|_X} \mathcal{A}_2$ defined in [GNR, 5.6]. We want to show that $f = f_\otimes$. Since both constructions commute with arbitrary base change, it suffices to check this in the split case, which has been established in Remark 4.2. □

As in Lemma 4.4(ii), we now consider the case $2n = (2m)^d$ and globalize the constructions of [KMRT, §15.B] into a morphism of stacks. We consider the stack $\mathfrak{C}_m^{d-\text{ét}}$, which is equivalent to the stack $\mathfrak{F}(\mathbf{PSp}_{2m})^{d-\text{ét}}$ of Appendix B, but using Remark B.7 we consider it to have objects $(X' \to X, (\mathcal{A}, \sigma))$ where $(\mathcal{A}, \sigma)$ is an Azumaya $\mathcal{O}|_T$–algebra of degree $2m$ with symplectic involution. This is equivalent to the stack of $(\mathbf{PSp}_{2m}^d) \rtimes \mathbb{S}_d$–torsors and we take the split object to be $\mathcal{M}' = (S^{\sqcup d} \to S, (\mathrm{M}_{2m}(\mathcal{O}), \sigma_m)|_{S^{\sqcup d}})$, which has automorphism sheaf $(\mathbf{PSp}_{2m}^d) \rtimes \mathbb{S}_d$. For an object $(X' \to X, (\mathcal{A}, \sigma)) \in \mathfrak{C}_m^{d-\text{ét}}$, denote by

$$\mathcal{E}' = \mathcal{I}som(\mathcal{M}'|_X, (X' \to X, (\mathcal{A}, \sigma)))$$

the corresponding $((\mathbf{PSp}_{2m}^d) \rtimes \mathbb{S}_d)|_X$–torsor. Using the morphism $\underline{h}'$ of (4.5.2), we can define a morphism of stacks

$$\Psi' \colon \mathfrak{C}_m^{d-\text{ét}} \to \mathfrak{D}_n \tag{4.7.1}$$
$$(X' \to X, (\mathcal{A}, \sigma)) \mapsto (X, \mathcal{E}' \wedge^{\mathcal{A}ut(\mathcal{M}')|_X} (\mathrm{M}_{2n}(\mathcal{O}), \sigma_q, f_q)|_X)).$$

4.8. **Proposition.** *Let $m \in \mathbb{Z}$ and let $n \in \mathbb{Z}$ such that $2n = (2m)^d$. Consider the morphism Ψ' of (4.7.1) defined above.*

(i) *The induced group sheaf homomorphism*

$$\Psi'_{\mathcal{M}'} \colon (\mathbf{PSp}_{2m})^d \rtimes \mathbb{S}_d \to \mathbf{PGO}_q$$

is $\underline{h}'$.

(ii) *The image of an object $(X' \to X, (\mathcal{A}, \sigma))$ is a quadratic triple of the form*

$$\Psi'(X' \to X, (\mathcal{A}, \sigma)) \cong (N_{X'/X}(\mathcal{A}), \sigma_N, f_N).$$

Proof. (i): This follows by an analogous argument as in the proof of Lemma 4.7(i).

(ii): By construction, we know that $\Psi'(X' \to X, (\mathcal{A}, \sigma)) = (X, (\mathcal{A}', \sigma_N, f_N))$ is a quadratic triple for some Azumaya $\mathcal{O}|_X$–algebra $\mathcal{A}'$. Since $\underline{h}'$ is a restriction of $\underline{\mathrm{Seg}}'$, for functoriality reasons we know from Corollary 3.8 that $\mathcal{A}' \cong N_{X'/X}(\mathcal{A})$ as Azumaya $\mathcal{O}|_X$–algebras. □

In the special case when $\underline{h}'$ is an isomorphism, we obtain an equivalence of stacks by Theorem 1.19. We compose this with the equivalence of stacks

$$\begin{aligned} \rho\colon \mathfrak{A}_1^2 &\to \mathfrak{C}_2^{2-\text{ét}} \\ (X' \to X, \mathcal{A}) &\mapsto (X' \to X, (\mathcal{A}, \sigma_{\mathcal{A}})), \end{aligned}$$

where $\mathfrak{A}_1^2$ is the stack of quaternion algebras over a degree 2 étale extension as defined at the beginning of Section 4, that equips a quaternion algebra $\mathcal{A}$ with its canonical symplectic involution $\sigma_{\mathcal{A}}$.

4.9. **Theorem.** *Assume that $m = 1$ and $d = 2$. The morphism*

$$\begin{array}{ccccc} \mathfrak{A}_1^2 & \xrightarrow{\ \rho\ } & \mathfrak{C}_2^{2-\text{ét}} & \xrightarrow{\ \Psi'\ } & \mathfrak{D}_2 \\ (X' \to X, \mathcal{A}) & \longmapsto & & \longmapsto & (X, (N_{X'/X}(\mathcal{A}), \sigma_N, f_N)) \end{array}$$

is an equivalence of gerbes. Furthermore, those stacks are equivalent to the following stacks over $\mathfrak{Sch}_S$.

(i) *The stack of $(\mathbf{PGL}_2 \times \mathbf{PGL}_2) \rtimes \mathbb{Z}/2\mathbb{Z}$-torsors.*
(ii) *The stack of $\mathbf{PGO}_q \cong \mathbf{PGO}_{\mathrm{M}_4(\mathcal{O}),\sigma_q,f_q}$-torsors.*
(iii) *The stack of adjoint semisimple group schemes of type $A_1 \times A_1$.*
(iv) *The stack of adjoint semisimple group schemes of type D_2.*

Proof. As noted above, by Theorem 1.19, Ψ' is an equivalence of stacks because $\underline{h}'$ of (4.5.2) is an isomorphism. The morphism ρ is the canonical equivalence and so their composition is an equivalence as well.

All semisimple group schemes of type $A_1 \times A_1$ are twisted forms of the split adjoint Chevalley group scheme of the same type, namely $\mathbf{PGL}_2 \times \mathbf{PGL}_2$, which has automorphism group $(\mathbf{PGL}_2 \times \mathbf{PGL}_2) \rtimes \mathbb{Z}/2\mathbb{Z}$. This provides the equivalence between (i) and (iii) since both are gerbes. In turn, the category $\mathfrak{A}_1^2$ is equivalent to (i) by Corollary B.4.

Similarly, the adjoint semisimple groups of type D_2 are twisted forms of the split adjoint Chevalley group of type D_2, which is $\mathbf{PGO}_q^+$, and its automorphism group is $\mathbf{PGO}_q$. As above, using again Theorem 1.19, this provides the equivalence between (ii), (iv), and $\mathfrak{D}_2$. □

4.10. **Remark.** The equivalence (iii) ⇔ (iv) in Theorem 4.9 implies that there is an isomorphism of S–group schemes $(\mathbf{SL}_2 \times \mathbf{SL}_2)/M \cong \mathbf{O}_4^+$ where M is the diagonal copy of $\boldsymbol{\mu}_2$. Such an isomorphism is constructed in [Con1, C.6.3].

4.11. **Remark.** Let $S = \mathrm{Spec}(\mathbb{Z})$ and let $\mathbb{F}$ be a field. Then, the fiber over $\mathrm{Spec}(\mathbb{F})$ of the gerbes $\mathfrak{A}_1^2$ and $\mathfrak{D}_2$ are the groupoids A_1^2 and D_2 of [KMRT,

§15] and the morphism between these fibers is the functor of [KMRT, §15.B]. Since any equivalence of gerbes gives rise to an equivalence of the fibres, Theorem 4.9 gives a proof of [KMRT, 15.7] which is different from the one in loc. cit. The analogous remark applies to Auel's result [A, 3.1], where it is assumed that 2 is invertible over S.

Appendix A. Twisted Sheaves and Weil Restriction

Here we prove a general lemma about composing the pullback, i.e., restriction, and pushforward of a sheaf on $\mathfrak{Sch}_S$ with respect to a finite étale cover $T \to S$ of degree d using contracted products as in Section 1.7. As a corollary, this describes the Weil restriction, when it exists, of $Y \times_S T$ for an S–scheme Y. We refer to [BLR, §7.6] for details on the Weil restriction.

Since T is a degree d étale cover, it is a twisted form of $S^{\sqcup d} = \sqcup_{i=1}^{d} S$. In particular, the sheaf

$$\begin{aligned} \mathit{Isom}(S^{\sqcup d}, T)\colon \mathfrak{Sch}_S &\to \mathfrak{Sets} \\ X &\mapsto \mathrm{Isom}_X(X^{\sqcup d}, T \times_S X) \end{aligned}$$

is a torsor for the group $\mathcal{A}ut(S^{\sqcup d}) = \mathbb{S}_d$.

A.1. **Lemma.** *Let $f\colon T \to S$ be a degree d étale cover. Let $\mathcal{F}\colon \mathfrak{Sch}_S \to \mathfrak{Sets}$ be any sheaf and equip $\mathcal{F}^d$ with the left action of $\mathbb{S}_d$ by permutations. Then, there is a canonical isomorphism of sheaves of sets*

$$\phi\colon \mathit{Isom}(S^{\sqcup d}, T) \wedge^{\mathbb{S}_d} \mathcal{F}^d \xrightarrow{\sim} f_*(\mathcal{F}|_T).$$

Proof. The contracted product $\mathit{Isom}(S^{\sqcup d}, T) \wedge^{\mathbb{S}_d} \mathcal{F}^d$ is the sheaf associated with the presheaf on $\mathfrak{Sch}_S$

$$X \mapsto \big(\mathrm{Isom}(X^{\sqcup d}, T \times_S X) \times \mathcal{F}^d(X)\big)/\sim$$

where the equivalence relation is given by $(\varphi\sigma, x) \sim (\varphi, \sigma x)$ for all $\varphi \in \mathrm{Isom}(X^{\sqcup d}, T \times_S X)$, $x \in \mathcal{F}^d(X)$, and $\sigma \in \mathbb{S}_d(X)$. We show that there is an injection from this presheaf into $f_*(\mathcal{F}|_T)$ which is locally surjective, and therefore will induce the desired isomorphism of sheaves.

