



HAL
open science

Tight High Probability Bounds for Linear Stochastic Approximation with Fixed Stepsize

Alain Durmus, Eric Moulines, Alexey Naumov, Sergey Samsonov, Kevin Scaman, Hoi-To Wai

► **To cite this version:**

Alain Durmus, Eric Moulines, Alexey Naumov, Sergey Samsonov, Kevin Scaman, et al.. Tight High Probability Bounds for Linear Stochastic Approximation with Fixed Stepsize. Advances in Neural Information Processing Systems, Dec 2021, Online, France. pp.30063 - 30074, 10.5555/3540261.3542562 . hal-04399029

HAL Id: hal-04399029

<https://hal.science/hal-04399029>

Submitted on 12 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Tight High Probability Bounds for Linear Stochastic Approximation with Fixed Stepsize

Alain Durmus*

ENS Paris-Saclay

alain.durmus@ens-paris-saclay.fr

Eric Moulines

Ecole Polytechnique

and HSE University

eric.moulines@polytechnique.edu

Alexey Naumov

HSE University

anaumov@hse.ru

Sergey Samsonov

HSE University

svsamsonov@hse.ru

Kevin Scaman

INRIA, DI/ENS, PSL Research University

kevin.scaman@gmail.com

Hoi-To Wai

The Chinese University of Hong Kong

htwai@se.cuhk.edu.hk

Abstract

This paper provides a non-asymptotic analysis of linear stochastic approximation (LSA) algorithms with fixed stepsize. This family of methods arises in many machine learning tasks and is used to obtain approximate solutions of a linear system $\bar{A}\theta = \bar{b}$ for which \bar{A} and \bar{b} can only be accessed through random estimates $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$. Our analysis is based on new results regarding moments and high probability bounds for products of matrices which are shown to be tight. We derive high probability bounds on the performance of LSA under weaker conditions on the sequence $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$ than previous works. However, in contrast, we establish polynomial concentration bounds with order depending on the stepsize. We show that our conclusions cannot be improved without additional assumptions on the sequence of random matrices $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$, and in particular that no Gaussian or exponential high probability bounds can hold. Finally, we pay a particular attention to establishing bounds with sharp order with respect to the number of iterations and the stepsize and whose leading terms contain the covariance matrices appearing in the central limit theorems.

1 Introduction

This paper provides a detailed analysis of Linear Stochastic Approximation (LSA) schemes which aim at finding a solution θ^* for a linear system of the form $\bar{A}\theta = \bar{b}$. In particular, we analyze LSA with a fixed stepsize $\alpha > 0$ which consists in defining a sequence of estimates $\{\theta_n : n \in \mathbb{N}\}$ for θ^* by the recursion

$$\theta_{n+1} = \theta_n - \alpha\{\mathbf{A}_{n+1}\theta_n - \mathbf{b}_{n+1}\}, \quad n \in \mathbb{N}, \quad (1)$$

where $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$ is a sequence of i.i.d. random variables used as proxy for $\bar{A} \in \mathbb{R}^{d \times d}$ and $\bar{b} \in \mathbb{R}^d$ which are typically unknown. This class of algorithms and the corresponding setting have a long history and important applications in signal processing such as channel equalization and echo cancellation [3, 22]. It has renewed interests in machine learning and computational statistics

*Authors listed in alphabetical order.

especially for least-square estimation, Reinforcement learning (RL) and Q -learning [4, 7, 38, 35]. The recursion (1) has already been studied in depth in several works which derive asymptotic [31, 22, 6, 3] and non-asymptotic [33, 25, 2, 19, 20, 5, 23, 34, 9, 12] guarantees.

However, in most cases, there is a consistent gap between these two types of analyses. While asymptotic analysis gives important insights on the qualitative convergence of (1) based on statistical key quantities of the problem on hand, they do not provide finite-time convergence, or high probability bounds, necessary to obtain non-asymptotic confidence sets, see [26, 10] and the references therein. On the other hand, non-asymptotic studies are in general too coarse and lose significant statistical information in their derivation. Further, their upper bounds are generally loose when used in predicting the actual performance of LSA. We aim at filling this gap and provide conditions on $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$ ensuring tight high probability bounds on the sequence $\{\theta_n : n \in \mathbb{N}\}$.

This problem has been addressed in several contributions but at the expense of strong conditions on the sequence $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$. [13] provided concentration bounds for non-linear stochastic algorithms under a log-Sobolev condition which turns out to be hard to verify for most applications except for the Euler-Maruyama discretization scheme applied to Stochastic Differential Equation. [27] derived concentration inequalities but assuming that the innovations in (1) are uniformly bounded. In contrast, we aim at giving simple and mild conditions ensuring high probability bounds. More precisely, one of our key contributions (Theorem 1) is to show that under mild conditions on the sequence $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$, for any $\delta \in (0, 1)$, $n \in \mathbb{N}$ and $u \in \mathbb{S}^{d-1}$,

$$\mathbb{P}\left(|u^\top(\theta_n - \theta^*)| \leq c\{\sqrt{\alpha u^\top \Sigma u} + \alpha\}\sqrt{\log(1/\delta)} + c\{\rho_\alpha^n + \alpha p_0^2\}\delta^{-\frac{1}{p_0}}\right) \geq 1 - \delta, \quad (2)$$

where $\rho_\alpha \in (0, 1)$, $c > 0$ is a constant independent of n, α, δ , and $p_0 = o(\alpha^{-1/4})$. In the above, Σ is the unique solution of the Lyapunov equation which naturally appears in central limit theorems for LSA with diminishing stepsize. In addition, we show that the bound we get is tight with respect to α and δ in the case where we only assume that $-\mathbb{E}[\mathbf{A}_1] = -\bar{A}$ is Hurwitz. Indeed, we provide counterexamples illustrating that for a fixed stepsize α and under the conditions that we consider, logarithmic dependence in $1/\delta$ cannot hold in (2) but only a polynomial one. Regarding the dependence with respect to α , we extend [28] and show that for α small enough, $\{\theta_n : n \in \mathbb{N}\}$ admits a unique stationary distribution π_α and establish a central limit theorem for this family of distribution as $\alpha \downarrow 0$ at rate $\sqrt{\alpha}$ and with asymptotic covariance matrix Σ appearing in (2).

Finally, our proofs rely on a new analysis of product of matrices, extending the recent work of [17]. In particular, we establish conditions ensuring uniform bounds in n of the p -th moments of $\mathbf{Y}_n \cdots \mathbf{Y}_1$, where $\{\mathbf{Y}_n : n \in \mathbb{N}^*\}$ is a sequence of independent matrices whose expected values have a spectral radius less than 1. In comparison to existing results, the main challenge that we address here is that the random matrices $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$ are not assumed to be almost surely symmetric.

The paper is organized as follows. Section 2 formally discusses the assumptions on LSA for our analysis. Section 3 presents the moment bound for product of random matrices. Using this result, Section 4 shows the high probability concentration inequality (2) and Section 5 shows the tightness of the bounds by deriving a central limit theorem for LSA.

Notations Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_- = \mathbb{Z} \setminus \mathbb{N}^*$. Let $d \in \mathbb{N}^*$ and Q be a symmetric positive definite $d \times d$ matrix. For $x \in \mathbb{R}^d$, we denote $\|x\|_Q = \{x^\top Q x\}^{1/2}$. For brevity, we set $\|x\| = \|x\|_{I_d}$. We denote $\|A\|_Q = \max_{\|x\|_Q=1} \|Ax\|_Q$, and the subscriptless norm $\|A\| = \|A\|_I$ is the standard spectral norm. We denote the condition number of Q as $\kappa_Q = \lambda_{\min}^{-1}(Q)\lambda_{\max}(Q)$. We denote $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. Let A_1, \dots, A_N be d -dimensional matrices. We denote $\prod_{\ell=i}^j A_\ell = A_j \cdots A_i$ if $i \leq j$ and with the convention $\prod_{\ell=i}^j A_\ell = I_d$ if $i > j$. We say that a centered random variable (r.v.) X is sub-Gaussian with variance factor σ^2 and we denote $X \in \text{SG}(\sigma^2)$ if for all $\lambda \in \mathbb{R}$, $\log \mathbb{E}[e^{\lambda X}] \leq \lambda^2 \sigma^2 / 2$. We define the Wasserstein distance of order 2 between two probabilities measure μ and ν on \mathbb{R}^d as $W_2(\mu, \nu) = \inf_{\zeta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^{2d}} \|x - y\|^2 d\zeta(x, y)$, where $\Pi(\mu, \nu)$ is the set of probability measures on $(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))$ with marginals μ and ν respectively. Denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of all probability measures on \mathbb{R}^d with the finite second moment.

2 Linear Stochastic Approximation: Setting and Assumptions

Consider the LSA recursion (1) with a deterministic initial point θ_0 . The main assumption required in this paper is as follows:

A1. $\{(\mathbf{A}_n, \mathbf{b}_n)\}_{n \in \mathbb{N}^*}$ is an i.i.d. sequence satisfying the following conditions.

- (i) $\mathbb{E}[\mathbf{b}_1] = \bar{\mathbf{b}}$ and there exists $C_b > 0$ such that, for any $u \in \mathbb{S}^{d-1}$, $u^\top (\mathbf{b}_1 - \bar{\mathbf{b}}) \in \text{SG}(C_b^2)$.
- (ii) There exists $C_A > 0$ such that $\|\mathbf{A}_1\| \leq C_A$ almost surely.
- (iii) The matrix $-\bar{A} = -\mathbb{E}[\mathbf{A}_1]$ is Hurwitz, i.e. for any eigenvalue λ of \bar{A} , $\text{Re}(\lambda) > 0$.

Both conditions A1-(i), (ii) are standard in analysis of LSA, e.g., in [11, 34, 24]. Meanwhile, A1-(iii) guarantees the existence of a unique solution θ^* to $\bar{A}\theta = \bar{\mathbf{b}}$. It is also a sufficient and necessary condition for the solution of the ordinary differential equation $\dot{\theta}_t = -\bar{A}\theta_t$ to converge exponentially to θ^* [18, Lemma 4.1.2]. The same kind of result holds for the discrete system $\theta_{n+1}^d - \theta_n^d = -\alpha \bar{A} \theta_n^d$.

Proposition 1. Assume that $-\bar{A}$ is a Hurwitz matrix. Then there exists a unique positive definite matrix Q satisfying the Lyapunov equation $\bar{A}^\top Q + Q\bar{A} = \text{I}$. In addition, setting

$$a = \|Q\|^{-1}/2, \quad \text{and} \quad \alpha_\infty = (1/2)\|\bar{A}\|_Q^{-2}\|Q\|^{-1}, \quad (3)$$

then for any $\alpha \in [0, \alpha_\infty]$, we get $\|\text{I} - \alpha\bar{A}\|_Q^2 \leq 1 - a\alpha$. If in addition $\alpha \leq \|Q\|^2$ then $1 - a\alpha \geq 1/2$.

This result is well known but its proof can be found in Appendix A.1 for completeness. The above proposition implies that the discrete system converges exponentially as $\|\theta_{n+1}^d - \theta_n^d\| \leq \sqrt{\kappa_Q}(1 - a\alpha)^{n/2}\|\theta_0^d\|$ for $\alpha \in (0, \alpha_\infty)$.

Recall that the aim of this paper is to derive high probability bounds on $u^\top \{\theta_n - \theta^*\}$ for any $n \in \mathbb{N}$, $u \in \mathbb{S}^{d-1}$. Below, we present a counterexample to show that under only A1, if $\alpha > 0$ is fixed, then there exists $\bar{p} > 0$ such that $\lim_{n \rightarrow +\infty} \mathbb{E}[\|\theta_n - \theta^*\|^p] = +\infty$ for $p \geq \bar{p}$. As a corollary, it is impossible to obtain exponential high probability bounds for $\{\|\theta_n - \theta^*\| : n \in \mathbb{N}\}$.

Example 1. Consider (1) with $d = 1$ taking $\mathbf{b}_n = 0$ for any $n \in \mathbb{N}^*$ and for $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$ an i.i.d. sequence of biased Rademacher r.v.s with parameter $q_A \in (1/2, 1)$:

$$\mathbf{A}_n = \begin{cases} 1 & \text{with probability } q_A, \\ -1 & \text{with probability } 1 - q_A. \end{cases} \quad (4)$$

This choice is associated with $\theta^* = 0$ and corresponds to the recursion: $\theta_n = \prod_{k=1}^n (1 - \alpha \mathbf{A}_k) \theta_0$, for some $\theta_0 \neq 0$. For any $p \geq 1$ and $\alpha \in (0, 1)$, we have by definition,

$$\mathbb{E}[|\theta_n|^p] = \{q_A(1 - \alpha)^p + (1 - q_A)(1 + \alpha)^p\}^n |\theta_0|^p.$$

Using the lower bounds $(1 - \alpha)^p \geq 1 - \alpha p$ and $(1 + \alpha)^p \geq 1 + \alpha p + p(p - 1)\alpha^2/2$, we get for any $p \geq 1$ and $\alpha \in (0, 1)$,

$$\mathbb{E}[|\theta_n|^p] \geq \{1 - p\alpha[(2q_A - 1) - (p - 1)\alpha(1 - q_A)/2]\}^n |\theta_0|^p.$$

If $\alpha \in (0, 1)$ is fixed, then for any $p > \bar{p}_{q_A, \alpha} = 1 + 2(2q_A - 1)/[\alpha(1 - q_A)]$, we have $\lim_{n \rightarrow +\infty} \mathbb{E}[|\theta_n|^p] = +\infty$. On the other hand, if $\alpha \in (0, 2(2q_A - 1)/(1 - q_A))$, then $\lim_{n \rightarrow +\infty} \mathbb{E}[\theta_n^2] = 0$. Therefore $\{\theta_n : n \in \mathbb{N}\}$ converges in distribution to the Dirac measure at 0 which corresponds to the unique stationary distribution of this sequence as a Markov chain. In such a case, this distribution admit p moments for any $p \geq 0$.