First, for any $\sigma \in \mathbb{S}_d(X)$, we have an isomorphism of schemes $\sigma_X\colon X^{\sqcup d} \to X^{\sqcup d}$ given by permutation. Since $\mathcal{F}$ is a sheaf, $\mathcal{F}(X^{\sqcup d}) \cong \mathcal{F}(X)^d$ and the permutation automorphism $\sigma\colon \mathcal{F}^d(X) \to \mathcal{F}^d(X)$ given by the action is the same as the morphism $\mathcal{F}(\sigma_X^{-1})$, where the inverse occurs since $\mathcal{F}$ is contravariant. Therefore, we view $\mathcal{F}^d(X)$ as $\mathcal{F}(X^{\sqcup d})$ where $\mathbb{S}^d$ acts on $X^{\sqcup d}$. Now, consider the canonical map of presheaves defined over X by

$$\begin{aligned} (\mathrm{Isom}(X^{\sqcup d}, T \times_S X) \times \mathcal{F}(X^{\sqcup d}))/\sim &\to \mathcal{F}(T \times_S X) = f_*(\mathcal{F}|_T)(X) \\ (\varphi, x) &\mapsto \mathcal{F}(\varphi^{-1})(x). \end{aligned}$$

If $\mathcal{F}(\varphi_1^{-1})(x_1) = \mathcal{F}(\varphi_2^{-1})(x_2)$, this means that

$$x_2 = (\mathcal{F}(\varphi_2) \circ \mathcal{F}(\varphi_1^{-1}))(x_1) = \mathcal{F}(\varphi_1^{-1}\varphi_2)(x_1),$$

but since $\varphi_1^{-1}\varphi_2 \in \mathbb{S}_d(X)$, this means that $x_2 = \varphi_2^{-1}\varphi_1 \cdot x_1$ using the action of $\mathbb{S}_d$ on $\mathcal{F}^d$. Therefore, under the relation $\sim$,

$$(\varphi_2, x_2) = (\varphi_2, \varphi_2^{-1}\varphi_1(x_1)) = (\varphi_2\varphi_2^{-1}\varphi_1, x_1) = (\varphi_1, x_1)$$

and so the map of presheaves is injective. If there exists and element $\varphi \in \mathrm{Isom}(X^{\sqcup d}, T \times_S X)$, i.e., $\mathit{Isom}(S^{\sqcup d}, T)(X) \neq \emptyset$, then for all $x \in \mathcal{F}(T \times_S X)$ we have

$$(\varphi, \mathcal{F}(\varphi)(x)) \mapsto x,$$

hence the map is surjective wherever $\mathit{Isom}(S^{\sqcup d}, T)$ has a point. Finally, since $\mathit{Isom}(S^{\sqcup d}, T)$ is a torsor, there is a cover of S over which it has points and so the map of presheaves is locally surjective. Hence, it is an isomorphism of sheaves as claimed. □

A.2. **Remark.** Lemma A.1 produces an isomorphism of sheaves of groups, abelian groups, rings, etc. whenever $\mathcal{F}$ has such a structure and the structure on $\mathcal{F}^d$ is given by component wise operations in $\mathcal{F}$.

The torsor $\mathit{Isom}(S^{\sqcup d}, T)$ corresponds to a cohomology class $[T] \in H^1(S, \mathbb{S}_d)$. Under the group homomorphism $\mathbb{S}_d \to \mathit{Aut}(\mathcal{F}^d)$ induced by the action of $\mathbb{S}_d$ on $\mathcal{F}^d$, the sheaf $\mathit{Isom}(S^{\sqcup d}, T) \wedge^{\mathbb{S}_d} \mathcal{F}^d$ corresponds to the image of $[T]$ via the map

$$H^1(S, \mathbb{S}_d) \to H^1(S, \mathit{Aut}(\mathcal{F}^d)).$$

Since $\mathbb{S}_d$ is an affine group scheme, by [M, §4, 4.3(a)] the torsor $\mathit{Isom}(S^{\sqcup d}, T)$ is representable by an S–scheme $\widetilde{T} \to S$ which is therefore also an $\mathbb{S}_d$–torsor. For any scheme Y which has a left action of $\mathbb{S}_d$, the associated sheaf $\mathrm{Hom}_S(\_\_, Y)$ also has a left action of $\mathbb{S}_d$. If the sheaf $\mathit{Isom}(S^{\sqcup d}, T) \wedge^{\mathbb{S}_d} \mathrm{Hom}_S(\_\_, Y)$ is representable, we denote the representing scheme by $\widetilde{T} \wedge_S^{\mathbb{S}_d} Y$ and call it the contracted product of schemes. For example, for $X \in \mathfrak{Sch}_S$ we have $\widetilde{T} \wedge_S^{\mathbb{S}_d} X^{\sqcup d} \cong X \times_S T$.

A.3. **Corollary.** *Let $Y \to S$ be an S–scheme such that the Weil restriction $R_{T/S}(Y \times_S T)$ exists as a scheme. Then there is an isomorphism*

$$\widetilde{T} \wedge_S^{\mathbb{S}_d} Y^d \xrightarrow{\sim} R_{T/S}(Y \times_S T),$$

where $\mathbb{S}_d$ permutes the factors of $Y^d = Y \times_S \ldots \times_S Y$.

Proof. We apply Lemma A.1 to $h_Y = \mathrm{Hom}_S(\_\_, Y)$. The sheaf h_Y^d is represented by the scheme Y^d. Applying the lemma we get an isomorphism

$$\mathit{Isom}(S^{\sqcup d}, T) \wedge^{\mathbb{S}_d} h_Y^d \xrightarrow{\sim} f_*(h_Y|_T).$$

We have $f_*(h_Y|_T) = \mathrm{Hom}_T(\_\_ \times_S T, Y \times_S T)$ and by definition the Weil restriction $R_{T/S}(Y \times_S T)$ exists if and only if this sheaf is representable, in which case it is represented by the Weil restriction. Hence by assumption these sheaves are representable, and so the left hand sheaf is represented by $\widetilde{T} \wedge_S^{\mathbb{S}_d} Y^d$. The claimed isomorphism then follows from the Yoneda Lemma. □

A.4. **Remark.** Since $T \to S$ is a finite étale cover, equivalently since $\widetilde{T} \to S$ is a Galois cover, the discussion at the end of [BLR, 6.2 B] says that a sufficient condition for $R_{T/S}(Y \times_S T)$ to exist is that the morphism $Y \times_S T \to T$ be quasi-projective. This will occur if $Y \to S$ is quasi-projective by [St, Tag 0B3G].

Appendix B. Cohomology of Semi-direct Products with Permutation Groups

Let $\mathbf{G}\colon \mathfrak{Sch}_S \to \mathfrak{Grp}$ be a sheaf of groups. Recall that $\mathbb{S}_d$ denotes the constant group sheaf associated to the abstract permutation group on d letters, and consider the semi-direct product $\mathbf{G}^d \rtimes \mathbb{S}_d$ defined by

$$((g_1, \dots, g_d), 1)(1, \sigma) = (1, \sigma)((g_{\sigma(1)}, \dots, g_{\sigma(d)}), 1)$$

for $\sigma \in \mathbb{S}_d$ and $(g_1, \dots, g_d) \in \mathbf{G}^d$. Our goal is to describe the gerbe $\mathfrak{Tors}(\mathbf{G}^d \rtimes \mathbb{S}_d)$ of $(\mathbf{G}^d \rtimes \mathbb{S}_d)$-torsors for the flat topology. We will show that it is equivalent to $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$, which we define to be the S–stack as follows.

(i) The objects are pairs $(T \to X, \mathcal{P})$ where $X \in \mathfrak{Sch}_S$, $T \to X$ is a finite étale cover of degree d, and $\mathcal{P}$ is a $\mathbf{G}|_T$-torsor over T.

(ii) The morphisms are triples $(f, g, \varphi)\colon (T' \to X', \mathcal{P}') \to (T \to X, \mathcal{P})$ where f and g are scheme morphisms such that

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

is a pullback diagram of schemes and $\varphi\colon \mathcal{P}' \xrightarrow{\sim} g^*(\mathcal{P})$ is a $\mathbf{G}|_{T'}$–torsor isomorphism. Composition of morphisms is given by

$$(f, g, \varphi) \circ (h, j, \psi) = (f \circ h, g \circ j, j^*(\varphi) \circ \psi).$$

(iii) The structure functor sends $(T \to X, \mathcal{P}) \mapsto X$ and $(f, g, \varphi) \mapsto f$.

It is clear that $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ is a fibered category over $\mathfrak{Sch}_S$. Since any pullback diagram as above where $f = \mathrm{Id}_X$ must have g be an isomorphism, we see that $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ is fibered in groupoids. However, it is perhaps not as clear that $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ is a stack. We argue this now, in fact showing that it is a gerbe.

B.1. **Lemma.** *The fibered category $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ defined above is a gerbe.*

Proof. To first see this is indeed a stack, we can view it as a composition of stacks. First, let $\mathfrak{A} \to \mathfrak{Sch}_S$ be the fibered category of étale cover of degree d, i.e., the objects are such étale morphisms $T \to X$ for some $X \in \mathfrak{Sch}_S$ and morphisms are pairs $(f, g)\colon (T' \to X') \to (T \to X)$ where f and g are

scheme morphisms making

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

a pullback diagram of schemes. This makes $\mathfrak{A}$ fibered in groupoids since if $f = \mathrm{Id}_X$, then g must be an isomorphism in order to have a pullback diagram. The homomorphism presheaves in $\mathfrak{A}$ are sheaves by [St, Tag 040L]. Finite étale morphisms of degree d are affine morphisms and they are characterized by a local condition, namely $T \to X$ is finite étale of degree d if and only if there exists a cover $\{X_i \to X\}_{i\in I}$ such that $T \times_X X_i \cong X_i^{\sqcup d}$ for each $i \in I$. Affine morphisms satisfy descent by [Ols, 4.4.7] and additionally [Vis, Prop 2.36] shows that the resulting glued morphism will be finite étale since it restricts to finite étale morphisms over an fppf cover (in fact, the result is stated more generally in [Vis] for an fpqc cover). Therefore, finite étale morphisms of degree d also satisfy descent. Thus, $\mathfrak{A}$ is a stack. Further, since finite étale morphisms of degree d are all locally isomorphic to $S^{\sqcup d} \to S$, we see $\mathfrak{A}$ is a gerbe.

The gerbe $\mathfrak{A}$ inherits the structure of a site from $\mathfrak{Sch}_S$ according to [St, Tag 06NU]. Since all morphisms in $\mathfrak{A}$ are cartesian by Lemma 1.10, the covers will be families of the form $\{(f_i, g_i)\colon (T_i \to X_i) \to (T \to X)\}_{i\in I}$ where $\{f_i\colon X_i \to X\}_{i\in I}$ is an fppf covering. Since the morphisms in $\mathfrak{A}$ define pullback diagrams, this means that there are isomorphisms $T_i \xrightarrow{\sim} T \times_X X_i$ and $\{g_i\colon T_i \to T\}_{i\in I}$ is also an fppf cover.

Now, we can view

$$\begin{aligned} \mathfrak{F}(\mathbf{G})^{d-\text{ét}} &\to \mathfrak{A} \\ (T \to X, \mathcal{P}) &\mapsto (T \to X) \\ (f, g, \varphi) &\mapsto (f, g) \end{aligned}$$

as a fibered category over $\mathfrak{A}$. This is clearly a stack since for two $\mathbf{G}|_T$–torsors $\mathcal{P}_1$ and $\mathcal{P}_2$ their sheaf of torsor isomorphisms respects fppf covers of the form $\{T_i \to T\}_{i\in I}$ and torsors allow gluing over such fppf covers.

Thus, we may invoke [St, Tag 09WX] and conclude that $p\colon \mathfrak{F}(\mathbf{G})^{d-\text{ét}} \to \mathfrak{Sch}_S$ is a stack fibered in groupoids. To see it is a gerbe, let $(T_1 \to X, \mathcal{P}_1)$ and $(T_2 \to X, \mathcal{P}_2)$ be two objects in the same fiber. We first choose a cover $\{X_i \to X\}_{i\in I}$ which splits both étale covers, that is $T_1 \times_X X_i \cong X_i^{\sqcup d}$ and $T_2 \times_X X_i \cong X_i^{\sqcup d}$. We then have local objects $(X_i^{\sqcup d} \to X_i, \mathcal{P}_1|_{X_i^{\sqcup d}})$ and $(X_i^{\sqcup d} \to X_i, \mathcal{P}_2|_{X_i^{\sqcup d}})$, each consisting of the data of d separate $\mathbf{G}|_{X_i}$–torsors. All $2d$ of these torsors can be made locally isomorphic over some further cover of each X_i and therefore our original objects are locally isomorphic. Additionally, since the fiber $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}(S)$ contains the object $(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}})$ involving the split étale extension and the trivial torsor, objects exist in all fibers. Hence $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ is a gerbe and we are done. $\square$

B.2. **Remark.** The condition that $T \to X$ be finite étale of degree d can be replaced by any condition (p) on affine morphisms which satisfies base change and descent, i.e., the conditions (BC) and (DESC) in [GW, App. C]. These morphisms will then also form an intermediate stack $\mathfrak{A}$ and the above argument will show that the corresponding fibered category $\mathfrak{F}(\mathbf{G})^{(p)}$ is a stack. This is done with quasi-coherent modules in place of $\mathbf{G}$–torsors in Appendix C.8.

We denote $S^{\sqcup d} = \sqcup_{i=1}^{d} S$ and we call the object $(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}})$ the split object of $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}(S)$. The trivial torsor $\mathbf{G}|_{S^{\sqcup d}}$ is described as follows. A morphism $X \to S^{\sqcup d}$ induces a decomposition $X = \sqcup_{i=1}^{d} X_i$ for S–schemes X_i by taking preimages of the factors of $S^{\sqcup d}$, and then $\mathbf{G}|_{S^{\sqcup d}}(X) = \mathbf{G}(X_1) \times \ldots \times \mathbf{G}(X_d)$. In particular, if $\pi\colon S^{\sqcup d} \to S$ is the canonical morphism, then $\mathbf{G}^d = \pi_*(\mathbf{G}|_{S^{\sqcup d}})$.

B.3. **Lemma.** *Consider the split object* $(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}}) \in \mathfrak{F}(\mathbf{G})^{d-\text{ét}}(S)$. *We have that*

$$\mathcal{A}ut(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}}) \xrightarrow{\sim} \mathbf{G}^d \rtimes \mathbb{S}_d.$$

Proof. Let $X \in \mathfrak{Sch}_S$. A section $\rho \in \mathcal{A}ut(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}})(X)$ is a morphism of the form $\rho = (\mathrm{Id}_X, g, \varphi)$ where $g\colon X^{\sqcup d} \xrightarrow{\sim} X^{\sqcup d}$ is a X–scheme automorphism and $\varphi\colon \mathbf{G}|_{X^{\sqcup d}} \xrightarrow{\sim} g^*(\mathbf{G}|_{X^{\sqcup d}})$ is a $\mathbf{G}|_{X^{\sqcup d}}$–torsor isomorphism. Since g must be an X–automorphism of $X^{\sqcup d}$, it is a permutation of the components and therefore corresponds to some $\sigma_g \in \mathbb{S}_d(X)$. The pullback of $\mathbf{G}|_{X^{\sqcup d}}$ with respect to g is described by the same permutation. In particular, for a scheme $\sqcup_{i=1}^{d} X_i$ over $X^{\sqcup d}$, we have

$$g^*(\mathbf{G}|_{X^{\sqcup d}})(\sqcup_{i=1}^{d} X_i) = \mathbf{G}(X_{\sigma_g^{-1}(1)}) \times \ldots \times \mathbf{G}(X_{\sigma_g^{-1}(d)}).$$

Thus, we have a canonical isomorphism of sheaves over $X^{\sqcup d}$,

$$\begin{aligned} \rho_g\colon g^*(\mathbf{G}|_{X^{\sqcup d}}) &\to \mathbf{G}|_{X^{\sqcup d}} \\ (x_{\sigma_g^{-1}(1)}, \ldots, x_{\sigma_g^{-1}(d)}) &\mapsto (x_1, \ldots, x_d). \end{aligned}$$

Therefore, the map $\rho_g \circ \varphi\colon \mathbf{G}|_{X^{\sqcup d}} \xrightarrow{\sim} g^*(\mathbf{G}|_{X^{\sqcup d}}) \xrightarrow{\sim} \mathbf{G}|_{X^{\sqcup d}}$ is an automorphism of $\mathbf{G}|_{X^{\sqcup d}}$ and is in $\mathcal{A}ut_{\mathbf{G}|_{X^{\sqcup d}}\text{-tors}}(\mathbf{G}|_{X^{\sqcup d}}) = \mathbf{G}(X)^d$. We now check that

$$\begin{aligned} \mathcal{A}ut(S^{\sqcup d} \to S, \mathbf{G}|_{S^{\sqcup d}})(X) &\to (\mathbf{G}^d \rtimes \mathbb{S}_d)(X) \\ (\mathrm{Id}_X, g, \varphi) &\mapsto (1, \sigma_g)(\rho_g \circ \varphi, 1) \end{aligned}$$

is an isomorphism of groups. Indeed, bijectivity is clear and so we check it is a homomorphism. Multiplication in the domain is given by

$$(\mathrm{Id}_X, g, \varphi) \circ (\mathrm{Id}_X, j, \psi) = (\mathrm{Id}_X, g \circ j, j^*(\varphi) \circ \psi).$$

Note that $\sigma_{g\circ j} = \sigma_g \sigma_j$ and that $\rho_{g\circ j} = \rho_j \circ j^*(\rho_g)$. A short computation shows that for $\phi \in \mathbf{G}(X)^d$, we have $\rho_j \circ j^*(\phi) \circ \rho_j^{-1} = \sigma_j(\phi)$. Therefore,

$$(\mathrm{Id}_X, g, \varphi) \circ (\mathrm{Id}_X, j, \psi) \mapsto (1, \sigma_g)(\rho_g \circ \varphi, 1)(1, \sigma_j)(\rho_j \circ \psi, 1)$$

$$\begin{aligned}&=(1,\sigma_g\sigma_j)(\rho_j\circ j^*(\rho_g\circ\varphi)\circ\rho_j^{-1},1)(\rho_j\circ\psi,1)\\&=(1,\sigma_{g\circ j})(\rho_j\circ j^*(\rho_g)\circ j^*(\varphi)\circ\psi,1)\\&=(1,\sigma_{g\circ j})(\rho_{g\circ j}\circ j^*(\varphi)\circ\psi,1)\end{aligned}$$

as required. Hence, this is a group isomorphism and it is clear these isomorphisms assemble into our desired isomorphism of sheaves $\mathcal{A}ut(S^{\sqcup d}\to S,\mathbf{G}|_{S^{\sqcup d}})\xrightarrow{\sim}\mathbf{G}^d\rtimes\mathbb{S}_d$. □

B.4. **Corollary.** *The morphism*

$$\begin{aligned}\mathfrak{F}(\mathbf{G})^{d-\text{ét}}&\to\mathfrak{Tors}(\mathbf{G}^d\rtimes\mathbb{S}_d)\\(T\to X,\mathcal{P})&\mapsto\mathcal{I}som((X^{\sqcup d}\to X,\mathbf{G}|_{X^{\sqcup d}}),(T\to X,\mathcal{P}))\end{aligned}$$

is an equivalence of gerbes.

Proof. This follows immediately from Proposition 1.16(ii) and Lemma B.3 since $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ is a gerbe and therefore $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}=\mathfrak{Forms}(S^{\sqcup d}\to S,\mathbf{G}|_{S^{\sqcup d}})$. □

For the next lemma, we recall that $H^1(S,\mathbb{S}_d)$ classifies the S-isomorphism classes of finite étale covers of S of degree d. For such a cover $T\to S$, we denote its S-isomorphism class by $[T]$.

B.5. **Lemma.** *We have a decomposition*

$$H^1(S,\mathbf{G}^d\rtimes\mathbb{S}_d)\xrightarrow{\sim}\coprod_{[T]\in H^1(S,\mathbb{S}_d)}H^1(T,\mathbf{G}|_T)/\operatorname{Aut}_S(T)\tag{B.5.1}$$

where each $\operatorname{Aut}_S(T)$ *acts on* $H^1(T,\mathbf{G}|_T)$ *by base change.*

Proof. Lemma B.4 shows that the set $H^1(S,\mathbf{G}^d\rtimes\mathbb{S}_d)$ classifies the objects of the fiber $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}(S)$ so that there is a decomposition

(B.5.2)

$$H^1(S,\mathbf{G}^d\rtimes\mathbb{S}_d)\xrightarrow{\sim}\coprod_{[T]\in H^1(S,\mathbb{S}_d)}\left\{\begin{array}{c}\text{Isomorphism classes } [(T'\to S,\mathcal{P})]\\ \text{where } [T']=[T]\in H^1(S,\mathbb{S}_d)\end{array}\right\}.$$

We fix a finite étale cover $T\to S$ of degree d. There is a surjection on isomorphism classes

$$\begin{aligned}H^1(T,\mathbf{G}|_T)&\twoheadrightarrow\left\{\begin{array}{c}\text{Isomorphism classes } [(T'\to S,\mathcal{P})]\\ \text{where } [T']=[T]\in H^1(S,\mathbb{S}_d)\end{array}\right\}\\[\mathcal{P}]&\mapsto[(T\to S,\mathcal{P})]\end{aligned}$$

and two isomorphism classes of $\mathbf{G}|_T$–torsors $[\mathcal{P}_1]$ and $[\mathcal{P}_2]$ provide the same isomorphism class in $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}(S)$ if and only if there exists $g\in\operatorname{Aut}_S(T)$ such that $\mathcal{P}_1\xrightarrow{\sim}g^*(\mathcal{P}_2)$ as $\mathbf{G}|_T$–torsors. Thus, the above map induces a bijection

$$H^1(T,\mathbf{G}|_T)/\operatorname{Aut}_S(T)\xrightarrow{\sim}\left\{\begin{array}{c}\text{Isomorphism classes } [(T'\to S,\mathcal{P})]\\ \text{where } [T']=[T]\in H^1(S,\mathbb{S}_d)\end{array}\right\}\tag{B.5.3}$$

where $\mathrm{Aut}_S(T)$ acts (by right action) on $H^1(T, \mathbf{G}|_T)$ by base change. Combining (B.5.2) and (B.5.3) yields the desired decomposition. □

B.6. **Remark.** The set $H^1(S, \mathbf{G}^d \rtimes \mathbb{S}_d)$ is described by a decomposition with respect to the fibers of $H^1(S, \mathbf{G}^d \rtimes \mathbb{S}_d) \to H^1(S, \mathbb{S}_d)$ in [Gil, 2.6.3]. In the special case of Lemma B.5, we prefer a direct approach.