However, this result is specific to this particular case and does not hold if only A1 holds. Consider $\{\theta_n : n \in \mathbb{N}\}$ defined by (1) with $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$ given in (4) and $\{\mathbf{b}_n : n \in \mathbb{N}^*\}$ be an i.i.d. sequence of zero-mean Gaussian random variables with unit variance independent of $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$. We show in Appendix A.2 that there exists $\alpha_{2, \infty}$ such that for any $\alpha \in (0, \alpha_{2, \infty}]$, the Markov chain $\{\theta_n : n \in \mathbb{N}\}$ admits a unique invariant distribution π_α for any $\alpha > 0$. Further, for any $\alpha \in (0, \alpha_{2, \infty}]$ there exists $p_\alpha \geq 1$ such that $\int_{\mathbb{R}} |\theta|^p d\pi_\alpha(\theta) = +\infty$ for any $p \geq p_\alpha$.

It is, however, possible to obtain any p -th moment uniform bound for $\{\|\theta_n - \theta^*\| : n \in \mathbb{N}\}$ by strengthening A1-(iii) to:

A2. There exist $\tilde{\alpha} \in (0, 1)$, $\tilde{\alpha}_\infty > 0$ and a definite positive d -dimensional matrix \tilde{Q} such that almost surely, for any $\alpha \in (0, \tilde{\alpha}_\infty]$, $\|\mathbf{I} - \alpha \mathbf{A}_1\|_{\tilde{Q}} < 1 - \tilde{\alpha}\alpha$.

Examples such that **A2** holds include regularized linear regression, where we take $\mathbf{A}_1 = \lambda \mathbf{I} + \mathbf{a}_1 \mathbf{a}_1^\top$, for some $\lambda > 0$ and under the assumption that $\|\mathbf{a}_1\|$ is bounded almost surely. The LSA recursion **(1)** approximates the solution to $(\lambda \mathbf{I} + \mathbb{E}[\mathbf{a}_1 \mathbf{a}_1^\top])\theta = \bar{b}$ which is guaranteed to have a unique solution.

On the other hand, examples where **A2** does not hold are common. For this, we consider TD(0) learning with linear function approximation. For a Markov Reward Process with \mathbf{X} as the state space, $P : \mathbf{X} \times \mathcal{X} \rightarrow [0, 1]$ as the transition probability, $R : \mathbf{X} \rightarrow \mathbb{R}$ as the reward function, and $\gamma \in (0, 1)$ as a discount factor, TD(0) learning is described as in **(1)** with

$$\mathbf{A}_n = \phi(x_n) \{ \phi(x_n) - \gamma \phi(x'_n) \}^\top, \quad \mathbf{b}_n = R(x_n) \phi(x_n), \quad (5)$$

where $\phi : \mathbf{X} \rightarrow \mathbb{R}^d$ is a feature map. A typical setting is when x_n is drawn from the stationary distribution of P and $x'_n \sim P(x_n, \cdot)$. It is easy to verify **A1** provided that $\|\phi(x)\|$, $R(x)$ are bounded for all $x \in \mathbf{X}$ [36]. However, **A2** is violated as \mathbf{A}_n is only rank-one.

Our next endeavor is to establish moment estimates on the product below:

$$\Gamma_{m:n}^{(\alpha)} = \prod_{i=m}^n (\mathbf{I} - \alpha \mathbf{A}_i), \quad m, n \in \mathbb{N}^*, \quad m \leq n. \quad (6)$$

We also define its expected value as $G_{m:n}^{(\alpha)} = \mathbb{E}[\Gamma_{m:n}^{(\alpha)}] = (\mathbf{I} - \alpha \bar{A})^{n-m+1}$.

The above product naturally appears after re-centering the LSA recursion **(1)**. For any $n \in \mathbb{N}^*$,

$$\theta_n - \theta^* = (\mathbf{I} - \alpha \mathbf{A}_n) \{ \theta_{n-1} - \theta^* \} + \alpha \varepsilon_n, \quad \varepsilon_n = \mathbf{b}_n - \bar{b} - \{ \mathbf{A}_n - \bar{A} \} \theta^*. \quad (7)$$

An easy induction implies that

$$\theta_n - \theta^* = \tilde{\theta}_n^{(\text{tr})} + \tilde{\theta}_n^{(\text{fl})}, \quad \tilde{\theta}_n^{(\text{tr})} = \Gamma_{1:n}^{(\alpha)} \{ \theta_0 - \theta^* \}, \quad \tilde{\theta}_n^{(\text{fl})} = \alpha \sum_{j=1}^n \Gamma_{j+1:n}^{(\alpha)} \varepsilon_j. \quad (8)$$

The decomposition **(8)** highlights the two sources of error in the estimation of θ^* by $\{\theta_n : n \in \mathbb{N}\}$ which will be separately tackled: $\{\tilde{\theta}_n^{(\text{tr})} : n \in \mathbb{N}\}$ corresponds to the transient (or bias) term and $\{\tilde{\theta}_n^{(\text{fl})} : n \in \mathbb{N}\}$ to the fluctuation term. Both errors are controlled by the product of matrices $\Gamma_{m:n}^{(\alpha)}$, thereby motivating the study of the moment bound on $\Gamma_{1:n}^{(\alpha)}$ as we present next.

3 Moment and High-probability Bounds for Products of Random Matrices

Recall from Proposition **1** that the expected value $G_{1:n}^{(\alpha)} = \mathbb{E}[\Gamma_{1:n}^{(\alpha)}]$ decays exponentially with n , here we expect a similar phenomenon to hold for the moment bound of $\Gamma_{1:n}^{(\alpha)}$. Precisely, in this section, we show that if p is fixed, then there exists $\alpha_p > 0$ such that for any $\alpha \in (0, \alpha_p]$, the p -th moment of $\Gamma_{m:n}^{(\alpha)}$ decays exponentially with $n - m$.

To facilitate our discussions, we introduce the following notations. For $B \in \mathbb{R}^{d \times d}$, we denote by $(\sigma_\ell(B))_{\ell=1}^d$ its singular values. For $p \geq 1$, the Schatten p -norm is denoted by $\|B\|_p = \{ \sum_{\ell=1}^d \sigma_\ell^p(B) \}^{1/p}$. For $p, q \geq 1$ and random matrix \mathbf{X} , we write $\|\mathbf{X}\|_{p,q} = \{ \mathbb{E}[\|\mathbf{X}\|_p^q] \}^{1/q}$.

In the following, we present the main technical result on the product of general random matrices. The proof is based on the framework introduced in [17].

Proposition 2. Let $\{\mathbf{Y}_\ell : \ell \in \mathbb{N}\}$ be an independent sequence and P be a positive definite matrix. Assume that for each $\ell \in \mathbb{N}$ there exist $m_\ell \in (0, 1)$ and $\sigma_\ell > 0$ such that $\|\mathbb{E}[\mathbf{Y}_\ell]\|_p^2 \leq 1 - m_\ell$ and $\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_p \leq \sigma_\ell$ almost surely. Define $\mathbf{Z}_n = \prod_{\ell=0}^n \mathbf{Y}_\ell = \mathbf{Y}_n \mathbf{Z}_{n-1}$, for $n \geq 1$ and starting from \mathbf{Z}_0 . Then, for any $2 \leq q \leq p$ and $n \geq 1$,

$$\|\mathbf{Z}_n\|_{p,q}^2 \leq \kappa_P \prod_{\ell=1}^n (1 - m_\ell + (p-1)\sigma_\ell^2) \|P^{1/2} \mathbf{Z}_0 P^{-1/2}\|_{p,q}^2, \quad (9)$$

where $\kappa_P = \lambda_{\min}^{-1}(P) \lambda_{\max}(P)$.

Proof of Proposition 2. Let $2 \leq q \leq p$. Consider the following decomposition $\mathbf{Z}_n = \mathbf{Y}_n \mathbf{Z}_{n-1} = (\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]) \mathbf{Z}_{n-1} + \mathbb{E}[\mathbf{Y}_n] \mathbf{Z}_{n-1}$, Therefore, we obtain for any $n \in \mathbb{N}$,

$$f_P(\mathbf{Z}_n) = \mathbf{A}_n + \mathbf{B}_n, \quad \mathbf{A}_n = f_P((\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]) \mathbf{Z}_{n-1}), \quad \mathbf{B}_n = f_P(\mathbb{E}[\mathbf{Y}_n]) f_P(\mathbf{Z}_{n-1}),$$

where $f_P : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$ is defined for any $B \in \mathbb{R}^{d \times d}$ by $f_P(B) = P^{1/2} B P^{-1/2}$. Since $\mathbb{E}[\mathbf{A}_n | \mathbf{B}_n] = 0$, [17, Proposition 4.3] (see Proposition 10 in Appendix B) implies that

$$\|f_P(\mathbf{Z}_n)\|_{p,q}^2 \leq \|\mathbf{B}_n\|_{p,q}^2 + (p-1) \|\mathbf{A}_n\|_{p,q}^2. \quad (10)$$

It remains to bound the two terms on the right hand side. To this end, we use [16, Theorem 6.20] which implies that for any $B_1, B_2 \in \mathbb{R}^{d \times d}$,

$$\|B_1 B_2\|_{p,q} \leq \|B_1\| \|B_2\|_{p,q}. \quad (11)$$

As a result and using that for any $B \in \mathbb{R}^{d \times d}$, $\|B\|_P = \|f_P(B)\|$, and $\|\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]\|_P \leq \sigma_n$ we get

$$\begin{aligned} \|\mathbf{A}_n\|_{p,q} &= (\mathbb{E} [\|f_P(\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]) f_P(\mathbf{Z}_{n-1})\|_p^q])^{1/q} \\ &\leq (\mathbb{E} [\|\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]\|_P^q \|f_P(\mathbf{Z}_{n-1})\|_p^q])^{1/q} \leq \sigma_n \|f_P(\mathbf{Z}_{n-1})\|_{p,q}. \end{aligned} \quad (12)$$

Similarly, applying $\|\mathbb{E}[\mathbf{Y}_n]\|_P^2 \leq 1 - m_n$

$$\begin{aligned} \|\mathbf{B}_n\|_{p,q}^2 &= (\mathbb{E} [\|f_P(\mathbb{E}[\mathbf{Y}_n]) f_P(\mathbf{Z}_{n-1})\|_p^q])^{2/q} \\ &\leq (\mathbb{E} [\|\mathbb{E}[\mathbf{Y}_n]\|_P^q \|f_P(\mathbf{Z}_{n-1})\|_p^q])^{2/q} \leq (1 - m_n) \|f_P(\mathbf{Z}_{n-1})\|_{p,q}^2. \end{aligned} \quad (13)$$

Combining (12) and (13) in (10) yields for any $n \in \mathbb{N}^*$, $\|f_P(\mathbf{Z}_n)\|_{p,q}^2 \leq (1 - m_n + (p-1)\sigma_n^2) \|f_P(\mathbf{Z}_{n-1})\|_{p,q}^2 \leq \prod_{i=1}^n (1 - m_i + (p-1)\sigma_i^2) \|f_P(\mathbf{Z}_0)\|_{p,q}^2$. The proof is then completed upon using (11) which implies that $\|\mathbf{Z}_n\|_{p,q} = \|P^{-1/2} f_P(\mathbf{Z}_n) P^{1/2}\|_{p,q} \leq \sqrt{\kappa_P} \|f_P(\mathbf{Z}_n)\|_{p,q}$. \square

In order to bound $\Gamma_{1:n}^{(\alpha)}$ using Proposition 2, we identify the latter with $\mathbf{Y}_\ell = \mathbf{I} - \alpha \mathbf{A}_\ell$, $\ell \geq 1$, $\mathbf{Y}_0 = \mathbf{I}$. As $-\bar{A}$ is Hurwitz, applying Proposition 1 yields $\|\mathbb{E}[\mathbf{Y}_\ell]\|_Q^2 = \|\mathbf{I} - \alpha \bar{A}\|_Q^2 \leq 1 - \alpha a$. Further, A 1-(ii) ensures that almost surely,

$$\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_Q = \alpha \|\mathbf{A}_\ell - \bar{A}\|_Q \leq 2\alpha \sqrt{\kappa_Q} C_A = b_Q \alpha.$$

Therefore, (9) holds with $m_\ell = \alpha a$ and $\sigma_\ell = b_Q \alpha$. Noting that as $\|\mathbf{I}\|_p = d^{1/p}$, we obtain the following corollary.

Corollary 1. *Assume A1-(ii)-(iii). Then, for any $\alpha \in [0, \alpha_\infty]$, $2 \leq q \leq p$, and $n \in \mathbb{N}$,*

$$\mathbb{E}^{1/q} \left[\|\Gamma_{1:n}^{(\alpha)}\|^q \right] \leq \|\Gamma_{1:n}^{(\alpha)}\|_{p,q} \leq \sqrt{\kappa_Q} d^{1/p} (1 - \alpha a + (p-1) b_Q^2 \alpha^2)^{n/2}, \quad (14)$$

where α_∞ was defined in (3), and

$$b_Q = 2\sqrt{\kappa_Q} C_A. \quad (15)$$

Note that Corollary 1 shows $\sup_{n \in \mathbb{N}} \mathbb{E}[\|\Gamma_{1:n}^{(\alpha)}\|^p] < +\infty$ for any $\alpha \in (0, \alpha_{p,\infty}]$, where

$$\alpha_{p,\infty} = \alpha_\infty \wedge a / (2b_Q^2(p-1)). \quad (16)$$

This kind of condition relating the choice α with the order p of the moment to be bounded is necessary as illustrated in Example 1. The above corollary further leads to the high-probability bound:

Corollary 2. *Assume A1-(ii)-(iii). Then, for any $\alpha \in (0, \alpha_\infty)$ where α_∞ was defined in (3), $\delta \in (0, 1)$ and $n \in \mathbb{N}$, with probability at least $1 - \delta$,*

$$\|\Gamma_{1:n}^{(\alpha)}\| \leq \sqrt{\kappa_Q} \exp \left[-(a\alpha - \alpha^2 b_Q^2 n) / 2 + b_Q \alpha \sqrt{2n \log(d/\delta)} \right].$$

Proof. The result follows from combining Corollary 1 with $p = q$ and Lemma 1 in Appendix B applied with $A = (-\log(\kappa_Q) + \alpha a n + b_Q^2 \alpha^2 n) / 2$, $B = \alpha^2 b_Q^2 n / 2$ and $C = d$, $p_0 = 2$, $p_1 = +\infty$. \square

The result which we obtain in Corollary 2 is tight with respect to δ , as illustrated via the following example that continues from Example 1.