B.7. **Remark.** The stack $\mathfrak{F}(\mathbf{G})^{d-\text{ét}}$ can equivalently be defined to have objects which are pairs $(T \to X, \mathcal{E})$ where $X \in \mathfrak{Sch}_S$ and $T \to X$ is a degree d étale cover as before, but where $\mathcal{E}$ is any twisted form of a designated split sheaf $\mathcal{E}_0$ (of rings, modules, algebras, etc.) on $\mathfrak{Sch}_T$ whose automorphism group is $\mathbf{G}|_T$.

Appendix C. Quasi-coherent sheaves on $\mathfrak{Sch}_S$

Following [St, Tag 03DK], an $\mathcal{O}$–module $\mathcal{E}$ is called *quasi-coherent* if for all $X \in \mathfrak{Sch}_S$ there is a covering $\{X_i \to X\}_{i \in I}$ such that for each $i \in I$ there is an exact sequence of $\mathcal{O}|_{X_i}$–modules

$$\bigoplus_{j \in J_i} \mathcal{O}|_{X_i} \to \bigoplus_{k \in K_i} \mathcal{O}|_{X_i} \to \mathcal{E}|_{X_i} \to 0$$

for some index sets J_i and K_i. If all K_i can be taken to be finite sets, we say $\mathcal{E}$ is *finitely generated*. If both J_i and K_i can be finite for all $i \in I$, then we say $\mathcal{E}$ is *finitely presented*. In particular, finite locally free $\mathcal{O}$–modules are quasi-coherent. Since quasi-coherence is a local condition, an $\mathcal{O}$–module $\mathcal{E}$ is quasi-coherent if and only if $\mathcal{E}|_T$ is a quasi-coherent $\mathcal{O}|_T$–module for all $T \in \mathfrak{Sch}_S$.

By [St, Tag 03DX], there is an equivalence between this (site wide) notion of quasi-coherent $\mathcal{O}$–module and the classical notion of a quasi-coherent $\mathcal{O}_S$–module on the locally ringed space S. Given a classical quasi-coherent sheaf E on S, it can be extended to a quasi-coherent $\mathcal{O}$–module by setting $\mathcal{E}(T) = g^*(E)(T)$. Conversely, for any quasi-coherent $\mathcal{O}$–module $\mathcal{E}$, there exists a classical quasi-coherent sheaf E on S such that for $X \in \mathfrak{Sch}_S$ with structure morphism $g\colon X \to S$, we have $\mathcal{E}(T) = g^*(E)(T)$. Then, E is simply the restriction of $\mathcal{E}$ to the small Zariski site consisting of open subschemes of S. For such a pair, we use the notation $E_{\text{fppf}} = \mathcal{E}$ and $\mathcal{E}_{\text{small}} = E$ (in [St] they write E^a for E_{fppf}). These of course satisfy $(E_{\text{fppf}})_{\text{small}} = E$ and $(\mathcal{E}_{\text{small}})_{\text{fppf}} = \mathcal{E}$.

The following is a key characterization of quasi-coherent $\mathcal{O}$–modules in terms of their restriction to $\mathfrak{Aff}_S$ under the equivalence of Lemma 1.3.

C.1. **Lemma** ([St, Tag 0GZV (1) ⇔ (7)])**.** *Let $\mathcal{M}\colon \mathfrak{Aff}_S \to \mathfrak{Ab}$ be a presheaf of $\mathcal{O}$–modules. Then, $\mathcal{M}$ is a quasi-coherent $\mathcal{O}$–module (in particular it is a sheaf) if and only if for every morphism $f\colon V \to U$ in $\mathfrak{Aff}_S$, the morphism*

$$\rho_f\colon \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \to \mathcal{M}(V)$$
$$m \otimes s \mapsto s \cdot m|_V$$

is an isomorphism of $\mathcal{O}(V)$–modules.

C.2. **Lemma** ([St, 0GNC(6)]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be quasi-coherent $\mathcal{O}$–modules. If $\mathcal{M}_1$ is finite locally free, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}_1, \mathcal{M}_2)$ is quasi-coherent as well.*

When working on the small Zariski site or small étale site, it is sufficient to assume $\mathcal{M}_1$ is finitely presented in Lemma C.2, see [St, Tag 01I8(3)] and [St, Tag 0GNB(6)]. However, this is not sufficient when working on a big site. The key difference is that all structure morphisms in these small sites are flat, while structure morphisms in a big site are arbitrary scheme maps.

Quasi-coherence interacts well with pullbacks.

C.3. **Lemma** ([St, Tag 03LC (1)]). *Let $g\colon X \to S$ be a morphism of schemes and let E be a quasi-coherent sheaf on S. Then, we have an equality $(g^*(E))_{\text{fppf}} = g^*(E_{\text{fppf}})$.*

In particular, if $\mathcal{E}$ is a quasi-coherent $\mathcal{O}$–module, then $g^*\mathcal{E} = (g^*(\mathcal{E}_{\text{small}}))_{\text{fppf}}$ and so pullbacks of quasi-coherent $\mathcal{O}$–modules are quasi-coherent. In order to have a similar result for pushforwards we assume that g is affine.

C.4. **Lemma.** *Let $g\colon X \to S$ be an affine morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}|_X$–module. Then, $g_*\mathcal{F} = (g_*(\mathcal{F}_{\text{small}}))_{\text{fppf}}$, which in particular means that $g_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}$–module.*

Equivalently, if F is a quasi-coherent sheaf on X, then we have $g_*(F_{\text{fppf}}) = (g_*(F))_{\text{fppf}}$.

Proof. This follows immediately from [St, Tag 02KG] which shows that, since g is affine, for any other morphism $h\colon S' \to S$ there is a commutative diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{h'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ S' & \xrightarrow{h} & S \end{array}$$

and $h^*(g_*(\mathcal{F}_{\text{small}})) = g'_*(h'^*(\mathcal{F}_{\text{small}}))$. The global sections of these sheaves are

$$h^*(g_*(\mathcal{F}_{\text{small}}))(S') = (g_*(\mathcal{F}_{\text{small}}))_{\text{fppf}}(S'), \text{ and}$$
$$g'_*(h'^*(\mathcal{F}_{\text{small}}))(S') = (h'^*(\mathcal{F}_{\text{small}}))(X \times_S S') = \mathcal{F}(X \times_S S') = (g_*\mathcal{F})(S')$$

and so $g_*\mathcal{F} = (g_*(\mathcal{F}_{\text{small}}))_{\text{fppf}}$ as claimed. □

C.5. **Remark.** Since our $\mathcal{O}|_X$–modules are sheaves with respect to the fppf topology, it is not sufficient in Lemma C.4 to only assume that g is quasi-compact and quasi-separated as is done in [St, Tag 01LC] for classical (small) quasi-coherent modules on X. An example demonstrating this is given in [St, Tag 03LC (2)].

We may construct a quasi-coherent $\mathcal{O}$–module from a collection of $\mathcal{O}(U)$–modules for every $U \in \mathfrak{Aff}_S$, given that the $\mathcal{O}(U)$–modules satisfy compatibility conditions with respect to tensor products similar to those in Lemma C.1.

C.6. **Lemma.** *Assume that for each $U \in \mathfrak{Aff}_S$ we are given an $\mathcal{O}(U)$–module M_U and for each morphism $f\colon V \to U$ in $\mathfrak{Aff}_S$ we are given an isomorphism of $\mathcal{O}(V)$–modules $\rho_f\colon M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} M_V$ such that the following conditions hold.*

(i) $\rho_{\mathrm{Id}_U}\colon M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \xrightarrow{\sim} M_U$ *is the canonical isomorphism, and*

(ii) *for morphisms $g\colon V' \to V$ and $f\colon V \to U$ in $\mathfrak{Aff}_S$, the following diagram commutes*

$$\begin{array}{ccc} (M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(V)) \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & \xrightarrow{\text{can}} & M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(V') \\ \downarrow{\scriptstyle \rho_f \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \rho_{f\circ g}} \\ M_V \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & \xrightarrow{\quad\rho_g\quad} & M_{V'} \end{array}$$

where can *is the canonical isomorphism.*

Then, there exists a unique quasi-coherent $\mathcal{O}$–module $\mathcal{M}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ such that for $U \in \mathfrak{Aff}_S$, we have $\mathcal{M}(U) = M_U$ and for each morphism $f\colon V \to U$ in $\mathfrak{Aff}_S$, the restriction morphism is $\mathcal{M}(f) = \rho_f \circ (\mathrm{Id} \otimes 1_{\mathcal{O}(V)})$.

Proof. We begin by defining a presheaf on $\mathfrak{Aff}_S$. Namely, we define

$$\begin{aligned} \mathcal{M}'\colon \mathfrak{Aff}_S &\to \mathfrak{Ab} \\ U &\mapsto M_U \\ (f\colon V \to U) &\mapsto \rho_f \circ (\mathrm{Id} \otimes 1_{\mathcal{O}(V)}). \end{aligned}$$

Conditions (i) and (ii) certify that this is a well-defined presheaf. Furthermore, using the $\mathcal{O}(U)$–action on each M_U gives $\mathcal{M}'$ the structure of a presheaf of $\mathcal{O}$–modules. By construction, for each morphism $f\colon V \to U$ the map

$$\begin{aligned} \mathcal{M}'(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) &\to \mathcal{M}'(V) \\ m \otimes s &\mapsto s \cdot \mathcal{M}'(f)(m) \end{aligned}$$

is simply ρ_f since

$$s \cdot \mathcal{M}'(f)(m) = s \cdot \rho_f(m \otimes 1) = \rho_f(m \otimes s)$$

where we use the $\mathcal{O}(V)$–linearity of ρ_f. Hence, these maps are isomorphisms and so Lemma C.1 says that $\mathcal{M}'$ is a quasi-coherent $\mathcal{O}$–module. Applying Lemma 1.3, we obtain a unique quasi-coherent $\mathcal{O}$–module $\mathcal{M}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ whose restriction to $\mathfrak{Aff}_S$ is $\mathcal{M}'$. This finishes the proof. □

In the remainder of the paper, in order to mirror the notation appearing, for example, in Lemma C.6, we write the following. Given a map of rings $f\colon R \to Q$, there is a base change functor $b_f\colon \mathfrak{Mod}_R \to \mathfrak{Mod}_Q$ defined by the tensor product, $b_f(M) = M \otimes_R Q$. If we have another map of rings $f'\colon R' \to Q'$ as well as functors $\mathcal{F}_{R'/R}\colon \mathfrak{Mod}_{R'} \to \mathfrak{Mod}_R$ and $\mathcal{F}_{Q'/Q}\colon \mathfrak{Mod}_{Q'} \to \mathfrak{Mod}_Q$, we write

$$\mathcal{F}_{R'/R}(\_) \otimes_R Q = b_f \circ \mathcal{F}_{R'/R}$$

$$\mathcal{F}_{Q'/Q}(\_ \otimes_{R'} Q') = \mathcal{F}_{Q'/Q} \circ b_{f'}$$

for the two functors $\mathfrak{Mod}_{R'} \to \mathfrak{Mod}_Q$. Additionally, we will write

$$\theta \colon \mathcal{F}_{R'/R}(\_) \otimes_R Q \xrightarrow{\sim} \mathcal{F}_{Q'/Q}(\_ \otimes_{R'} Q')$$

to denote that θ is a natural isomorphism between the functors $b_f \circ \mathcal{F}_{R'/R}$ and $\mathcal{F}_{Q'/Q} \circ b_{f'}$.