Example (Continuation of Example 1). Consider $\{\theta_n : n \in \mathbb{N}\}$ defined by (1) with $\{\mathbf{A}_n : n \in \mathbb{N}^*\}$ given in (4) and $\mathbf{b}_n = 0$ for any $n \in \mathbb{N}^*$. Define

$$\varphi_q(\alpha) = q_A \log\left(\frac{1+\alpha}{1-\alpha}\right) - \log(1+\alpha), \quad \bar{\alpha}_q = \sup\{\bar{\alpha} > 0 : \varphi_q(\alpha) > 0, \forall \alpha \in (0, \bar{\alpha})\}. \quad (17)$$

Note that $\varphi_q(\alpha) \sim \alpha(2q_A - 1)$ as $\alpha \downarrow 0$. Therefore since $q_A > 1/2$, $\{\bar{\alpha} > 0 : \varphi_q(\alpha) > 0 \text{ for any } \alpha \in (0, \bar{\alpha})\} \neq \emptyset$ and $\bar{\alpha}_q$ is well-defined. Consider also $\tilde{\varphi}_q(\alpha) = \varphi_q(\alpha) \log^{-1}[(1+\alpha)/(1-\alpha)]$. Then, we show in Appendix B that for any $\bar{\delta} \in (e^{-2n\tilde{\varphi}_q(\alpha)}, 1)$ and $\underline{\delta} \in (e^{-n\tilde{\varphi}_q(\alpha)/(q_A(1-q_A)) - 2^{-1} \log(n)}, 1)$,

$$\mathbb{P}\left(\theta_n \geq \exp\left(-\varphi_q(\alpha)n + \log\left(\frac{1+\alpha}{1-\alpha}\right) \sqrt{\frac{n \log(1/\bar{\delta})}{2}}\right)\right) \leq \bar{\delta}, \quad (18)$$

$$\mathbb{P}\left(\theta_n \geq \exp\left(-\varphi_q(\alpha)n + \log\left(\frac{1+\alpha}{1-\alpha}\right) \sqrt{nq_A(1-q_A) \log(1/\underline{\delta}) + \frac{n \log(n)}{2}}\right)\right) \geq \underline{\delta}. \quad (19)$$

Note that the bound given by (18) and (19) shows that the tail distribution associated with θ_n behaves as a log-normal one. Indeed, if ξ is a zero-mean one dimensional Gaussian distribution with unit variance, then an easy computation shows that for any $\sigma > 0$, $\mathbb{P}(e^{\sigma\xi} \geq t) \sim (2\pi\sigma^2)^{-1/2} \log^{-1}(t) \exp(-(2\sigma^2)^{-1}t^2)$ as $t \rightarrow \infty$, therefore, to have $\mathbb{P}(e^{\sigma\xi} \geq t_\delta) \leq \delta$ for a small $\delta > 0$, then t_δ has to be of order $\exp(\sigma\sqrt{\log(1/\delta)})$.

We conclude the section with a complementary result of Corollary 1 that does not require A1-(ii):

Proposition 3. Assume A1-(iii), $\|\mathbf{A}_1 - \bar{\mathbf{A}}\| \in \text{SG}(C'_A)$ for some $C'_A > 0$. Then, for any $\alpha \in (0, \alpha_\infty)$ where α_∞ was defined in (3), $2 \leq q \leq p$, and $n \in \mathbb{N}$,

$$\mathbb{E}^{1/q} \left[\|\Gamma_{1:n}^{(\alpha)}\|^q \right] \leq \|\Gamma_{1:n}^{(\alpha)}\|_{p,q} \leq \sqrt{\kappa_Q} d^{1/p} (1 - a\alpha + q(p-1)(b'_Q)^2 \alpha^2)^{n/2}, \quad (20)$$

where $b'_Q = 2\sqrt{\kappa_Q} C'_A$.

The proof is similar to that of Proposition 2 and it can be found in Appendix B.

4 Finite-time High-probability Bounds for LSA

Relying on the results established in Section 3 and the decomposition (8), we derive in this section high probability bounds on $u^\top \{\theta_n - \theta^*\}$ for any $n \in \mathbb{N}$ and $u \in \mathbb{S}^{d-1}$, where $\{\theta_n : n \in \mathbb{N}\}$ is defined in (1). We begin our study with the transient term $\tilde{\theta}_n^{(\text{tr})}$ defined in (8). The proof of the following statement is given in Appendix C.1.

Proposition 4. Assume A1 and let $p_0 \geq 2$. Then, for any $n \in \mathbb{N}^*$, $\alpha \in (0, \alpha_{p_0, \infty})$, where $\alpha_{p_0, \infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1)$ it holds with probability at least $1 - \delta$ that

$$|u^\top \Gamma_{1:n}^{(\alpha)}(\theta_0 - \theta^*)| \leq \sqrt{\kappa_Q} d^{1/p_0} (1 - a\alpha/4)^n \|\theta_0 - \theta^*\| \delta^{-1/p_0},$$

where a was defined in (3).

Proposition 4 only provides a polynomial high probability bound with respect to δ . This is due to the fact that only polynomial moments of $\|\Gamma_{1:n}^{(\alpha)}\|$ up to a maximal order are uniformly bounded in the number of iterations n .

We now turn on the fluctuation term $\tilde{\theta}_n^{(\text{fl})}$ defined in (8). We note that under A1, the sequence $\{\varepsilon_n : n \in \mathbb{N}\}$ defined in (7) is i.i.d.. From this observation and following [12], we consider the decomposition

$$\tilde{\theta}_n^{(\text{fl})} = \alpha \sum_{j=1}^n \Gamma_{j+1:n}^{(\alpha)} \varepsilon_j = J_n^{(\alpha, 0)} + H_n^{(\alpha, 0)}, \quad (21)$$

where $\{(J_n^{(\alpha,0)}, H_n^{(\alpha,0)}) : n \in \mathbb{N}\}$ are defined by induction for $n \geq 0$ as:

$$\begin{aligned} J_{n+1}^{(\alpha,0)} &= (\mathbf{I} - \alpha \bar{A}) J_n^{(\alpha,0)} + \alpha \varepsilon_{n+1}, & J_0^{(\alpha,0)} &= 0, \\ H_{n+1}^{(\alpha,0)} &= (\mathbf{I} - \alpha \mathbf{A}_n) H_n^{(\alpha,0)} - \alpha (\mathbf{A}_{n+1} - \bar{A}) J_n^{(\alpha,0)}, & H_0^{(\alpha,0)} &= 0. \end{aligned} \quad (22)$$

The latter recurrence can be written as

$$J_n^{(\alpha,0)} = \alpha \sum_{j=1}^n G_{j+1:n}^{(\alpha)} \varepsilon_j, \quad H_n^{(\alpha,0)} = -\alpha \sum_{j=1}^n \Gamma_{j+1:n}^{(\alpha)} (\mathbf{A}_j - \bar{A}) J_{j-1}^{(\alpha,0)}.$$

Note that $J_n^{(\alpha,0)}$ is a linear statistics of the random variables $\{\varepsilon_j : j \in \{1, \dots, n\}\}$ which are centered and i.i.d. under A 1. In our next results, we show that $J_n^{(\alpha,0)}$ is the leading term as the stepsize $\alpha \downarrow 0$. Denote for any $n \in \mathbb{N}^*$ and $\alpha > 0$, the covariance matrix of $J_n^{(\alpha,0)}$ as

$$\Sigma_n^\alpha = \text{Cov}(J_n^{(\alpha,0)}). \quad (23)$$

Proposition 5. *Assume A 1. Then for any $n \in \mathbb{N}^*$, $\alpha \in (0, \alpha_\infty]$, where α_∞ is defined in (3), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1)$, it holds with probability at least $1 - \delta$,*

$$|u^\top J_n^{(\alpha,0)}| < D_1 \sqrt{\{u^\top \Sigma_n^\alpha u\} \log(2/\delta)} + \alpha \sqrt{1 + \log(1/(a\alpha))} D_2 \log^{3/2}(2/\delta), \quad (24)$$

where $D_1 = 60\sqrt{3}e^{4/3}$ and D_2 is defined in (50).

The proof of Proposition 5 is postponed to Appendix C.2.

We analyze further the covariance associated with $J_n^{(\alpha,0)}$ and its dependence with respect to n and α . First, note that for any $\alpha \in (0, \alpha_{2,\infty}]$, $\{\Sigma_n^\alpha : n \in \mathbb{N}^*\}$ converges to $\alpha \Sigma^\alpha$ as $n \rightarrow \infty$ where $\Sigma^\alpha = \alpha \sum_{k=0}^\infty G_{1:k} \Sigma_\varepsilon G_{1:k}^\top$ is the unique solution of the Ricatti equation

$$\bar{A} \Sigma^\alpha + \Sigma^\alpha \bar{A}^\top - \alpha \bar{A} \Sigma^\alpha \bar{A}^\top = \Sigma_\varepsilon, \quad \text{with } \Sigma_\varepsilon = \mathbb{E}[\varepsilon_1 \varepsilon_1^\top]. \quad (25)$$

Indeed, using Proposition 1, we easily get that for any $n \geq 0$,

$$\|\Sigma_n^\alpha - \alpha \Sigma^\alpha\| \leq \alpha^2 \sum_{k>n} \|G_{1:k}\|^2 \|\Sigma_\varepsilon\| \leq \alpha a^{-1} \kappa_Q \|\Sigma_\varepsilon\| (1 - \alpha a)^n. \quad (26)$$

We now give an expansion of Σ^α with respect to α . It is well-known that as $\alpha \downarrow 0$, Σ^α converges to Σ , the unique solution of the Lyapunov equation (see [32, Lemma 9.1])

$$\bar{A} \Sigma + \Sigma \bar{A}^\top = \Sigma_\varepsilon. \quad (27)$$

Our next result states this convergence is of the order of the stepsize α .

Proposition 6. *Assume that A 1-(iii) holds. Then, for any $\alpha \in (0, \alpha_\infty]$, where α_∞ is defined in (3),*

$$\|\Sigma^\alpha - \Sigma\|_Q \leq \alpha a^{-1} \|\bar{A} \Sigma \bar{A}^\top\|_Q,$$

where Σ^α and Σ are defined in (25) and (27) respectively and a is given in (3).

The proof is given in Appendix C.3. The last step in bounding $\tilde{\theta}_n^{(\text{fl})}$ is to consider $H_n^{(\alpha,0)}$. We proceed similarly to (22) and consider the decomposition $H_n^{(\alpha,0)} = J_n^{(\alpha,1)} + H_n^{(\alpha,1)}$, where $\{(J_n^{(\alpha,1)}, H_n^{(\alpha,1)}) : n \in \mathbb{N}\}$ are defined by induction for $n \geq 0$ as:

$$\begin{aligned} J_{n+1}^{(\alpha,1)} &= (\mathbf{I} - \alpha \bar{A}) J_n^{(\alpha,1)} - \alpha (\mathbf{A}_{n+1} - \bar{A}) J_n^{(\alpha,0)}, & J_0^{(\alpha,1)} &= 0, \\ H_{n+1}^{(\alpha,1)} &= (\mathbf{I} - \alpha \mathbf{A}_{n+1}) H_n^{(\alpha,1)} - \alpha (\mathbf{A}_{n+1} - \bar{A}) J_n^{(\alpha,1)}, & H_0^{(\alpha,1)} &= 0. \end{aligned} \quad (28)$$

In our next results, we bound each term of this decomposition separately.

Proposition 7. *Assume A 1 and let $p_0 \geq 2$. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_{p_0,\infty})$, where $\alpha_{p_0,\infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1/2)$, with probability at least $1 - 2\delta$, it holds*

$$|u^\top J_n^{(\alpha,1)}| < e D_3 \alpha \log^2(1/\delta), \quad |u^\top H_n^{(\alpha,1)}| < D_4 \alpha p_0^2 \delta^{-1/p_0}, \quad (29)$$

where D_3 and D_4 are given in (58) and (61), respectively.

The proof of Proposition 7 is postponed to Appendix C.4. Now we are ready to combine the previous bounds and to state the main result of this section.

Theorem 1. *Assume A 1 and let $p_0 \geq 2$. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_{p_0, \infty})$, where $\alpha_{p_0, \infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1/4)$, with probability at least $1 - 4\delta$, it holds*

$$\alpha^{-1/2} |u^\top (\theta_n - \theta^*)| < D_1 \sqrt{\{u^\top \Sigma^\alpha u\} \log(2/\delta)} + \alpha^{1/2} q^{(1)}(\alpha, \delta) + (1 - a\alpha/4)^n \Delta^{(1)}(\alpha, \delta), \quad (30)$$

where Σ^α is the unique solution of (25), $D_1 = 60\sqrt{3}e^{4/3}$, a is defined in (3),

$$\begin{aligned} q^{(1)}(\alpha, \delta) &= (eD_3 \log^2(1/\delta) + \sqrt{1 + \log(1/a\alpha)} D_2 \log^{3/2}(2/\delta)) + D_4 p_0^2 \delta^{-1/p_0}, \\ \Delta^{(1)}(\alpha, \delta) &= D_1 \sqrt{a^{-1} \kappa_Q \|\Sigma_\varepsilon\| \log(2/\delta)} + \sqrt{\kappa_Q} d^{1/p_0} \|\theta_0 - \theta^*\| \alpha^{-1/2} \delta^{-1/p_0}, \end{aligned} \quad (31)$$

where κ_Q and Σ_ε are defined in (3) and (25) respectively.