The results above allow us to assemble certain families of functors between module categories into a functor between categories of quasi-coherent sheaves in the following technical lemma. We will use the concept of an affine morphism for which we refer to [St, Tag 01S6]. In particular, affine morphisms are stable under base change by [St, Tag 01SD] and so for an affine morphism $f \colon T \to S$ and any affine scheme $U \in \mathfrak{Sch}_S$, the fiber product $U \times_S T$ is an affine scheme. Recall as well that $\mathfrak{QCoh}(S)$ denotes the category of quasi-coherent $\mathcal{O}$–modules on $\mathfrak{Sch}_S$. The category $\mathfrak{QCoh}(T)$ is then the category of quasi-coherent $\mathcal{O}|_T$–modules, which are functors on $\mathfrak{Sch}_T$.

C.7. **Lemma.** *Let $T \to S$ be an affine morphism (as in* [St, Tag 01S6]*) of schemes such that the following holds. Assume that we have a family of covariant functors*

$$\mathcal{F}_U \colon \mathfrak{Mod}_{\mathcal{O}(T \times_S U)} \to \mathfrak{Mod}_{\mathcal{O}(U)}$$

for each $U \in \mathfrak{Aff}_S$. Further, suppose that for each morphism $f \colon V \to U$ in $\mathfrak{Aff}_S$, we have an isomorphism of functors

$$\theta_f \colon \mathcal{F}_U(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}_V(\_ \otimes_{\mathcal{O}(T \times_S U)} \mathcal{O}(T \times_S V))$$

satisfying the following conditions. For each pair of morphisms $g \colon V' \to V$ and $f \colon V \to U$ in $\mathfrak{Aff}_S$, the following diagrams commute.

$$\begin{array}{ccc} \mathcal{F}_U(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) & & \\ \downarrow{\scriptstyle \theta_{\mathrm{Id}_U}} & \searrow^{\mathrm{can}} & \\ \mathcal{F}_U(\_ \otimes_{\mathcal{O}(T \times_S U)} \mathcal{O}(T \times_S U)) & \xrightarrow{\mathcal{F}_U(\mathrm{can})} & \mathcal{F}_U(\_) \end{array} \tag{C.7.1}$$

and

$$\begin{array}{ccc} (\mathcal{F}_U(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)) \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & \xrightarrow{\mathrm{can}} & \mathcal{F}_U(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V') \\ \downarrow{\scriptstyle \theta_f \otimes \mathrm{Id}} & & \\ \mathcal{F}_V(\_ \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)) \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & & \downarrow{\scriptstyle \theta_{f \circ g}} \\ \downarrow{\scriptstyle \theta_g} & & \\ \mathcal{F}_{V'}((\_ \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)) \otimes_{\mathcal{O}(V_T)} \mathcal{O}(V'_T)) & \xrightarrow{\mathcal{F}_{V'}(\mathrm{can})} & \mathcal{F}_{V'}(\_ \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V'_T)). \end{array} \tag{C.7.2}$$

For brevity in the diagrams, we abuse notation by using can *to denote various canonical isomorphisms. We also denote $T \times_S U = U_T$ and likewise for V_T and V'_T. Then, there is a functor $\mathcal{F}_{T/S} \colon \mathfrak{QCoh}(T) \to \mathfrak{QCoh}(S)$ defined as*

follows. For each quasi-coherent $\mathcal{O}|_T$–module $\mathcal{M}$, the $\mathcal{O}$–module $\mathcal{F}_{T/S}(\mathcal{M})$ has the following properties:

(i) $\mathcal{F}_{T/S}(\mathcal{M})(U) = \mathcal{F}_U(\mathcal{M}(T \times_S U))$ *for all* $U \in \mathfrak{Aff}_S$, *and*
(ii) *for each morphism* $f\colon V \to U$ *in* $\mathfrak{Aff}_S$, *the associated restriction map* $\mathcal{F}_{T/S}(\mathcal{M})(f)$ *is defined by*

$$\begin{array}{ccc} \mathcal{F}_U(\mathcal{M}(U_T)) & \xrightarrow{\mathrm{Id}\otimes 1} & \mathcal{F}_U(\mathcal{M}(U_T)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \\ & \searrow^{\mathcal{F}_{T/S}(\mathcal{M})(f)} & \downarrow{\scriptstyle \theta_f(\mathcal{M}(U_T))} \\ & & \mathcal{F}_V\big(\mathcal{M}(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)\big) \\ & & \downarrow{\scriptstyle \mathcal{F}_V(\rho_{f'})} \\ & & \mathcal{F}_V(\mathcal{M}(V_T)) \end{array}$$

where $f'\colon T \times_S V \to T \times_S U$ *is the pullback of* f *and*

$$\rho_{f'}\colon \mathcal{M}(T \times_S U) \otimes_{\mathcal{O}(T\times_S U)} \mathcal{O}(T \times_S V) \to \mathcal{M}(T \times_S V)$$

is the canonical isomorphism as in Lemma C.1 arising from $\mathcal{M}$ *being quasi-coherent and both* $T \times_S U$ *and* $T \times_S V$ *being affine schemes.*
(iii) *For each morphism* $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ *of quasi-coherent* $\mathcal{O}|_T$–*modules, the morphism* $\mathcal{F}_{T/S}(\varphi)\colon \mathcal{F}(\mathcal{M}_1) \to \mathcal{F}(\mathcal{M}_2)$ *is defined over* $U \in \mathfrak{Aff}_S$ *by*

$$\mathcal{F}_{T/S}(\varphi)(U) = \mathcal{F}_U(\varphi(T \times_S U))\colon \mathcal{F}_U(\mathcal{M}_1(T \times_S U)) \to \mathcal{F}_U(\mathcal{M}_2(T \times_S U)).$$

Proof. There are various compatibility conditions that need to be checked. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}|_T$–module. We begin by using Lemma C.6 to define the sheaf $\mathcal{F}_{T/S}(\mathcal{M})$. Because $\mathcal{M}$ is quasi-coherent, by Lemma C.1 it comes with standard isomorphisms $\rho_f\colon \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{M}(V)$ for each morphism $f\colon V \to U$ in $\mathfrak{Aff}_S$.

Now, for each $U \in \mathfrak{Aff}_S$, we set $M_U = \mathcal{F}_U(\mathcal{M}(T \times_S U))$. For a morphism $f\colon V \to U$ in $\mathfrak{Aff}_S$, we define the isomorphism $\phi_f\colon M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} M_V$ to be

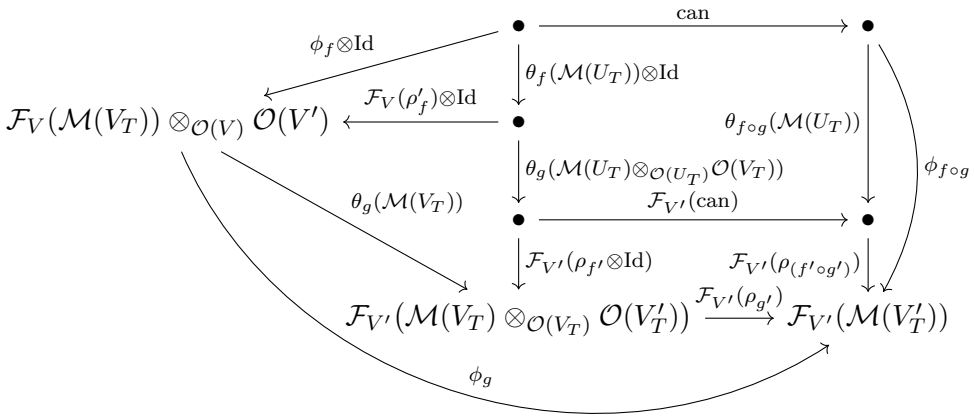
$$\rho_f = \mathcal{F}_V(\rho_{f'}) \circ \theta_f(\mathcal{M}(U_T))$$

where $f'\colon T \times_S V \to T \times_S U$ is the pullback of f. Because the map $\rho_{\mathrm{Id}_{U_T}}\colon \mathcal{M}(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(U_T) \to \mathcal{M}(U_T)$ is the canonical isomorphism, the commutativity of (C.7.1) implies that ϕ_{Id_U} is the canonical isomorphism, as required in Lemma C.6.

For morphisms $g\colon V' \to V$ and $f\colon V \to U$ in $\mathfrak{Aff}_S$, passing $\mathcal{M}(T \times_S U)$ into diagram (C.7.2) yields

$$\begin{array}{ccc}
 & & \mathcal{F}_U(\mathcal{M}(U_T)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V') \\
 & \overset{\text{can}}{\nearrow} & \\
(\mathcal{F}_U(\mathcal{M}(U_T)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)) \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & & \\
\downarrow{\scriptstyle \theta_f(\mathcal{M}(U_T))\otimes \mathrm{Id}} & & \\
\mathcal{F}_V\big(\mathcal{M}(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)\big) \otimes_{\mathcal{O}(V)} \mathcal{O}(V') & & \downarrow{\scriptstyle \theta_{f\circ g}(\mathcal{M}(U_T))} \\
\downarrow{\scriptstyle \theta_g(\mathcal{M}(U_T)\otimes_{\mathcal{O}(U_T)}\mathcal{O}(V_T))} & & \\
\mathcal{F}_{V'}\big((\mathcal{M}(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)) \otimes_{\mathcal{O}(V_T)} \mathcal{O}(V'_T)\big) & & \\
 & \underset{\mathcal{F}_{V'}(\text{can})}{\searrow} & \\
 & & \mathcal{F}_{V'}(\mathcal{M}(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V'_T)).
\end{array}$$

This diagram can be extended to



where the bullets represent the entries in the previous diagram. The triangle in the top left of the diagram commutes by definition of ϕ_f. The triangle below it commutes because θ_g is a natural transformation. The bottom most square commutes because $\rho_{(f'\circ g')} \circ \mathrm{can} = \rho_{g'} \circ (\rho_{f'} \otimes \mathrm{Id})$ since $\mathcal{M}$ is quasi-coherent, and this has simply been passed through $\mathcal{F}_{V'}$. Finally, the portions of the diagram involving the curved arrows commute by definition of ϕ_g and $\phi_{f\circ g}$. Ultimately, this shows that $\phi_{f\circ g} \circ \mathrm{can} = \phi_g \circ (\phi_f \otimes \mathrm{Id})$ as required by Lemma C.6. Hence, applying Lemma C.6 produces a quasi-coherent $\mathcal{O}$–module $\mathcal{F}_{T/S}(\mathcal{M})$ with properties (i) and (ii) of the statement.