Proof. The proof follows from the decomposition

$$u^\top (\theta_n - \theta^*) = u^\top \Gamma_{1:n}^{(\alpha)}(\theta_0 - \theta^*) + u^\top J_n^{(\alpha, 0)} + u^\top J_n^{(\alpha, 1)} + u^\top H_n^{(\alpha, 1)},$$

where $J_n^{(\alpha, 0)}$, $J_n^{(\alpha, 1)}$ and $H_n^{(\alpha, 1)}$ are defined in (22)-(28), the union bound and Proposition 4, Proposition 5, (26) and Proposition 7. \square

We now discuss the high probability bound (30). First, note that the term $\Delta^{(1)}(\alpha, \delta)$, and in particular the initial condition vanishes exponentially fast in the number of iterations n . In addition, $q^{(1)}(\alpha, \delta)$ and $\Delta^{(1)}(\alpha, \delta)$ are of order δ^{-1/p_0} as $\delta \rightarrow 0$ and therefore (30) provides polynomial high probability bounds on LSA. However, this conclusion is expected as illustrated in Example 1. Finally, the discussion of (30) with respect to α is postponed to the next section.

Under A2 we can provide a better bound for $H_n^{(\alpha, 1)}$.

Proposition 8. *Assume A1 and A2. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_\infty \wedge \tilde{\alpha}_\infty)$, where α_∞ is defined in (3), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1/2)$, with probability at least $1 - 2\delta$, it holds*

$$|u^\top J_n^{(\alpha, 1)}| < eD_3 \alpha \log^2(1/\delta), \quad |u^\top H_n^{(\alpha, 1)}| < eD_5 \alpha \log^2(1/\delta), \quad (32)$$

where D_3 and D_5 are given in (58) and (62) respectively.

As a result, we can establish exponential high probability bounds with respect to δ .

Theorem 2. *Assume A1 and A2. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_\infty \wedge \tilde{\alpha}_\infty)$, $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1/4)$, with probability at least $1 - 4\delta$, it holds*

$$\alpha^{-1/2} |u^\top (\theta_n - \theta^*)| < D_1 \sqrt{\{u^\top \Sigma^\alpha u\} \log(2/\delta)} + \alpha^{1/2} q^{(2)}(\alpha, \delta) + (1 - \alpha\tilde{a})^{n/2} \Delta^{(2)}(\alpha, \delta),$$

where $D_1 = 60\sqrt{3}e^{4/3}$, Σ^α is solution of (25),

$$\begin{aligned} q^{(2)}(\alpha, \delta) &= e(D_3 + D_5) \log^2(1/\delta) + \sqrt{1 + \log(1/\tilde{a}\alpha)} D_2 \log^{3/2}(2/\delta), \\ \Delta^{(2)}(\alpha, \delta) &= D_1 \sqrt{\tilde{a}^{-1} \kappa_{\tilde{Q}} \|\Sigma_\varepsilon\| \log(2/\delta)} + \kappa_{\tilde{Q}}^{1/2} \|\theta_0 - \theta^*\| \alpha^{-1/2}, \end{aligned} \quad (33)$$

where Σ_ε is defined in (25).

Proof. The proof follows the lines of Theorem 1 with Proposition 8 used instead of Proposition 7. \square

5 Optimality of the derived bounds with respect to α : analysis of $(\theta_n)_{n \in \mathbb{N}}$ as a Markov chain

In this section, we study the sequence $\{\theta_n : n \in \mathbb{N}\}$ defined in (1) as a Markov chain. This perspective will allow us to show that the bounds that we derived in Theorem 1 are near-Berstein high probability bounds with respect to the stepsize α . Denote by R_α the Markov kernel associated with $\{\theta_n : n \in \mathbb{N}\}$. First, we show that if α is small enough then R_α is geometrically ergodic with respect to the Wasserstein distance of order 2 denoted by W_2 and give a representation of its stationary distribution as an infinite sum.

Theorem 3. *Assume A1. Then, for any $\alpha \in (0, \alpha_{2,\infty})$, where $\alpha_{2,\infty}$ is defined in (16), R_α admits a unique stationary distribution $\pi_\alpha \in \mathcal{P}_2(\mathbb{R}^d)$ and for any $n \in \mathbb{N}$,*

$$W_2^2(\delta_\theta R_\alpha^n, \pi_\alpha) \leq \sqrt{\kappa_Q d(1 - a\alpha/2)^n} \int_{\mathbb{R}^d} \|\tilde{\theta} - \theta\|^2 d\pi_\alpha(\tilde{\theta}). \quad (34)$$

Further, if $\{(\mathbf{A}_k, \mathbf{b}_k) : k \in \mathbb{N}_-\}$ is any sequence of i.i.d. random variables with the same distribution as $(\mathbf{A}_1, \mathbf{b}_1)$, then the following limit exists almost surely and in L^2 and has distribution π_α :

$$\theta_\infty^{(\alpha)} = \lim_{n \rightarrow -\infty} \theta_n^{(\alpha, \leftarrow)}, \quad \theta_n^{(\alpha, \leftarrow)} = \alpha \sum_{k=n}^1 \Gamma_{k:0} \mathbf{b}_{k-1}, \quad \Gamma_{k:0} = \prod_{i=k}^0 (\mathbf{I}_d - \alpha \mathbf{A}_i). \quad (35)$$

The proof is postponed to Appendix E.1. Based on Theorem 1, we easily get concentration bounds for the family of distributions $\{\pi_\alpha : \alpha \in (0, \alpha_{2,\infty})\}$ around θ^* .

Theorem 4. *Assume A1 and let $p_0 \geq 2$. Then, for any $\alpha \in (0, \alpha_{p_0,\infty})$, where $\alpha_{p_0,\infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1/4)$, with probability at least $1 - 4\delta$, it holds*

$$\alpha^{-1/2} |u^\top (\theta_\infty^{(\alpha)} - \theta^*)| < D_1 \sqrt{\{u^\top \Sigma u\} \log(2/\delta)} + \alpha^{1/2} [a^{-1/2} \|\bar{A} \Sigma \bar{A}^\top\|_Q^{1/2} + q^{(1)}(\alpha, \delta)], \quad (36)$$

where Σ is the unique solution of (27), $D_1 = 60\sqrt{3}e^{2/3}$, a is defined in (3), and $q^{(1)}(\alpha, \delta)$ in (31).

Proof. The proof follows from Theorem 1, the Portmanteau theorem [21, Theorem 13.16], and the fact that convergence in W_2 implies weak convergence. \square

Our results is only polynomial in δ and we cannot expect improving this dependency as illustrated in Example 1 for fixed α . The leading term in (36) as $\alpha \downarrow 0$ is $\sqrt{D_1 \{u^\top \Sigma u\}}$. In our next result, we establish a central limit theorem for the family $(\theta_\infty^{(\alpha)})_{\alpha \in (0, \alpha_{2,\infty}]}$ where Σ plays the role of the asymptotic covariance matrix. As a result, (36) is a Bernstein-type high probability bound with respect to α and therefore (36) is sharp. Define for any $\alpha \in (0, \alpha_{2,\infty}]$,

$$\tilde{\theta}_\infty^{(\alpha)} = \alpha^{-1/2} \{\theta_\infty^{(\alpha)} - \theta^*\}. \quad (37)$$

Theorem 5. *Assume A1. Then, the family $\{\tilde{\theta}_\infty^{(\alpha)} : \alpha \in (0, \alpha_{2,\infty}]\}$ converges in law as $\alpha \downarrow 0$ to a zero-mean Gaussian random variable with covariance matrix Σ defined by (27).*

Note that this result was established in [28, Theorem 1] for general stochastic approximation schemes but under stronger conditions on the sequence $\{\varepsilon_n : n \in \mathbb{N}^*\}$. In particular, it is assumed that the distribution of ε_1 admits a density with respect to the Lebesgue measure. We relax this condition and provide a new proof for this result. In particular, our strategy to establish Theorem 5 is to consider the decomposition (21) of $\{\theta_n : n \in \mathbb{N}\}$ with $\theta_0 = 0$, since in such case $\theta_n = \tilde{\theta}_n^{(f)}$ for any $n \in \mathbb{N}$. Define $\{J_n^{(\alpha, \leftarrow)} : n \in \mathbb{N}_-\}$ by

$$J_n^{(\alpha, \leftarrow)} = \alpha \sum_{k=n}^1 G_{k:0} \varepsilon_{k-1}, \quad G_{k:0} = \prod_{i=k}^0 (\mathbf{I} - \alpha \bar{A}). \quad (38)$$

Note that for any $n \in \mathbb{N}$, $\theta_{-n+1}^{(\alpha, \leftarrow)}$ has the same distribution as $\theta_n^{(\alpha)}$ starting from $\theta_0 = 0$ and $J_n^{(\alpha, 0)}$ as $J_{-n+1}^{(\alpha, \leftarrow)}$. In contrast to $J_n^{(\alpha, 0)}$, $J_{-n+1}^{(\alpha, \leftarrow)}$ admits a limit in L^2 and almost surely denoted by $J_\infty^{(\alpha, \leftarrow)}$. Then, we get for any $u \in \mathbb{S}^{d-1}$, $\alpha \in (0, \alpha_{2,\infty}]$, bounded and Lipschitz functions $f : \mathbb{R} \rightarrow \mathbb{R}$, with Lipschitz constant smaller than 1, by the Lebesgue dominated convergence theorem

$$\begin{aligned} & |\mathbb{E}[f(u^\top \tilde{\theta}_\infty^{(\alpha, \leftarrow)})] - \mathbb{E}[f(\alpha^{-1/2} u^\top J_\infty^{(\alpha, \leftarrow)})]| \\ &= \lim_{n \rightarrow +\infty} |\mathbb{E}[f(\alpha^{-1/2} u^\top [\theta_{-n+1}^{(\alpha, \leftarrow)} - \theta^*])] - \mathbb{E}[f(\alpha^{-1/2} u^\top J_{-n+1}^{(\alpha, \leftarrow)})]| \\ &= \lim_{n \rightarrow +\infty} |\mathbb{E}[f(\alpha^{-1/2} u^\top [\theta_n^{(\alpha)} - \theta^*])] - \mathbb{E}[f(\alpha^{-1/2} u^\top J_n^{(\alpha, 0)})]| \leq \limsup_{n \rightarrow +\infty} \mathbb{E}[\|\alpha^{-1/2} u^\top H_n^{(\alpha, 0)}\|]. \end{aligned}$$

Using the decomposition $H_n^{(\alpha, 0)} = J_n^{(\alpha, 1)} + H_n^{(\alpha, 1)}$, where $\{(J_n^{(\alpha, 1)}, H_n^{(\alpha, 1)}) : n \in \mathbb{N}\}$ are defined in (28) and plugging the bounds provided by Proposition 11 and Proposition 12 in Appendix C.4 shows that

$$\limsup_{\alpha \rightarrow 0} |\mathbb{E}[f(u^\top \tilde{\theta}_\infty^{(\alpha, \leftarrow)})] - \mathbb{E}[f(\alpha^{-1/2} u^\top J_\infty^{(\alpha, \leftarrow)})]| = 0.$$

Therefore by the Cramer Wold device and the Portmanteau theorem [21, Theorem 13.16], Theorem 5 follows from the next result.

Proposition 9. *Assume A1. Then, for any $u \in \mathbb{S}^{d-1}$, $\{\alpha^{-1/2} u^\top J_\infty^{(\alpha, \leftarrow)} : \alpha \in (0, \alpha_{2, \infty}]\}$ converges in distribution to the zero-mean Gaussian distribution with variance $u^\top \Sigma u$ where Σ is given in (27).*

The proof is postponed to Appendix E.2.

6 Conclusion

In this paper, we provided a novel non-asymptotic analysis of LSA algorithms with fixed stepsize. For any $\delta \in (0, 1)$, we obtain bounds on the sequence $\{\|\theta_n - \theta^*\| : n \in \mathbb{N}\}$ that holds with probability at least $1 - \delta$. The bounds are proven to be tight with respect to the stepsize, and we show that these bounds necessarily have polynomial dependency in δ . Importantly, our results do not require the matrices \mathbf{A}_n to be symmetric but only Hurwitz, which enables one to apply them to various scenarios such as reinforcement learning. Future work includes extending our high probability bounds to a larger panel of random noise, e.g., with heavy tailed distribution, Markovian dependency, as well as Polyak-Ruppert averaging.

References

- [1] R. B. Ash. *Information theory*. Tracts in Pure & Applied Mathematics. John Wiley & Sons Inc, 1966. ISBN 0470034459,9780470034453.
- [2] F. Bach and E. Moulines. Non-strongly-convex smooth stochastic approximation with convergence rate $o(1/n)$. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013. URL <https://proceedings.neurips.cc/paper/2013/file/7fe1f8abaad094e0b5cb1b01d712f708-Paper.pdf>.
- [3] A. Benveniste, M. Métivier, and P. Priouret. *Adaptive algorithms and stochastic approximations*, volume 22. Springer Science & Business Media, 2012.
- [4] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and distributed computation: numerical methods*. 2003.
- [5] J. Bhandari, D. Russo, and R. Singal. A finite time analysis of temporal difference learning with linear function approximation. In *Conference On Learning Theory*, pages 1691–1692, 2018.
- [6] V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.
- [7] L. Bottou, F. E. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *Siam Review*, 60(2):223–311, 2018.
- [8] S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford University Press, 2013.
- [9] S. Chen, A. Devraj, A. Busic, and S. Meyn. Explicit mean-square error bounds for monte-carlo and linear stochastic approximation. In *International Conference on Artificial Intelligence and Statistics*, pages 4173–4183. PMLR, 2020.
- [10] X. Chen, J. D. Lee, X. T. Tong, Y. Zhang, et al. Statistical inference for model parameters in stochastic gradient descent. *The Annals of Statistics*, 48(1):251–273, 2020.
- [11] G. Dalal, B. Szörényi, G. Thoppe, and S. Mannor. Finite sample analyses for TD(0) with function approximation. In *Thirty-Second AAAI Conference on Artificial Intelligence*, 2018.
- [12] A. Durmus, E. Moulines, A. Naumov, S. Samsonov, and H.-T. Wai. On the stability of random matrix product with markovian noise: Application to linear stochastic approximation and td learning, 2021.
- [13] N. Frikha, S. Menozzi, et al. Concentration bounds for stochastic approximations. *Electronic Communications in Probability*, 17, 2012.

- [14] S. Guo, F. Qi, and H. M. Srivastava. Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic. *Integral Transforms and Special Functions*, 18(11):819–826, 2007. doi: 10.1080/10652460701528933. URL <https://doi.org/10.1080/10652460701528933>.
- [15] P. Hall and C. Heyde. *Martingale Limit Theory and Its Application*. Academic Press, 1980.
- [16] F. Hiai and D. Petz. *Introduction to Matrix Analysis and Applications*. Universitext. Springer International Publishing, 2014. ISBN 9783319041506.
- [17] D. Huang, J. Niles-Weed, J. A. Tropp, and R. Ward. Matrix concentration for products. *arXiv preprint arXiv:2003.05437*, 2020.
- [18] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Number 223 in Operator Theory: Advances and Applications. Springer, 2012. ISBN 978-3-0348-0398-4. doi: 10.1007/978-3-0348-0399-1. 10.1007/978-3-0348-0399-1.
- [19] P. Jain, S. M. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Accelerating stochastic gradient descent for least squares regression. In *Conference On Learning Theory*, pages 545–604. PMLR, 2018.
- [20] P. Jain, D. Nagaraj, and P. Netrapalli. Making the last iterate of sgd information theoretically optimal. In A. Beygelzimer and D. Hsu, editors, *Proceedings of the Thirty-Second Conference on Learning Theory*, volume 99 of *Proceedings of Machine Learning Research*, pages 1752–1755, Phoenix, USA, 25–28 Jun 2019. PMLR.
- [21] A. Klenke. *Probability Theory: A Comprehensive Course*. Universitext. Springer London, 2013. ISBN 9781447153603.
- [22] H. Kushner and G. G. Yin. *Stochastic approximation and recursive algorithms and applications*, volume 35. Springer Science & Business Media, 2003.
- [23] C. Lakshminarayanan and C. Szepesvari. Linear stochastic approximation: How far does constant step-size and iterate averaging go? In A. Storkey and F. Perez-Cruz, editors, *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pages 1347–1355. PMLR, 2018.
- [24] O. Macchi and E. Eweda. Second-order convergence analysis of stochastic adaptive linear filtering. *IEEE Transactions on Automatic Control*, 28(1):76–85, 1983.
- [25] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on optimization*, 19(4):1574–1609, 2009.
- [26] Y. Nesterov and J.-P. Vial. Confidence level solutions for stochastic programming. *Automatica*, 44(6):1559–1568, 2008.
- [27] B. Pepin. Concentration inequalities for additive functionals: A martingale approach. *Stochastic Processes and their Applications*, 135:103–138, 2021.
- [28] G. Pflug. Stochastic Minimization with Constant Step-Size: Asymptotic Laws. *SIAM Journal on Control and Optimization*, 24(4):655–666, 1986. doi: 10.1137/0324039. URL <https://doi.org/10.1137/0324039>.
- [29] I. Pinelis. *An Approach to Inequalities for the Distributions of Infinite-Dimensional Martingales*, pages 128–134. Birkhäuser Boston, Boston, MA, 1992. ISBN 978-1-4612-0367-4. doi: 10.1007/978-1-4612-0367-4_9. URL https://doi.org/10.1007/978-1-4612-0367-4_9.
- [30] I. Pinelis. Optimum Bounds for the Distributions of Martingales in Banach Spaces. *The Annals of Probability*, 22(4):1679 – 1706, 1994. doi: 10.1214/aop/1176988477. URL <https://doi.org/10.1214/aop/1176988477>.
- [31] B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM journal on control and optimization*, 30(4):838–855, 1992.
- [32] A. S. Poznyak. *Advanced Mathematical Tools for Automatic Control Engineers: Deterministic Techniques*. Elsevier, Oxford, 2008.
- [33] A. Rakhlin, O. Shamir, and K. Sridharan. Making gradient descent optimal for strongly convex stochastic optimization. In *Proceedings of the 29th International Conference on Machine Learning*, pages 1571–1578, 2012.

- [34] R. Srikant and L. Ying. Finite-Time Error Bounds For Linear Stochastic Approximation and TD Learning. In *Conference on Learning Theory*, 2019.
- [35] R. S. Sutton. Learning to predict by the methods of temporal differences. *Machine Learning*, 3(1):9–44, Aug 1988. ISSN 1573-0565. doi: 10.1007/BF00115009.
- [36] J. N. Tsitsiklis and B. Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE Transactions on Automatic Control*, 42(5):674–690, May 1997. ISSN 2334-3303. doi: 10.1109/9.580874.
- [37] C. Villani. *Optimal transport : old and new*. Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 2009. ISBN 978-3-540-71049-3.
- [38] C. J. Watkins and P. Dayan. Q-learning. *Machine learning*, 8(3-4):279–292, 1992.

A Proofs of Section 2

A.1 Proofs of Proposition 1

The existence and uniqueness of Q follows from [32, Lemma 9.1, p. 140]. Regarding the second statement, note that for any $x \in \mathbb{R}^d \setminus \{0\}$, we have

$$\frac{x^\top (I - \alpha \bar{A})^\top Q (I - \alpha \bar{A}) x}{x^\top Q x} = 1 - \alpha \frac{\|x\|^2}{x^\top Q x} + \alpha^2 \frac{x^\top \bar{A}^\top Q \bar{A} x}{x^\top Q x}.$$

Hence, we get that for all $\alpha \in [0, \alpha_\infty]$,

$$1 - \alpha \frac{\|x\|^2}{x^\top Q x} + \alpha^2 \frac{x^\top \bar{A}^\top Q \bar{A} x}{x^\top Q x} \leq 1 - \alpha \|Q\|^{-1} + \alpha^2 \|\bar{A}\|_Q^2 \leq 1 - (1/2) \|Q\|^{-1} \alpha.$$

The proof is completed using that for any matrix $\bar{A} \in \mathbb{R}^{d \times d}$, $\|\bar{A}\|_Q \leq \kappa_Q^{1/2} \|\bar{A}\|$.

A.2 Proof for Example 1

The existence and uniqueness of the stationary distribution π_α is a consequence of Theorem 3 noting that A1 is satisfied for the particular case that we consider. We now show the second statement. Let $\alpha \in (0, \alpha_{2,\infty})$. First, note that since \mathbf{b}_1 is a zero-mean Gaussian random variables with unit variance independent of \mathbf{A}_1 , we have for any $p \geq 1$,

$$\begin{aligned} \mathbb{E}[\theta_1^{2p}] &= \sum_{k=0}^{2p} \binom{2p}{k} \mathbb{E}[\theta_0^{2p-k}] \mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p-k}] \mathbb{E}[\mathbf{b}_1^k] \\ &= \sum_{k=0}^p \binom{2p}{2k} \mathbb{E}[\theta_0^{2(p-k)}] \mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2(p-k)}] \mathbb{E}[\mathbf{b}_1^{2k}] \geq \mathbb{E}[\theta_0^{2p}] \mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p}]. \end{aligned}$$

This shows that taking θ_0 with distribution π_α that if $\int_{\mathbb{R}} |\theta|^{2p} d\pi_\alpha(\theta) < +\infty$, then it is necessary that $\mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p}] \leq 1$. However, using that $\mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p}] = \{q_A(1 - \alpha)^{2p} + (1 - q_A)(1 + \alpha)^{2p}\}$ and $(1 - \alpha)^{2p} \geq 1 - 2\alpha p$ and $(1 + \alpha)^{2p} \geq 1 + 2\alpha p + 2p(2p - 1)\alpha^2/2$, we get for any $p \geq 1$, $\mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p}] \geq \{1 - 2p\alpha[(2q_A - 1) - (2p - 1)\alpha(1 - q_A)/2]\}$, therefore $\mathbb{E}[(1 - \alpha \mathbf{A}_1)^{2p}] \leq 1$ does not hold for $2p > \bar{p}_{q,\alpha} = 1 + 2(2q_A - 1)/[\alpha(1 - q_A)]$.

B Technical and supporting results for Section 3

Proposition 10 ([17, Proposition 4.3]). *Consider two random matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{d \times d}$ that satisfy $\mathbb{E}[\mathbf{Y}|\mathbf{X}] = 0$. Then for $2 \leq q \leq p$,*

$$\|\mathbf{X} + \mathbf{Y}\|_{p,q}^2 \leq \|\mathbf{X}\|_{p,q}^2 + C_p \|\mathbf{Y}\|_{p,q}^2,$$

where $C_p = p - 1$.

Lemma 1. Let $A \in \mathbb{R}$, $B > 0$, $C \geq 1$, $p_0, p_1 \in \mathbb{R}$ such that $1 \leq p_0 \leq p_1 < +\infty$, and X a real random variable satisfying, for any $p \in [p_0, p_1]$,

$$\mathbb{E}[|X|^p] \leq C \exp(-Ap + Bp^2). \quad (39)$$

Then, for all $\delta \in (0, 1]$, we have, with probability at least $1 - \delta$,

$$|X| \leq \exp\left(-A + Bp_0 + 2\sqrt{B \log(C/\delta)} + \log(C/\delta)/p_1\right), \quad (40)$$

with the convention $c/\infty = 0$ for $c > 0$. In addition if (39) is satisfied for any $p \geq p_0$, then with probability at least $1 - \delta$,

$$|X| \leq \exp\left(-A + Bp_0 + 2\sqrt{B \log(C/\delta)}\right).$$

Proof. Note that by the monotone convergence theorem, it is sufficient to show (39). By Markov's inequality, we have, for any $t > 0$ and $p \in [p_0, p_1]$,

$$\mathbb{P}(|X| \geq t) \leq \mathbb{E}[|X|^p]/t^p \leq C \exp(-p(\log(t) + A - Bp)). \quad (41)$$

Taking $t = \exp(-A + 2Ba^*)$ for $a^* \in \mathbb{R}$ and maximizing over $p \in [p_0, p_1]$, we obtain

$$\mathbb{P}(|X| \geq \exp(-A + 2Ba^*)) \leq C \exp(-Bp(2a^* - p)) \leq C \exp(-B\phi(a^*)), \quad (42)$$

where

$$\phi(a^*) = \max_{p \in [p_0, p_1]} p(2a^* - p) = (2p_0 a^* - p_0^2) \mathbb{1}_{(-\infty, p_0]}(a^*) + (a^*)^2 \mathbb{1}_{(p_0, p_1)}(a^*) + (2p_1 a^* - p_1^2) \mathbb{1}_{[p_1, +\infty)}(a^*).$$

Note that for any $t \in \mathbb{R}$, the inverse of ϕ is given by

$$\phi^{\leftarrow}(t) = \frac{p_0^2 + t}{2p_0} \mathbb{1}_{(-\infty, p_0^2]}(t) + t^{1/2} \mathbb{1}_{(p_0^2, p_1^2)}(t) + \frac{p_1^2 + t}{2p_1} \mathbb{1}_{[p_1^2, +\infty)}(t).$$

For $\delta > 0$, taking $a_\delta^* = \phi^{\leftarrow}(\log(C/\delta)/B)$ gives

$$\mathbb{P}(|X| \geq \exp[-A + 2B\phi^{\leftarrow}(\log(C/\delta)/B)]) \leq \delta.$$

The proof then follows from the fact that for any $t \in \mathbb{R}$, $\phi^{\leftarrow}(t) \leq p_0/2 + \sqrt{t} + t/(2p_1)$. \square

Proof of (18) and (19)

Let $\alpha \in (0, \bar{\alpha}_q)$. Note that by definition of $\{\theta_n : n \in \mathbb{N}\}$ with (4), for any $n \in \mathbb{N}$,

$$\theta_n = (1 - \alpha)^{N_n} (1 + \alpha)^{n - N_n}, \quad \text{where } N_n = \sum_{k=1}^n \mathbb{1}_{\{1\}}(Z_k).$$

Then, for any $\beta > 0$, we get

$$\begin{aligned} \mathbb{P}(\theta_n \geq e^{-\alpha\beta n}) &= \mathbb{P}(\log(\theta_n) \geq -\alpha\beta n) = \mathbb{P}\left(N_n \log\left(\frac{1 - \alpha}{1 + \alpha}\right) \geq -\alpha\beta n - n \log(1 + \alpha)\right) \\ &= \mathbb{P}\left(N_n \leq n \log^{-1}\left(\frac{1 + \alpha}{1 - \alpha}\right) \{\alpha\beta + \log(1 + \alpha)\}\right) \\ &= \mathbb{P}\left(N_n - q_A n \leq -n \left[q_A - \log^{-1}\left(\frac{1 + \alpha}{1 - \alpha}\right) \{\alpha\beta + \log(1 + \alpha)\}\right]\right). \end{aligned} \quad (43)$$

Let $\beta_{\alpha, q} = \alpha^{-1} [q_A \log\{(1 + \alpha)/(1 - \alpha)\} - \log(1 + \alpha)]$. Note that with the condition, $\alpha \in (0, \bar{\alpha}_q)$, $\beta_{\alpha, q} > 0$ and therefore for any $\beta \in (0, \beta_{\alpha, q})$,

$$x_{\alpha, \beta} = \left[q_A - \log^{-1}\left(\frac{1 + \alpha}{1 - \alpha}\right) \{\alpha\beta + \log(1 + \alpha)\} \right] \in (0, \tilde{\varphi}_q(\alpha)), \quad (44)$$

We now show (18). From (43), it follows using Hoeffding's inequality that for any $\beta \in (0, \beta_{\alpha, q})$,

$$\mathbb{P}(\theta_n \geq e^{-\alpha\beta n}) \leq e^{-2nx_{\alpha, \beta}^2}. \quad (45)$$

Hence, for $\bar{\delta} \in (e^{-2n\tilde{\varphi}_q^2(\alpha)}, 1)$, there exists $x \in (0, \tilde{\varphi}_q(\alpha))$ such that $e^{-2nx^2} = \bar{\delta}$ given by $x = \sqrt{\log(1/\bar{\delta})/2n}$, which corresponds by (44) to

$$\beta = \alpha^{-1} \left\{ q_A - \sqrt{\log(1/\bar{\delta})/(2n)} \right\} \log \left(\frac{1+\alpha}{1-\alpha} \right) - \alpha^{-1} \log(1+\alpha) \in (0, \beta_{\alpha,q}) .$$

This completes the proof of (18) using (45).