To finish constructing the functor $\mathcal{F}_{T/S}$, we need to define the image of morphisms. Let $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of quasi-coherent $\mathcal{O}|_T$–modules. Due to Lemma 1.3, it is sufficient to only define $\mathcal{F}_{T/S}(\varphi)$ over the schemes in $\mathfrak{Aff}_S$. For $U \in \mathfrak{Aff}_S$, we define $\mathcal{F}_{T/S}(\varphi)(U) = \mathcal{F}_U(\varphi(T \times_S U))$ as in condition (iii) of the statement. It is clear that if $\mathcal{F}_{T/S}(\varphi)$ is a well-defined morphism, then $\mathcal{F}_{T/S}$ will preserve identities and compositions and hence will be a functor. Therefore, we just need to verify that $\mathcal{F}_{T/S}(\varphi)$ is a well-defined natural transformation of functors. Let $f\colon V \to U$ be a morphism in $\mathfrak{Aff}_S$. Once again, we consider a large diagram

$$\begin{array}{ccc}
\mathcal{F}_U(\mathcal{M}_1(U_T)) & \xrightarrow{\mathcal{F}_U(\varphi(U_T))} & \mathcal{F}_U(\mathcal{M}_2(U_T)) \\
\downarrow{\scriptstyle \mathrm{Id}\otimes 1} & & \downarrow{\scriptstyle \mathrm{Id}\otimes 1} \\
\mathcal{F}_U(\mathcal{M}_1(U_T)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) & \xrightarrow{\mathcal{F}_U(\varphi(U_T))\otimes \mathrm{Id}} & \mathcal{F}_U(\mathcal{M}_2(U_T)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \\
\downarrow{\scriptstyle \theta_f(\mathcal{M}_1(U_T))} & & \downarrow{\scriptstyle \theta_f(\mathcal{M}_2(U_T))} \\
\mathcal{F}_V\big(\mathcal{M}_1(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)\big) & \xrightarrow{\mathcal{F}_V(\varphi(U_T)\otimes \mathrm{Id})} & \mathcal{F}_V\big(\mathcal{M}_2(U_T) \otimes_{\mathcal{O}(U_T)} \mathcal{O}(V_T)\big) \\
\downarrow{\scriptstyle \mathcal{F}_V(\rho_{1,f'})} & & \downarrow{\scriptstyle \mathcal{F}_V(\rho_{2,f'})} \\
\mathcal{F}_V(\mathcal{M}_1(V_T)) & \xrightarrow{\mathcal{F}_V(\varphi(V_T))} & \mathcal{F}_V(\mathcal{M}_2(V_T)).
\end{array}$$

Here, $\rho_{i,f}$ is the standard isomorphism from $\mathcal{M}_i$ being quasi-coherent. The top square is trivially commutative. The middle square commutes because θ_f is a natural transformation. The bottom square commutes because $\rho_{2,f'} \circ (\varphi(U_T) \otimes \mathrm{Id}) = \varphi(V_T) \circ \rho_{1,f'}$ due to φ being a morphism of $\mathcal{O}$–modules and this has been passed through $\mathcal{F}_V$. The compositions down each column are the restriction morphisms $\mathcal{F}_{T/S}(\mathcal{M}_i)(f)$ respectively, so the commutativity of the outermost paths shows that $\mathcal{F}_{T/S}(\varphi)$ is well-defined. This finishes the construction of $\mathcal{F}_{T/S}$ and hence concludes the proof. □

C.8. **Stack Morphism.** If we have compatible functors between module categories for an even wider ranger of morphisms, we can construct a morphism of stacks of quasi-coherent sheaves. The codomain of the stack morphism will be the substack $\mathfrak{QCoh}$ of $\mathfrak{Sh}$ consisting of quasi-coherent modules.

(i) The objects of $\mathfrak{QCoh}$ are pairs $(X, \mathcal{F})$ with $X \in \mathfrak{Sch}_S$ and $\mathcal{F}$ a quasi-coherent $\mathcal{O}|_X$–module on $\mathfrak{Sch}_X$.
(ii) The morphisms of $\mathfrak{QCoh}$ are pairs $(g, \varphi)\colon (X', \mathcal{F}') \to (X, \mathcal{F})$ where $g\colon X' \to X$ is a morphism of S–schemes and $\varphi\colon \mathcal{F}' \to g^*(\mathcal{F})$ is a morphism of $\mathcal{O}|_{X'}$–modules. Composition is given by $(g, \varphi)\circ(h, \psi) = (g \circ h, h^*(\varphi) \circ \psi)$.

Second, consider the stack of affine morphisms $p\colon \mathfrak{AffMor} \to \mathfrak{Sch}_S$ whose

(i) objects are affine morphisms $T' \to T$ where $T \in \mathfrak{Sch}_S$,

(ii) morphisms are pairs $(f,g)\colon (X' \to X) \to (T' \to T)$ where f and g are scheme morphisms making

$$\begin{array}{ccc} X' & \xrightarrow{g} & T' \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & T \end{array}$$

a fiber product diagram in $\mathfrak{Sch}_S$, and

(iii) the structure functor sends $(T' \to T) \mapsto T$ and $(f,g) \mapsto f$.

The pullback diagram condition on morphisms makes this fibered in groupoids. It is indeed a stack since $\mathcal{H}om(T_1' \to T, T_2' \to T) = \mathcal{I}som_T(T_1', T_2')$, which is a sheaf, and affine morphisms satisfy gluing by [Ols, 4.4.7].

Let $\mathfrak{I} \subseteq \mathfrak{AffMor}$ be any substack as defined in [Vis, 4.1.6]. If $\mathfrak{I}$ is a full subcategory, this is equivalent to choosing a family of affine morphisms which are stable under base change and allow descent. The substack $\mathfrak{I} \to \mathfrak{Sch}_S$ gives $\mathfrak{I}$ an inherited site structure as in [St, Tag 06NU]. In detail, the covers will be families of morphisms of the form $\{(f_i, g_i)\colon (T_i' \to T_i) \to (T' \to T)\}_{i\in I}$ where $\{f_i\colon T_i \to T\}_{i\in I}$ is an fppf cover (and every (f_i, g_i) is cartesian, but this holds by default since $\mathfrak{AffMor}$ is fibered in groupoids). Since morphisms in $\mathfrak{AffMor}$ must define pullback diagrams, this means that $T_i' \xrightarrow{\sim} T' \times_T T_i$ for each $i \in I$ and so $\{g_i\colon T_i' \to T'\}_{i\in I}$ is also an fppf cover.

Next, we define the stack of quasi-coherent sheaves over $\mathfrak{I}$, denoted $\mathfrak{QCoh}_{\mathfrak{I}}$, as follows.

(i) Objects are pairs $(h\colon T' \to T, \mathcal{M})$ with $h \in \mathfrak{I}$ and $\mathcal{M}$ a quasi-coherent $\mathcal{O}|_{T'}$–module.

(ii) Morphisms are triples $(f,g,\varphi)\colon (j\colon X' \to X, \mathcal{N}) \to (h\colon T' \to T, \mathcal{M})$ where $(f,g)\colon j \to h$ is a morphism in $\mathfrak{I}$ and $\varphi\colon \mathcal{N} \to g^*(\mathcal{M})$ is an isomorphism of $\mathcal{O}|_{X'}$–modules. Composition is given by

$$(f_1, g_1, \varphi_1) \circ (f_2, g_2, \varphi_2) = (f_1 \circ f_2, g_1 \circ g_2, g_2^*(\varphi_1) \circ \varphi_2).$$

(iii) The structure functor is $p\colon \mathfrak{QCoh}_{\mathfrak{I}} \to \mathfrak{I}$ which behaves as $(h\colon T' \to T, \mathcal{M}) \mapsto (h\colon T' \to T)$ and $(f,g,\varphi) \mapsto (f,g)$ on objects and morphisms respectively.

It is clear this is a stack since, for two objects $(T' \to T, \mathcal{M}_1)$ and $(T' \to T, \mathcal{M}_2)$ in the same fiber, $\mathcal{I}som_{\mathcal{O}|_{T'}}(\mathcal{M}_1, \mathcal{M}_2)$ is a sheaf and quasi-coherent modules permit gluing along fppf covers. Since $\mathfrak{I}$ is fibered in groupoids and we require that φ in a morphism of $\mathfrak{QCoh}_{\mathfrak{I}}$ be an isomorphism, the stack $\mathfrak{QCoh}_{\mathfrak{I}} \to \mathfrak{I}$ is also fibered in groupoids.

However, we ultimately want to consider $\mathfrak{QCoh}_{\mathfrak{I}}$ as a stack over $\mathfrak{Sch}_S$. By [St, 09WX], the composition $\mathfrak{QCoh}_{\mathfrak{I}} \to \mathfrak{I} \to \mathfrak{Sch}_S$ does produce a stack and it is also fibered in groupoids.

Throughout the remainder of this section, we let $\mathfrak{I}$ be a substack of $\mathfrak{AffMor}$ and assume we are given the following data:

(C.8.a) a functor $\mathcal{F}_h\colon \mathfrak{Mod}_{\mathcal{O}(U')} \to \mathfrak{Mod}_{\mathcal{O}(U)}$ for every object $h\colon U' \to U$ in $\mathfrak{I}$ for which $U, U' \in \mathfrak{Aff}_S$, and

(C.8.b) for every fiber product diagram

$$D = \begin{array}{ccc} V' & \xrightarrow{f'} & U' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ V & \xrightarrow{f} & U \end{array}$$

where U, U', V and hence V' are affine, $h, h' \in \mathfrak{I}$, and $f \in \mathfrak{Aff}_S$ is an arbitrary morphism, we are given an isomorphism of functors

$$\theta_D \colon \mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}_{h'}(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')).$$

Further, we assume that these functors satisfy the following compatibility conditions.

(C.8.c) For every fiber product diagram D of the form below, the diagram on the right commutes

$$D = \begin{array}{ccc} U' & = & U' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ U & = & U, \end{array} \qquad \begin{array}{ccc} \mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) & & \\ \downarrow{\scriptstyle \theta_D} & \searrow^{\text{can}} & \\ \mathcal{F}_h(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(U')) & \xrightarrow{\mathcal{F}_h(\text{can})} & \mathcal{F}_h(\_) \end{array}$$

where $h \in \mathfrak{I}$ and $U, U' \in \mathfrak{Aff}_S$.