We now show (19). Using [1, Lemma 4.7.2] and (43), it holds that for any $\beta \in (0, \beta_{\alpha,q})$,

$$\mathbb{P}(\theta_n \geq e^{-\alpha\beta n}) \geq \exp(-n\text{KL}(q_A - x_{\alpha,\beta}|q_A) - 2^{-1}\log(n)) , \quad (46)$$

where for any $\tilde{q} \in (0, 1)$,

$$\text{KL}(\tilde{q}|q_A) = \tilde{q} \log(\tilde{q}/q_A) + (1-\tilde{q}) \log((1-\tilde{q})/(1-q_A)) .$$

Note that for any $\tilde{q} \in (0, 1)$, $\tilde{q} \leq q_A$, using $\log(1+t) \leq t$ for any $t > -1$, we get

$$\text{KL}(\tilde{q}|q_A) \leq (q_A - \tilde{q})^2 / (q_A(1-q_A)) .$$

Therefore, plugging this result into (46) yields for any $\beta \in (0, \beta_{\alpha,q})$,

$$\mathbb{P}(\theta_n \geq e^{-\alpha\beta n}) \geq \exp(-nx_{\alpha,\beta}^2 / (q_A(1-q_A)) - 2^{-1}\log(n)) . \quad (47)$$

Hence, for $\underline{\delta} \in (e^{-n\tilde{\varphi}_q^2(\alpha)/(q_A(1-q_A)) - 2^{-1}\log(n)}, 1)$ there exists $x \in (0, \tilde{\varphi}_q(\alpha))$ such that $e^{-nx^2/(q_A(1-q_A)) - 2^{-1}\log(n)} = \underline{\delta}$, given by $x = \sqrt{2^{-1}\log(n) + q_A(1-q_A)\log(1/\underline{\delta})/n}$, which corresponds by (44) to

$$\beta = \alpha^{-1} \left\{ q_A - \sqrt{2^{-1}\log(n) + q_A(1-q_A)\log(1/\underline{\delta})/n} \right\} \log \left(\frac{1+\alpha}{1-\alpha} \right) - \alpha^{-1} \log(1+\alpha) .$$

This completes the proof of (19) using (47).

Proof of Proposition 3. It suffices to repeat the argument of Corollary 1. We need a version of Proposition 2 for the product $\mathbf{Z}_n = \prod_{\ell=0}^n \mathbf{Y}_\ell$ where $\{\mathbf{Y}_\ell : \ell \in \mathbb{N}\}$ are an independent and for each $\ell, q \in \mathbb{N}$ there exist $m_\ell \in (0, 1)$ and $\sigma_{\ell,q} > 0$ such that $\|\mathbb{E}[\mathbf{Y}_\ell]\|_Q^2 \leq 1 - m_\ell$ and $\mathbb{E}^{1/q}[\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_Q^q] \leq \sigma_{\ell,q}$. We use notations of $\mathbf{A}_n, \mathbf{B}_n$ from Proposition 2. Applying independence of \mathbf{Z}_{n-1} and \mathbf{Y}_n and $\mathbb{E}^{1/q}[\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_Q^q] \leq \sigma_{\ell,q}$ we estimate

$$\|\mathbf{A}_n\|_{p,q} \leq \left(\mathbb{E} \left[\|\mathbf{Y}_n - \mathbb{E}[\mathbf{Y}_n]\|_Q^q \|f_Q(\mathbf{Z}_{n-1})\|_p^q \right] \right)^{1/q} \leq \sigma_{n,q} \|f_Q(\mathbf{Z}_{n-1})\|_{p,q} . \quad (48)$$

The bound for $\|\mathbf{B}_n\|_{p,q}^2$ remains the same: $\|\mathbf{B}_n\|_{p,q}^2 \leq (1 - m_n) \|f_Q(\mathbf{Z}_{n-1})\|_{p,q}^2$. Combining this inequality with (48) and (10) yields for any $n \in \mathbb{N}^*$, $\|f_Q(\mathbf{Z}_n)\|_{p,q}^2 \leq (1 - m_n + (p-1)\sigma_{n,q}^2) \|f_Q(\mathbf{Z}_{n-1})\|_{p,q}^2 \leq \prod_{i=1}^n (1 - m_i + (p-1)\sigma_{i,q}^2) \|f_Q(\mathbf{Z}_0)\|_{p,q}^2$. The proof is then completed upon using (11) which implies that $\|\mathbf{Z}_n\|_{p,q} = \|Q^{-1/2} f_Q(\mathbf{Z}_n) Q^{1/2}\|_{p,q} \leq \sqrt{\kappa_Q} \|f_Q(\mathbf{Z}_n)\|_{p,q}$. Finally, it remains take $\mathbf{Y}_\ell = \mathbf{I} - \alpha \mathbf{A}_\ell, \ell \geq 1, \mathbf{Y}_0 = \mathbf{I}$. As $-\bar{A}$ is Hurwitz, applying Proposition 1 yields $\|\mathbb{E}[\mathbf{Y}_\ell]\|_Q^2 = \|\mathbf{I} - \alpha \bar{A}\|_Q^2 \leq 1 - a\alpha$. Further, since $\|\mathbf{A}_\ell - \bar{A}\| \in \text{SG}(C'_A)$ we get by Lemma 3

$$\mathbb{E}^{1/q}[\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_Q^q] = \alpha \mathbb{E}^{1/q}[\|\mathbf{A}_\ell - \bar{A}\|_Q^q] \leq 2\alpha \sqrt{\kappa_Q} C'_A = \alpha b'_Q \sqrt{q} .$$

Taking $m_\ell = a\alpha$ and $\sigma_{\ell,q} = b'_Q \alpha \sqrt{q}$ we get the claim of the proposition. \square

C Proofs of Section 4

For ease of presentation, we drop in this section the dependence of $J_n^{(\alpha,0)}, H_n^{(\alpha,0)}, J_n^{(\alpha,1)}, H_n^{(\alpha,1)}$ with respect to α and simply write $J_n^{(0)}, H_n^{(0)}, J_n^{(1)}, H_n^{(1)}$, respectively. We denote $\tilde{\mathbf{A}}_n = \mathbf{A}_n - \bar{A}$.

C.1 Proof of Proposition 4

Let $n \in \mathbb{N}^*$, $\alpha \in (0, \alpha_{p_0, \infty}]$, $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1)$. Under A1, applying Corollary 2 with $p = p_0$ yields

$$\begin{aligned} \mathbb{E}^{1/p_0}[|u^\top \Gamma_{1:n}^{(\alpha)}(\theta_0 - \theta^*)|^{p_0}] &\leq \mathbb{E}^{1/p_0}[\|\Gamma_{1:n}^{(\alpha)}\|^{p_0} \|\theta_0 - \theta^*\|] \\ &\leq \sqrt{\kappa_Q} d^{1/p_0} (1 - a\alpha + (p_0 - 1)b_Q^2 \alpha^2)^{n/2} \|\theta_0 - \theta^*\|. \end{aligned}$$

Since $\alpha \leq a/(2b_Q^2(p_0 - 1))$, using $(1 - t)^{1/2} \leq 1 - t/2$ for $t \in [0, 1]$, we get

$$\mathbb{E}^{1/p_0}[|u^\top \Gamma_{1:n}^{(\alpha)} \tilde{\theta}_0|^{p_0}] \leq \sqrt{\kappa_Q} d^{1/p_0} \|\theta_0 - \theta^*\| (1 - a\alpha/4)^n.$$

Applying Markov's inequality easily completes the proof.

C.2 Proof of Proposition 5

Let $n \in \mathbb{N}^*$, $\alpha \in (0, \alpha_{p_0, \infty}]$, $u \in \mathbb{S}^{d-1}$ and $\delta \in (0, 1)$. Using (22) and applying Rosenthal's inequality [30, Theorem 4.1]² for sum of centered independent random variables we get for any $p \geq 2$,

$$\mathbb{E}[|u^\top J_n^{(0)}|^p] \leq (60e)^p p^{p/2} \{u^\top \Sigma_n^\alpha u\}^{p/2} + \alpha^p 60^p p^p \mathbb{E} \left[\max_{\ell=1, \dots, n} |u^\top G_{\ell+1:n} \varepsilon_\ell|^p \right].$$

Applying Lemma 5, we obtain for any $p \geq 2$,

$$\mathbb{E}[|u^\top J_n^{(0)}|^p] \leq (60e)^p p^{p/2} \{u^\top \Sigma_n^\alpha u\}^{p/2} + (9\{1 + \log[1/(a\alpha)]\} \kappa_Q C_\varepsilon^2)^{p/2} \alpha^p 60^p p^{3p/2}, \quad (49)$$

where the constant C_ε is given in (63). Applying Markov's inequality, we get for any $p \geq 2$, $c_1, c_2 > 0$,

$$\begin{aligned} \mathbb{P}(|u^\top J_n^{(0)}| \geq c_1 \{u^\top \Sigma_n^\alpha u\}^{1/2} + c_2) &\leq \{c_1 \{u^\top \Sigma_n^\alpha u\}^{1/2} + c_2\}^{-p} \left[(60e)^p p^{p/2} \{u^\top \Sigma_n^\alpha u\}^{p/2} + (9\{1 + \log[1/(a\alpha)]\} \kappa_Q C_\varepsilon^2)^{p/2} \alpha^p 60^p p^{3p/2} \right] \\ &\leq (60e)^p p^{p/2} c_1^{-p} + (9\{1 + \log[1/(a\alpha)]\} \kappa_Q C_\varepsilon^2)^{p/2} \alpha^p 60^p p^{3p/2} c_2^{-p}. \end{aligned}$$

Taking $p = 3 \log(2/\delta)$, $c_1 = D_1 (\log(2/\delta))^{1/2}$ and $c_2 = \alpha \sqrt{1 + \log(1/(a\alpha))} D_2 \log^{3/2}(2/\delta)$ yields the statement, where

$$D_1 = 60\sqrt{3}e^{4/3}, \quad D_2 = 540\sqrt{3}e^{1/3} \kappa_Q^{1/2} C_\varepsilon. \quad (50)$$

C.3 Proof of Proposition 6

Lemma 2. Assume that A1-(iii) holds. Then, for any $\alpha \in (0, \alpha_\infty]$, where α_∞ is defined in (3),

$$\|\Sigma^\alpha - \Sigma\|_Q \leq \alpha a^{-1} \|\bar{A} \Sigma \bar{A}^\top\|_Q,$$

where Σ^α and Σ are defined in (25) and (27) respectively and a is given in (3).

Proof. Let $\alpha \in (0, \alpha_\infty]$. By definition, (25) and (27) imply

$$\bar{A}(\Sigma^\alpha - \Sigma) + (\Sigma^\alpha - \Sigma)\bar{A}^\top - \alpha \bar{A}(\Sigma^\alpha - \Sigma)\bar{A}^\top = \alpha \bar{A} \Sigma \bar{A}^\top,$$

which writes

$$\Sigma^\alpha - \Sigma - (I - \alpha \bar{A})(\Sigma^\alpha - \Sigma)(I - \alpha \bar{A})^\top = \alpha^2 \bar{A} \Sigma \bar{A}^\top.$$

This implies, by Proposition 1,

$$\|\Sigma^\alpha - \Sigma\|_Q \leq (1 - \alpha a) \|\Sigma - \Sigma\|_Q + \alpha^2 \|\bar{A} \Sigma \bar{A}^\top\|_Q,$$

Rearranging terms completes the proof. \square

²Note that the specific universal constants $C_{R,1} = 60e$ and $C_{R,2} = 60$ are not given in the statement, but a close inspection of the proof provide the given estimates.

C.4 Proof of Proposition 7

Proposition 7 is a direct consequence of the following statements.