(C.8.d) For all fiber product diagrams

$$D_f = \begin{array}{ccc} V' & \xrightarrow{f'} & U' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ V & \xrightarrow{f} & U \end{array}, \quad D_g = \begin{array}{ccc} W' & \xrightarrow{g'} & V' \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} \\ W & \xrightarrow{g} & V \end{array}, \text{ and } D_{f\circ g} = \begin{array}{ccc} W' & \xrightarrow{f'\circ g'} & U' \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h} \\ W & \xrightarrow{f\circ g} & U \end{array}$$

with $h \in \mathfrak{I}$ and $f, g \in \mathfrak{Aff}_S$ the diagram

$$\begin{array}{ccc} (\mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & \xrightarrow{\text{can}} & \mathcal{F}_h(\_) \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \\ \downarrow{\scriptstyle \theta_{D_f}\otimes \mathrm{Id}} & & \\ \mathcal{F}_{h'}(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) & & \downarrow{\scriptstyle \theta_{D_{f\circ g}}} \\ \downarrow{\scriptstyle \theta_{D_g}} & & \\ \mathcal{F}_{h''}((\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(V')) \otimes_{\mathcal{O}(V')} \mathcal{O}(W')) & \xrightarrow{\mathcal{F}_{h''}(\text{can})} & \mathcal{F}_{h''}(\_ \otimes_{\mathcal{O}(U')} \mathcal{O}(W')) \end{array}$$

commutes.

With these assumptions, we work up to a stack version of Lemma C.7. First, let $h \colon T' \to T$ be any morphism in $\mathfrak{I}$. For each $U \in \mathfrak{Aff}_T$ and each morphism $f \colon V \to U$ in $\mathfrak{Aff}_T$, we have fiber product diagrams

$$D_U = \begin{array}{ccc} T' \times_T U & \longrightarrow & T' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ U & \longrightarrow & T \end{array} \text{ and } D_f = \begin{array}{ccc} T' \times_T V & \xrightarrow{f'} & T' \times_T U \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} \\ V & \xrightarrow{f} & U. \end{array}$$

If we set $\mathcal{F}_U = \mathcal{F}_{h'}$ and $\theta_f = \theta_{D_f}$, then it is clear that the compatibility conditions assumed above specialize into the requirements of Lemma C.7. Therefore, we may apply the lemma to obtain a functor denoted $\mathcal{F}_{T'/T} \colon \mathfrak{QCoh}(T') \to \mathfrak{QCoh}(T)$. These functors are related to one another in the following way.

C.9. **Lemma.** *Let $h' \colon X' \to X$ and $h \colon T' \to T$ be two morphisms in $\mathfrak{I}$ and let $\mathcal{F}_{X'/X}$ and $\mathcal{F}_{T'/T}$ be the associated functors defined above. Then, for every fiber product diagram*

$$D = \begin{array}{ccc} X' & \xrightarrow{g} & T' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & T \end{array}$$

with f, g morphisms of $\mathfrak{Sch}_S$, we have an isomorphism of functors

$$\phi_D \colon \mathcal{F}_{X'/X} \circ g^* \xrightarrow{\sim} f^* \circ \mathcal{F}_{T'/T}.$$

Proof. Since D is a fiber product diagram, for $U \in \mathfrak{Aff}_X$ we get another fiber product diagram

$$D_U = \begin{array}{ccc} X' \times_X U & \xrightarrow{g_U} & T' \times_T U \\ \downarrow{\scriptstyle h'_U} & & \downarrow{\scriptstyle h_U} \\ U & = & U \end{array}$$

where $\tilde{g}$ is an isomorphism. Now, let $\mathcal{M} \in \mathfrak{QCoh}(T')$ with its canonical isomorphisms ρ_f for morphisms $f \in \mathfrak{Aff}_{T'}$. We have an isomorphism

$$\phi_D(\mathcal{M})(U) \colon \mathcal{F}_{X'/X}(g^*(\mathcal{M}))(U) \xrightarrow{\sim} \mathcal{F}_{T'/T}(\mathcal{M})(U) = f^*(\mathcal{F}_{T'/T}(\mathcal{M}))(U)$$

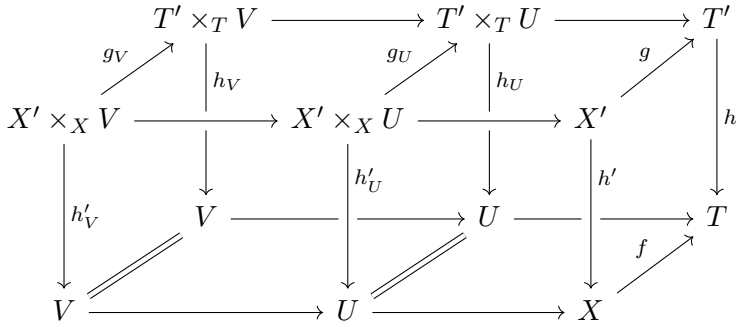
defined by

$$\begin{array}{ccc} \mathcal{F}_{h'_U}(\mathcal{M}(X' \times_X U)) & \xrightarrow{\mathcal{F}_{h'_U}(\rho_{g_U}^{-1})} & \mathcal{F}_{h'_U}(\mathcal{M}(T' \times_T U) \otimes_{\mathcal{O}(T' \times_T U)} \mathcal{O}(X' \times_X U)) \\ & \searrow^{\phi_D(\mathcal{M})(U)} & \downarrow{\scriptstyle \theta_{D_U}^{-1}} \\ & & \mathcal{F}_{h_U}(\mathcal{M}(T' \times_T U)) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \\ & & \downarrow{\scriptstyle \mathrm{can}} \\ & & \mathcal{F}_{h_U}(\mathcal{M}(T' \times_T U)). \end{array}$$

We check that these isomorphisms are compatible with the restriction along a morphism $V \to U$ of affine schemes in $\mathfrak{Aff}_X$, and hence via Lemma 1.3, give a well-defined isomorphism of sheaves

$$\phi_D(\mathcal{M}) \colon \mathcal{F}_{X'/X}(g^*(\mathcal{M})) \xrightarrow{\sim} f^*(\mathcal{F}_{T'/T}(\mathcal{M})).$$

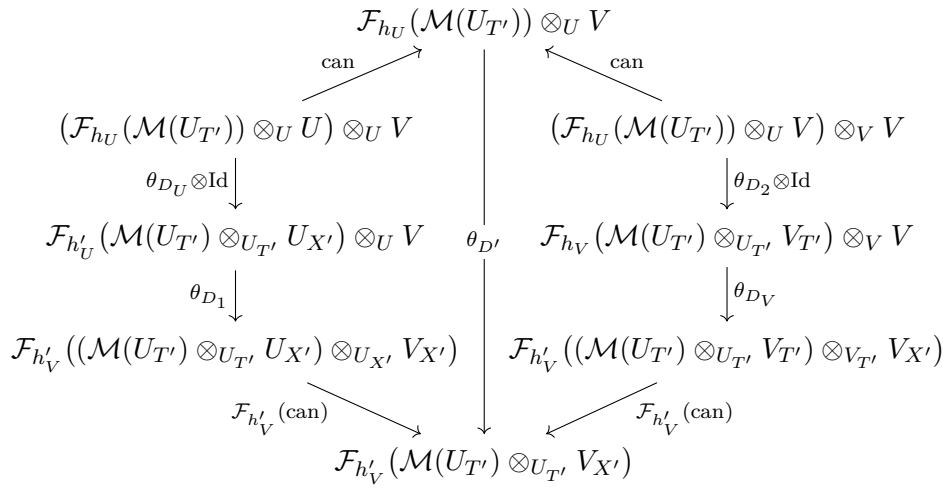
This also follows from constructing a large commutative diagram. First, we have a commutative diagram



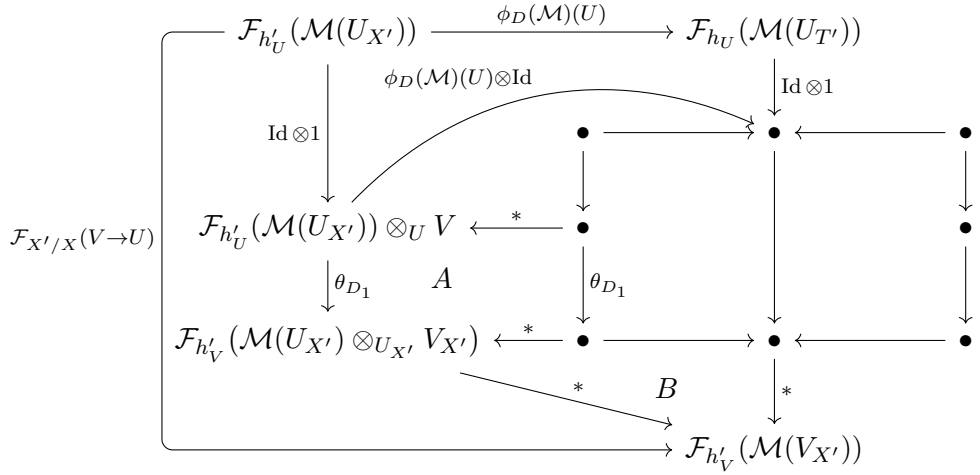
where the vertical faces are all pullback diagrams. The commutativity of the left cube means that we have an equality of pullback diagrams

$$
\begin{aligned}
D' = \quad & \begin{array}{ccccccc}
X' \times_X V & \longrightarrow & X' \times_X U & \xrightarrow{g_U} & T' \times_T U \\
\downarrow{\scriptstyle h'_V} & D_1 & \downarrow{\scriptstyle h'_U} & D_U & \downarrow{\scriptstyle h_U} \\
V & \longrightarrow & U & = & U
\end{array} \\
= \quad & \begin{array}{ccccccc}
X' \times_X V & \xrightarrow{g_V} & T' \times_T V & \longrightarrow & T' \times_T U \\
\downarrow{\scriptstyle h'_V} & D_V & \downarrow{\scriptstyle h_V} & D_2 & \downarrow{\scriptstyle h_U} \\
V & = & V & \longrightarrow & U.
\end{array}
\end{aligned}
$$

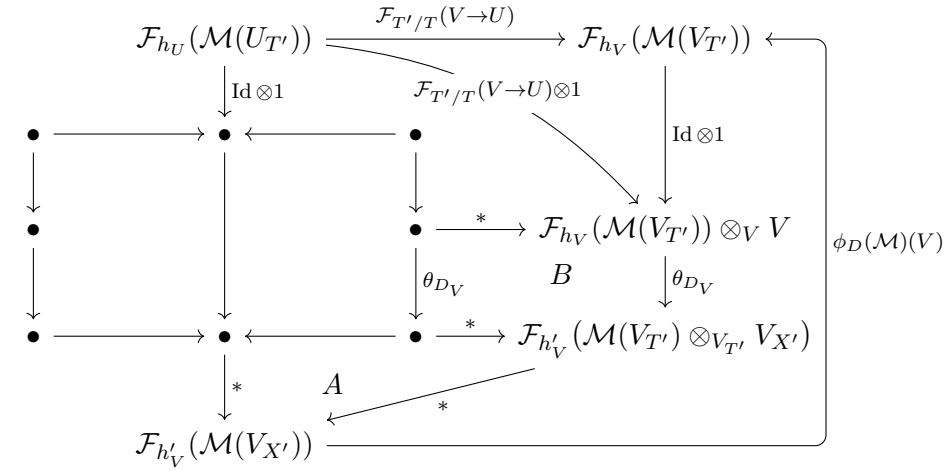
Therefore, our compatibility assumptions produce the following commutative diagram. To save space, we use the abbreviations $T' \times_T U = U_{T'}$, $X' \times_X U = U_{X'}$, and similarly for $V_{T'}$ and $V_{X'}$. We also use the abuse of notation $\_ \otimes_{\mathcal{O}(U)} \mathcal{O}(V) = \_ \otimes_U V$.