Proposition 11. Assume A1. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_\infty)$, where $\alpha_{p_0, \infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$ and $p \geq 2$,

$$\mathbb{E}[|u^\top J_n^{(1)}|^p] \leq D_3^p \alpha^p p^{2p}, \quad (51)$$

where D_3 is given in (58). Moreover, for any $\delta \in (0, 1)$ with probability at least $1 - \delta$,

$$|u^\top J_n^{(1)}| \leq eD_3 \alpha \log^2(1/\delta). \quad (52)$$

Proposition 12. Assume A1 and let $p_0 \geq 2$. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_{p_0, \infty})$, where $\alpha_{p_0, \infty}$ is defined in (16), $u \in \mathbb{S}^{d-1}$,

$$\mathbb{E}[|u^\top H_n^{(1)}|^{p_0}] \leq D_4^{p_0} \alpha^{p_0} p_0^{2p_0}, \quad (53)$$

where D_4 is given in (61). Moreover, for any $\delta \in (0, 1)$ with probability at least $1 - \delta$,

$$|u^\top H_n^{(1)}| \leq D_4 \alpha p_0^2 \delta^{-1/p_0}. \quad (54)$$

Proof of Proposition 11. First, we note that (28) implies

$$J_n^{(1)} = \alpha^2 \sum_{\ell=1}^{n-1} S_{\ell+1:n} \varepsilon_\ell, \quad \text{with } S_{\ell+1:n} = \sum_{k=\ell+1}^n (I - \alpha \bar{A})^{n-k-1} \tilde{\mathbf{A}}_k (I - \alpha \bar{A})^{k-1-\ell}.$$

It is easy to check that the sequence $(\alpha^2 S_{\ell+1:n} \varepsilon_\ell, \mathfrak{F}_{\ell+1:n})_{\ell=1}^{n-1}$ is a martingale-difference, where $\mathfrak{F}_{\ell+1:n} = \sigma((\mathbf{A}_j, \mathbf{b}_j)_{j \in \{\ell+1, \dots, n\}})$. We may use Burkholder's inequality [15, Theorem 2.10] to get

$$\mathbb{E}[|u^\top J_n^{(1)}|^p] \leq (36p)^p \alpha^{2p} \mathbb{E} \left[\left(\sum_{\ell=1}^{n-1} (u^\top S_{\ell+1:n} \varepsilon_\ell)^2 \right)^{p/2} \right]. \quad (55)$$

Using the Minkowski inequality,

$$\mathbb{E}[|u^\top J_n^{(1)}|^p] \leq (36p)^p \alpha^{2p} \left(\sum_{\ell=1}^{n-1} \mathbb{E}^{2/p} [(u^\top S_{\ell+1:n} \varepsilon_\ell)^p] \right)^{p/2}. \quad (56)$$

Denote $V_{\ell+1}^\top = u^\top S_{\ell+1:n}$. Note that by Assumption A1-(ii) and Lemma 1, $\|(I - \alpha \bar{A})^{n-k-1} \tilde{\mathbf{A}}_k (I - \alpha \bar{A})^{k-1-\ell}\| \leq \kappa_Q C_A (1 - \alpha a)^{(n-\ell-2)/2}$. Applying [29, Theorem 3]³, we get for any $t \geq 0$

$$\mathbb{P}(\|V_{\ell+1}\| \geq t) \leq 2 \exp \left\{ -t^2 / (2\kappa_Q^2 C_A^2 (n - \ell) (1 - \alpha a)^{n-\ell-2}) \right\}.$$

Using Lemma 3,

$$\mathbb{E}^{2/p} [\|V_{\ell+1}\|^p] \leq 2\sqrt{2} C_A^2 \kappa_Q^2 (n - \ell) (1 - \alpha a)^{(n-\ell-2)p}. \quad (57)$$

Since $S_{\ell+1:n}$ and ε_ℓ are independent,

$$\mathbb{E}[|u^\top S_{\ell+1:n} \varepsilon_\ell|^p] \leq \mathbb{E}[\|V_{\ell+1}\|^p] \sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}[|u^\top \varepsilon_\ell|^p].$$

Assumption A1-(i), Lemma 3 and Lemma 5 imply, that for any $u \in \mathbb{S}^{d-1}$,

$$\mathbb{E}[|u^\top \varepsilon_\ell|^p] \leq p^{p/2} C_\varepsilon^p (2\sqrt{2})^{p/2}.$$

Combining this bound with (57),

$$\mathbb{E}[|u^\top S_{\ell+1:n} \varepsilon_\ell|^p] \leq p^p (2\sqrt{2})^p C_A^p \kappa_Q^p C_\varepsilon^p (n - \ell)^{p/2} (1 - \alpha a)^{(n-\ell-2)p/2}. \quad (58)$$

This inequality and (56) imply

$$\begin{aligned} \mathbb{E}[|u^\top J_n^{(1)}|^p] &\leq p^{2p} \alpha^{2p} (72\sqrt{2})^p C_A^p \kappa_Q^p C_\varepsilon^p \left(\sum_{\ell=1}^{n-1} (n - \ell) (1 - \alpha a)^{(n-\ell-2)p/2} \right)^{p/2} \\ &\leq \alpha^p D_3^p p^{2p}, \quad \text{where } D_3 = (72\sqrt{2}) C_A \kappa_Q C_\varepsilon a^{-1} (1 - \alpha a_\infty)^{-1}. \end{aligned} \quad (59)$$

³with $\mathcal{X} = \mathbb{R}^d$ equipped with the Euclidean norm $\|\cdot\|$. Note that $\|x\|, x \in \mathbb{R}^d$ is twice Gateaux differentiable and $\mathcal{X} \in D(A_1, A_2)$ with $A_1 = A_2 = 1$.

Now the equation (29) follows from Markov's inequality. Namely, for any $c_1 > 0$ it holds

$$\mathbb{P}\left(|u^\top J_n^{(1)}| \geq c_1 \alpha D_3\right) \leq \frac{\alpha^p D_3^p p^{2p}}{c_1^p \alpha^p D_3^p} = c_1^{-p} p^{2p}.$$

Taking $p = \log(1/\delta)$ and $c_1 = e \log^2(1/\delta)$, we obtain (29). \square

Proof of Proposition 12. With the decomposition (28), we represent

$$u^\top H_n^{(1)} = -\alpha \sum_{\ell=1}^n u^\top \Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(1)}.$$

Using Minkowski's inequality,

$$\mathbb{E}^{1/p}[|u^\top H_n^{(1)}|^p] \leq \alpha \sum_{\ell=1}^n \mathbb{E}^{1/p}[|u^\top \Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(1)}|^p].$$

Now, using the independence of $\Gamma_{\ell+1:n}^{(\alpha)}$, $\tilde{\mathbf{A}}_\ell$, $J_{\ell-1}^{(1)}$, and Item (ii),

$$\begin{aligned} \mathbb{E}^{1/p}[|u^\top \Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(1)}|^p] &\leq \mathbb{E}^{1/p}[\|u^\top \Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell\|] \sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p}[|v^\top J_{\ell-1}^{(1)}|] \\ &\leq 2 C_A \mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\|^p] \sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p}[|v^\top J_{\ell-1}^{(1)}|]. \end{aligned} \quad (60)$$

Hence, applying Corollary 2 to $\mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\|^p]$, and (59) to $\sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p}[|v^\top J_{\ell-1}^{(1)}|]$,

$$\mathbb{E}^{1/p}[|u^\top H_n^{(1)}|^p] \leq 2 C_A \sqrt{\kappa_Q} d^{1/p} D_3 \alpha^2 p^2 \sum_{\ell=1}^n (1 - a\alpha + (p-1)b_Q^2 \alpha^2)^{(n-\ell)/2}.$$

Since $p_0 - 1 \leq a/(2b_Q^2 \alpha)$, from the previous estimate it follows

$$\begin{aligned} \mathbb{E}^{1/p_0}[|u^\top H_n^{(1)}|^{p_0}] &\leq 2 C_A \sqrt{\kappa_Q} d^{1/p_0} D_3 \alpha^2 p_0^2 \sum_{\ell=1}^n (1 - \alpha a)^{(n-\ell)/2} \\ &\leq 4 C_A \sqrt{\kappa_Q} d^{1/p_0} D_3 \alpha p_0^2 / a. \end{aligned}$$

Hence,

$$\mathbb{E}[|u^\top H_n^{(1)}|^{p_0}] \leq D_4 \alpha^{p_0} p_0^{2p_0}, \text{ where } D_4 = 4 C_A \sqrt{\kappa_Q} d^{1/p_0} D_3 / a. \quad (61)$$

Using Markov's inequality, we get with probability at least $1 - \delta$,

$$|u^\top H_n^{(1)}| \leq D_4 \alpha p_0^2 / \delta^{1/p_0}.$$

\square

C.5 Proof of Proposition 8

The proof is along the same lines as the proof of $H_n^{(1)}$ in Appendix C.4, with the better bound for $\mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\|^p]$. For reader's convenience, we provide the proof below. Starting with equation (60), we note that under A2,

$$\mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\|^p] \leq \sqrt{\kappa_{\tilde{Q}}} (1 - \alpha \tilde{a})^{n-\ell}.$$

We also apply (59) to $\sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p}[|v^\top J_{\ell-1}^{(1)}|]$. Then

$$\mathbb{E}^{1/p}[|u^\top H_n^{(1)}|^p] \leq 2 \sqrt{\kappa_{\tilde{Q}}} C_A D_3 \alpha^2 p^2 \sum_{\ell=1}^n (1 - \alpha \tilde{a})^{n-\ell} \leq D_5 \alpha p^2,$$

where we have defined

$$D_5 = 2 \sqrt{\kappa_{\tilde{Q}}} C_A D_3 / \tilde{a}. \quad (62)$$

Now the equation (32) follows from Markov's inequality.

D Concentration results for sub-Gaussian random variables

Lemma 3. *Random variable $X \in \text{SG}(\sigma^2)$ for some $\sigma > 0$ if and only if for all $t \geq 0$ the condition $\mathbb{P}(|X| \geq t) \leq 2 \exp\{-t^2/(2\sigma^2)\}$ holds. In addition, in such a case, for any $p \geq 2$, we have*

$$\mathbb{E}[|X|^p] \leq \sqrt{2}e(2/e)^{p/2}p^{p/2}\sigma^p.$$

Proof. The first statement is well-known, see for example [8, Theorem 2.1]. We now show the second statement. By the Fubini theorem, $\mathbb{E}[|X|^p] = p \int_0^{+\infty} u^{p-1} \mathbb{P}(|X| > u) du$, we get

$$\mathbb{E}[|X|^p] = 2p \int_0^\infty u^{p-1} e^{-u^2/(2\sigma^2)} du = p2^{p/2}\sigma^p\Gamma(p/2),$$

using the change of variable $t = u^2/(2\sigma^2)$. Now, using an upper bound $\Gamma(p/2) \leq (p/2)^{(p-1)/2}e^{1-p/2}$ (see e.g. [14, Theorem 2]), and $p^{1/2} \leq 2^{p/2}$, we finally get

$$\mathbb{E}[|X|^p] \leq \sqrt{2}e(2/e)^{p/2}p^{p/2}\sigma^p.$$

□

Lemma 4. *Let $\{X_\ell : \ell \in \mathbb{N}\}$ be a sequence of random variables such that $X_\ell \in \text{SG}(\sigma^2)$ for any $\ell \in \mathbb{N}$ and some $\sigma^2 > 0$. Then for any $p \geq 2$,*

$$\mathbb{E} \left[\max_{\ell=1, \dots, n} \left(|X_\ell| / \sqrt{1 + \log \ell} \right)^p \right] \leq 3^p \sigma^p p^{p/2}.$$

Proof. Set $a_k = (1 + \log k)^{1/2}$ for $k \in \mathbb{N}^*$. Using the Fubini's theorem, $\mathbb{E}[|\xi|^p] = p \int_0^{+\infty} u^{p-1} \mathbb{P}(|\xi| > u) du$, the union bound, and Lemma 3, we get

$$\begin{aligned} \mathbb{E} \left[\max_{k=1, \dots, n} \{|X_k|^p / a_k^p\} \right] &= p \int_0^{+\infty} u^{p-1} \mathbb{P} \left(\max_{k=1, \dots, n} |X_k| \geq ua_k \right) du \\ &\leq 2^p \sigma^p + p \int_{2\sigma}^{+\infty} u^{p-1} \mathbb{P} \left(\max_{k=1, \dots, n} |X_k| \geq ua_k \right) du \\ &\leq 2^p \sigma^p + p \int_{2\sigma}^{+\infty} u^{p-1} \sum_{k=1}^n \mathbb{P}(|X_k| \geq ua_k) du \\ &\leq 2^p \sigma^p + 2p \int_{2\sigma}^{+\infty} u^{p-1} \sum_{k=1}^n \exp\{-u^2 a_k^2 / (2\sigma^2)\} du \\ &\leq 2^p \sigma^p + 2p\sigma^p \int_2^{+\infty} y^{p-1} \exp\{-y^2/2\} \left(\sum_{k=1}^n k^{-y^2/2} \right) dy \\ &\leq 2^p \sigma^p + \frac{\pi^2 p \sigma^p}{3} \int_2^{+\infty} y^{p-1} \exp\{-y^2/2\} dy \leq 2^p \sigma^p + \frac{\pi^2 p \sigma^p 2^{p/2-1}}{3} \Gamma(p/2). \end{aligned}$$

Using $\Gamma(p/2) \leq (p/2)^{(p-1)/2}e^{1-p/2}$ (see [14, Theorem 2]), we get

$$\mathbb{E} \left[\max_{k=1, \dots, n} \{|X_k|^p / a_k^p\} \right] \leq 2^p \sigma^p + \pi^2 \sigma^p p^{(p+1)/2} e^{1-p/2} / (3\sqrt{2}).$$

Since $\sqrt{p}e^{-p/2} \leq e^{-1/2}$ and $2^p \leq 4p^{p/2}$,

$$\mathbb{E} \left[\max_{k=1, \dots, n} \{|X_k|^p / a_k^p\} \right] \leq \sigma^p p^{p/2} (4 + \pi^2 e^{1/2} / (3\sqrt{2})) < 9\sigma^p p^{p/2}.$$

□

Lemma 5. *Assume A1. Then, for any $n \in \mathbb{N}^*$ and $v \in \mathbb{S}^{d-1}$, $v^\top \varepsilon_n$ defined by (7) is a sub-Gaussian random variable with parameter*

$$C_\varepsilon^2 = 2C_b^2 + 8C_A^2 \|\theta^*\|^2. \quad (63)$$

In addition, for any $n \in \mathbb{N}^*$, $p \geq 2$, $u \in \mathbb{S}^{d-1}$ and $\alpha \in (0, \alpha_\infty)$, it holds

$$\mathbb{E} \left[\max_{\ell=1, \dots, n} |u^\top G_{\ell+1:n}^{(\alpha)} \varepsilon_\ell|^p \right] \leq (9\kappa_Q p C_\varepsilon^2 \{1 + \log[1/(a\alpha)]\})^{p/2},$$

where α_∞ , a and κ_Q are defined in (3).

Proof. First we prove (63). Using the representation (7), for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\exp\{\lambda v^\top \varepsilon_n\}] &\leq \mathbb{E}[\exp\{\lambda v^\top (\mathbf{b}_n - \bar{b} - \{\mathbf{A}_n - \bar{A}\}\theta^*)\}] \\ &\leq \mathbb{E}^{1/2}[\exp\{2\lambda v^\top (\mathbf{b}_n - \bar{b})\}] \mathbb{E}^{1/2}[\exp\{2\lambda v^\top (\bar{A} - \mathbf{A}_n)\theta^*\}]. \end{aligned}$$

Note that A 1-(ii) implies $|v^\top (\bar{A} - \mathbf{A}_n)\theta^*| \leq 2C_A \|\theta^*\|$. Hence, using the Hoeffding inequality, $v^\top (\bar{A} - \mathbf{A}_n)\theta^* \in \text{SG}(4C_A^2 \|\theta^*\|^2)$. Combining this result with A 1-(i),

$$\mathbb{E}[\exp\{\lambda v^\top \varepsilon_n\}] \leq \exp\{\lambda^2 C_b^2\} \exp\{4\lambda^2 C_A^2 \|\theta^*\|^2\},$$

yielding the first statement of the lemma.

To prove the second part, let us denote $v_\ell = (\mathbf{I} - \alpha\bar{A})^{n-\ell} u / \|(\mathbf{I} - \alpha\bar{A})^{n-\ell} u\| \in \mathbb{S}^{d-1}$. Using Proposition 1,

$$\begin{aligned} \mathbb{E}[\max_{\ell=1, \dots, n} |u^\top G_{\ell+1:n}^{(\alpha)} \varepsilon_\ell|^p] &= \mathbb{E}[\max_{\ell=1, \dots, n} |v_\ell^\top \varepsilon_\ell|^p \|G_{\ell+1:n}^{(\alpha)} u\|^p] \\ &\leq \kappa_Q^{p/2} \mathbb{E}[\max_{\ell=1, \dots, n} |v_\ell^\top \varepsilon_\ell|^p (1 - \alpha a)^{p(n-\ell)/2}] \\ &\leq \kappa_Q^{p/2} \mathbb{E} \left[\max_{\ell=1, \dots, n} \frac{|v_\ell^\top \varepsilon_\ell|^p}{(1 + \log(n - \ell + 1))^{p/2}} \right] \left\{ \max_{x>0} (1 + \log(x+1)) e^{-\alpha a x} \right\}^{p/2} \\ &\leq \kappa_Q^{p/2} (9C_\varepsilon^2 p)^{p/2} \left\{ \max_{x>0} [(1 + \log(x+1)) e^{-\alpha a x}] \right\}^{p/2}, \end{aligned}$$

where in the last inequality we used Lemma 4. Set $f(x) = (1 + \log(x+1))e^{-cx}$ with $c = \alpha a \leq 1$ over $x > 0$. First, note that $f'(x) = e^{-cx}(1/(1+x) - c - c \log(x+1)) < 0$ for all $x > 1/c - 1$, and thus the maximum is attained for $x \in [0, 1/c - 1]$. Moreover, for any $x \leq 1/c - 1$, we have $f(x) \leq 1 + \log(1+x) \leq 1 + \log(1/c)$, leading to the desired result. \square

E Proof of Section 5

E.1 Proof of Theorem 3

Let $\alpha \in (0, \alpha_{2,\infty})$ and $\lambda_1, \lambda_2 \in \mathcal{P}_2(\mathbb{R}^d)$. By [37, Theorem 4.1], there exists a couple of random variables $\theta_0^{(1)}, \theta_0^{(2)}$ such that $W_2^2(\lambda_1, \lambda_2) = \mathbb{E}[\|\theta_0^{(1)} - \theta_0^{(2)}\|^2]$ independent of $\{(\mathbf{A}_n, \mathbf{b}_n) : n \in \mathbb{N}^*\}$. We introduce then a synchronous coupling between $\lambda_1 R_\alpha^n$ and $\lambda_2 R_\alpha^n$ as follows. Let $\{(\theta_n^{(1)}, \theta_n^{(2)}) : n \in \mathbb{N}\}$ starting from $\theta_0^{(1)}$ and $\theta_0^{(2)}$ respectively and for all $n \geq 0$,

$$\begin{aligned} \theta_{n+1}^{(1)} &= (\mathbf{I} - \alpha \mathbf{A}_{n+1}) \theta_n^{(1)} + \alpha \mathbf{b}_{n+1} \\ \theta_{n+1}^{(2)} &= (\mathbf{I} - \alpha \mathbf{A}_{n+1}) \theta_n^{(2)} + \alpha \mathbf{b}_{n+1}. \end{aligned} \tag{64}$$

Since for all $n \geq 0$, the distribution of $(\theta_n^{(1)}, \theta_n^{(2)})$ belongs to $\Pi(\lambda_1 R_\alpha^n, \lambda_2 R_\alpha^n)$, by definition of the Wasserstein distance we get for any $n \in \mathbb{N}$,

$$\begin{aligned} W_2(\lambda_1 R_\alpha^n, \lambda_2 R_\alpha^n) &\leq \mathbb{E}^{1/2}[\|\theta_n^{(1)} - \theta_n^{(2)}\|^2] = \mathbb{E}^{1/2}[\|\Gamma_{1:n}[\theta_0^{(1)} - \theta_0^{(2)}]\|^2] \\ &\leq D_2(1 - \alpha a/2)^{n/2} W_2(\lambda_1, \lambda_2), \end{aligned} \tag{65}$$

where we have used Corollary 1 for the last inequality. By [37, Theorem 6.16], the space $\mathcal{P}_2(\mathbb{R}^d)$ endowed with W_2 is a Polish space. Then, $(\lambda_1 R_\alpha^n)_{n \geq 0}$ is a Cauchy sequence and converges to a limit $\pi_\alpha^{\lambda_1} \in \mathcal{P}_2(\mathbb{R}^d)$, $\lim_{n \rightarrow +\infty} W_2(\lambda_1 R_\alpha^n, \pi_\alpha^{\lambda_1}) = 0$. We show that the limit $\pi_\alpha^{\lambda_1}$ does not depend

on λ_1 . Assume that there exists $\pi_\alpha^{\lambda_2}$ such that $\lim_{k \rightarrow +\infty} W_2(\lambda_2 R_\alpha^n, \pi_\alpha^{\lambda_2}) = 0$. By the triangle inequality

$$W_2(\pi_\alpha^{\lambda_1}, \pi_\alpha^{\lambda_2}) \leq W_2(\pi_\alpha^{\lambda_1}, \lambda_1 R_\alpha^n) + W_2(\lambda_1 R_\alpha^n, \lambda_2 R_\alpha^n) + W_2(\pi_\alpha^{\lambda_2}, \lambda_2 R_\alpha^n).$$

Thus by (65), taking the limits as $n \rightarrow +\infty$, we get $W_2(\pi_\alpha^{\lambda_1}, \pi_\alpha^{\lambda_2}) = 0$ and $\pi_\alpha^{\lambda_1} = \pi_\alpha^{\lambda_2}$. The limit is thus the same for all initial distributions and is denoted by π_α . Moreover, π_α is invariant for R_α . Indeed for all $k \in \mathbb{N}^*$, $W_2(\pi_\alpha R_\alpha, \pi_\alpha) \leq W_2(\pi_\alpha R_\alpha, \pi_\alpha R_\alpha^n) + W_2(\pi_\alpha R_\alpha^n, \pi_\alpha)$, Using (65) again, we get taking $n \rightarrow +\infty$, $W_2(\pi_\alpha R_\alpha, \pi_\alpha) = 0$ and $\pi_\alpha R_\alpha = \pi_\alpha$. The fact that π_α is the unique stationary distribution is straightforward by contradiction and using (65). (34) is a simple consequence of (65) taking $\lambda_2 = \pi_\alpha$.

It remains to show that $\theta_\infty^{(\alpha)}$ is well-defined and has distribution π_α . Since $\{(\mathbf{A}_k, \mathbf{b}_k) : k \in \mathbb{N}_-\}$ is i.i.d., $\sum_{n \leq -1} \mathbb{E}^{1/2}[\|\theta_n - \theta_{n+1}\|^2] = \sum_{n \leq -1} \mathbb{E}^{1/2}[\|\Gamma_{n:0} \mathbf{b}_{n-1}\|^2] = \sum_{n \leq -1} \mathbb{E}^{1/2}[\|\Gamma_{n:0}\|^2] \mathbb{E}^{1/2}[\|\mathbf{b}_{n-1}\|^2]$ and therefore A1-(i) combined with Corollary 1 ensures that this series is finite and therefore $(\theta_n)_{n \in \mathbb{N}_-}$ defined in (35) is a Cauchy sequence almost surely and in L^2 which ensures its convergence. Finally, assume now that $\{(\mathbf{A}_k, \mathbf{b}_k) : k \in \mathbb{N}_-\}$ is independent of $\{(\mathbf{A}_k, \mathbf{b}_k) : k \in \mathbb{N}^*\}$. To conclude it is then sufficient to note that if $\theta_0 = \theta_\infty^{(\alpha)}$, then θ_1 has the same distribution as $\theta_\infty^{(\alpha)}$ by definition of the recursion (1).

E.2 Proof of Proposition 9

Consider a sequence $\{\alpha_n : n \in \mathbb{N}\}$ converging to 0 such that for any $n \in \mathbb{N}$ $\alpha_n \in (0, \alpha_{2,\infty}]$, and let $u \in \mathbb{S}^{d-1}$. For ease of notation, we simply denote $G_{k:0}^{(n)} = G_{k:0}^{(\alpha_n)}$. Note that

$$\alpha_n^{-1/2} u^\top J_\infty^{(\alpha_n, \leftarrow)} = \sum_{k=-\infty}^1 \Delta M_{n,k}, \quad \Delta M_{n,k} = \alpha_n^{1/2} u^\top G_{k:0}^{(n)} \varepsilon_{k-1}.$$

By [15, Theorem 3.6], it is sufficient to show that

$$\sup_{k \leq 1} \Delta M_{n,k} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0 \tag{66}$$

$$\sum_{k \leq 1} \Delta M_{n,k}^2 \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} u^\top \Sigma u \tag{67}$$

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\sup_{k \leq 1} \Delta M_{n,k}^2] < +\infty. \tag{68}$$

First, by Markov inequality and Proposition 1 and A1-(i), we have for any $\eta > 0$ that

$$\begin{aligned} \mathbb{P}(\sup_{k \leq 1} \Delta M_{n,k} \geq \eta) &\leq \eta^{-4} \mathbb{E}[\sup_{k \leq 1} \Delta M_{n,k}^4] \leq \eta^{-4} \alpha_n^2 \sum_{k \leq 1} \mathbb{E}[\|G_{k:0}^{(n)}\|^4 \|\varepsilon_{k-1}\|^4] \\ &\leq \eta^{-4} \alpha_n^2 \mathbb{E}[\|\varepsilon_0\|^4] \kappa_Q^2 \sum_{k \leq 1} (1 - a\alpha_n)^{-2(k-1)} \leq \eta^{-4} \alpha_n^2 \mathbb{E}[\|\varepsilon_0\|^4] \kappa_Q^2 (1 - (1 - a\alpha_n)^2)^{-1}, \end{aligned}$$

which shows that (66) holds.

Denote by Σ_n the unique solution of the Ricatti equation (25) with $\alpha \leftarrow \alpha_n$. We get by Lemma 2 that there exists $C \geq 0$ such that for any $n \in \mathbb{N}$,

$$\|\Sigma - \Sigma_n\| \leq C\alpha_n.$$

Therefore, we obtain that

$$\left| \sum_{k \leq 1} \Delta M_{n,k}^2 - u^\top \Sigma u \right| \leq \alpha_n \left| \sum_{k \leq 1} ((G_{k:0}^{(n)})^\top u)^\top [\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon] (G_{k:0}^{(n)})^\top u \right| + C\alpha_n.$$

Then, to establish (67), it remains to show that

$$\alpha_n \left| \sum_{k \leq 1} ((G_{k:0}^{(n)})^\top u)^\top [\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon] (G_{k:0}^{(n)})^\top u \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \tag{69}$$

This follows from **A1-(ii)-(i)** and **Proposition 1** which shows that

$$\begin{aligned}
& \mathbb{E}[|\sum_{k \leq 1} ((G_{k:0}^{(n)})^\top u)^\top [\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon] (G_{k:0}^{(n)})^\top u|^2] \\
&= \sum_{k \leq 1} \mathbb{E} \left[|((G_{k:0}^{(n)})^\top u)^\top [\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon] (G_{k:0}^{(n)})^\top u|^2 \right] \\
&\leq \sum_{k \leq 1} \|G_{k:0}^{(n)}\|^4 \mathbb{E}[\|\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon\|^2] \leq \mathbb{E}[\|\varepsilon_0 \varepsilon_0^\top - \Sigma_\varepsilon\|^2] \kappa_Q^2 (1 - (1 - a\alpha_n)^2)^{-1}.
\end{aligned}$$

Therefore,

$$\lim_{n \rightarrow +\infty} \alpha_n^2 \mathbb{E} \left[\left| \sum_{k \leq 1} ((G_{k:0}^{(n)})^\top u)^\top [\varepsilon_{k-1} \varepsilon_{k-1}^\top - \Sigma_\varepsilon] (G_{k:0}^{(n)})^\top u \right|^2 \right] = 0,$$

which completes the proof of **(69)**.

Finally, we show **(68)** which follows from **A1-(ii)-(i)** and **Proposition 1**,

$$\begin{aligned}
\mathbb{E}[\sup_{k \leq 1} \Delta M_{n,k}^2] &\leq \sum_{k \leq 1} \mathbb{E}[\Delta M_{n,k}^2] = \alpha_n \sum_{k \leq 1} ((G_{k:0}^{(n)})^\top u)^\top \mathbb{E}[\varepsilon_{k-1} \varepsilon_{k-1}^\top] (G_{k:0}^{(n)})^\top u \\
&\leq \alpha_n \mathbb{E}[\varepsilon_0 \varepsilon_0^\top] \sum_{k \leq 1} \|G_{k:0}^{(n)}\|^2 \leq \mathbb{E}[\varepsilon_0 \varepsilon_0^\top] \kappa_Q / a.
\end{aligned}$$