we extend the left side of this diagram (denoting its entries with $\bullet$ as before) to



where arrows labelled "$*$" are the appropriate isomorphisms induced by the quasi-coherence of $\mathcal{M}$. Square A commutes since θ_{D_1} is a natural transformation and square B commutes because $\mathcal{M}$ is quasi-coherent. The remaining new squares commute by definition. We also extend the right side of the original diagram to



where the square A commutes due to the quasi-coherence of $\mathcal{M}$, the square B commutes because θ_{D_V} is a natural transformation, and again the other new squares commute by definition. Considering the expanded diagram in its totality, the outermost path is the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_{X'/X}(g^*(\mathcal{M}))(U) & \xrightarrow{\phi_D(\mathcal{M})(U)} & f^*(\mathcal{F}_{T'/T}(\mathcal{M}))(U) \\
\downarrow{\scriptstyle \mathcal{F}_{X'/X}(V\to U)} & & \downarrow{\scriptstyle \mathcal{F}_{T'/T}(V\to U)} \\
\mathcal{F}_{X'/X}(g^*(\mathcal{M}))(V) & \xrightarrow{\phi_D(\mathcal{M})(V)} & f^*(\mathcal{F}_{T'/T}(\mathcal{M}))(V).
\end{array}$$

Since this is commutative for all morphisms $V \to U$, we have a well defined isomorphism of sheaves $\phi_D(\mathcal{M})$.

Now, we must check that these isomorphisms are functorial in $\mathcal{M}$ and thus provide our desired isomorphism of functors ϕ_D. Luckily, this follows directly from the fact that the various $\mathcal{F}_h$ for $h \in \mathfrak{I}$ are functors. No more large diagrams are needed, and we are done. □

C.10. **Proposition.** *The functors $\mathcal{F}_{T'/T}$ assemble into a morphism of stacks $\mathfrak{F}\colon \mathfrak{QCoh}_{\mathfrak{I}} \to \mathfrak{QCoh}$ defined by*

(i) $\mathfrak{F}(h\colon T' \to T, \mathcal{M}) = (T, \mathcal{F}_{T'/T}(\mathcal{M}))$ *on objects, and*
(ii) *for a morphism $(f, g, \varphi)\colon (h'\colon X' \to X, \mathcal{N}) \to (h\colon T' \to T, \mathcal{M})$, we have a pullback diagram*

$$D = \begin{array}{ccc} X' & \xrightarrow{g} & T' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & T \end{array}$$

and we set $\mathfrak{F}(f, g, \varphi) = (f, \mathfrak{F}(\varphi))$ where

$$\mathfrak{F}(\varphi)\colon \mathcal{F}_{X'/X}(\mathcal{N}) \xrightarrow{\mathcal{F}_{X'/X}(\varphi)} \mathcal{F}_{X'/X}(g^*(\mathcal{M})) \xrightarrow{\phi_D(\mathcal{M})} f^*(\mathcal{F}_{T'/T}(\mathcal{M}))$$

is an $\mathcal{O}|_X$–module morphism between the appropriate modules.

Proof. We must argue that $\mathfrak{F}$ is a well defined functor and that it preserves cartesian morphisms. Since it is already clear that it respects the structure functors, this will be sufficient to conclude that $\mathfrak{F}$ is a morphism of stacks.

Consider the identity morphism of an object $(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}_{\mathcal{M}})\colon (h\colon T' \to T, \mathcal{M}) \to (h\colon T' \to T, \mathcal{M})$ and its associated fiber product diagram D. Our assumption (C.8.c) implies that the isomorphism ϕ_D constructed in Lemma C.9 is the identity. Therefore, $\mathfrak{F}(\mathrm{Id}_{\mathcal{M}}) = \mathrm{Id}_{\mathcal{F}_{T'/T}(\mathcal{M})}$ as required.

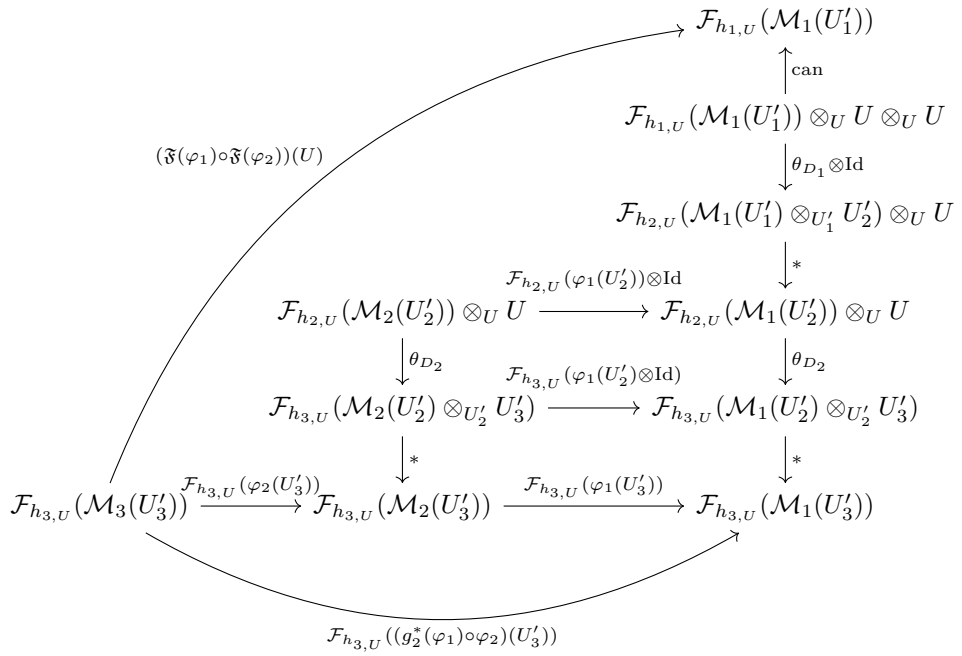
Assume we have a composition of morphisms

$$(h_3\colon T_3' \to T_3, \mathcal{M}_3) \xrightarrow{(f_2, g_2, \varphi_2)} (h_2\colon T_2' \to T_2, \mathcal{M}_2) \xrightarrow{(f_1, g_1, \varphi_1)} (h_1\colon T_1' \to T_1, \mathcal{M}_1).$$

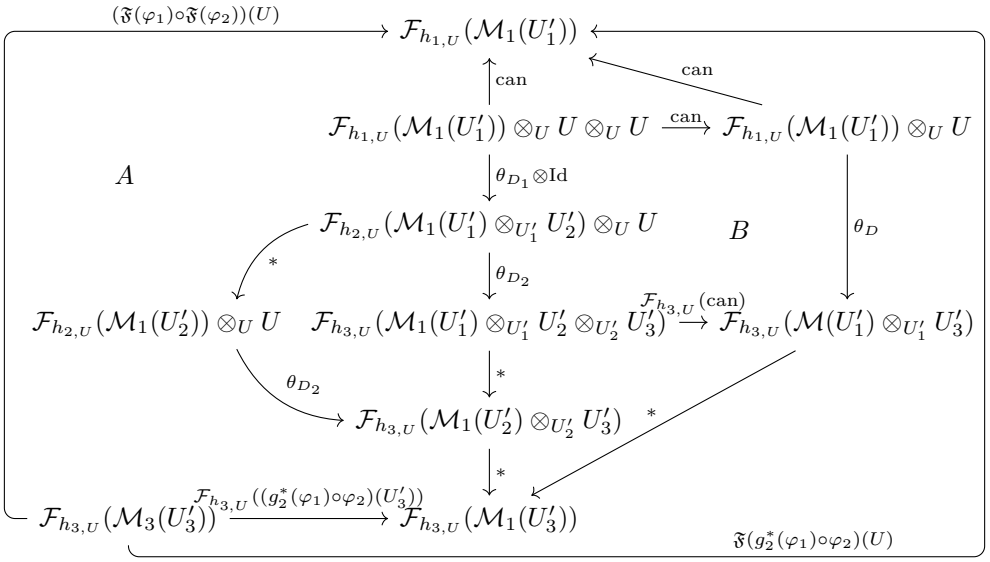
For $U \in \mathfrak{Aff}_{T_3}$, let $T_i' \times_{T_i} U = U_i'$. Then, we have fiber product diagrams

$$D = \begin{array}{ccccc} U_3' & \xrightarrow{g_{2,U}} & U_2' & \xrightarrow{g_{1,U}} & U_1' \\ \downarrow{\scriptstyle h_{3,U}} & D_2 & \downarrow{\scriptstyle h_{2,U}} & D_1 & \downarrow{\scriptstyle h_{1,U}} \\ U & = & U & = & U. \end{array}$$

The commutativity of the following diagram



implies the commutativity of face A in the diagram below



and face B commutes by assumption (C.8.d). This diagram implies that $(\mathfrak{F}(\varphi_1) \circ \mathfrak{F}(\varphi_2))(U) = \mathfrak{F}(g_2^*(\varphi_1) \circ \varphi_2)(U)$ for all $U \in \mathfrak{Aff}_{T_3}$ and therefore $\mathfrak{F}(\varphi_1) \circ \mathfrak{F}(\varphi_2) = \mathfrak{F}(g_2^*(\varphi_1) \circ \varphi_2)$ in general. This shows that $\mathfrak{F}$ respects composition and hence is a well-defined functor.

Finally, we address cartesian morphisms. Recall that by construction, all morphisms in $\mathfrak{QCoh}_{\mathfrak{I}}$ are cartesian. Since $\mathfrak{QCoh}$ is a substack of the stack $\mathfrak{Sh}$ of Example 1.12, a morphism $(f,\psi) \in \mathfrak{QCoh}$ is cartesian if and only if ψ is an isomorphism. Now, consider a morphism $(f,g,\varphi)\colon (h'\colon X' \to X, \mathcal{N}) \to (h\colon T' \to T, \mathcal{M})$ in $\mathfrak{QCoh}_{\mathfrak{I}}$. The map φ is an isomorphism by definition and the map ϕ_D constructed in Lemma C.9 is an isomorphism as well. Therefore, $\mathfrak{F}(\varphi) = \phi_D(\mathcal{M}) \circ \mathcal{F}_{X'/X}(\varphi)$ is an isomorphism and thus $\mathfrak{F}(f,g,\varphi) = (f, \mathfrak{F}(\varphi))$ is a cartesian morphism. This concludes the proof. □

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