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Phan van Tin

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# ON A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

ABSTRACT. We consider a generalized derivative nonlinear Schrödinger equation. In [13, Theorem 2], the author showed that there exists a modified wave operator with suitable regular small asymptotic states given at infinity. In this paper, we prove existence of wave operator under an explicit smallness of the given asymptotic states. Our method bases on studying the associated system used in [17]. Moreover, we show that if initial data is small in  $H^2(\mathbb{R})$  then the associated solution scatters up to Gauge transformation in sense of Theorem 1.3.

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## 1. INTRODUCTION

We consider the following equation

$$iu_t + \Delta u + i|u|^{2\sigma}u = 0, \tag{1.1}$$

for given  $\sigma > 0$  and  $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  is unknown function.

Local well posedness of (1.1) was studied in [8] on general domain. More precisely, for  $\sigma \geq 1$ , the authors proved that (1.1) is local existence in  $H^1$  and global existence if initial data is small enough. Furthermore, in the case  $0 < \sigma < 1$ , there exists a solution in  $(C_w \cap L^\infty)(\mathbb{R}, H^1)$  for given  $u_0 \in H^1$ .

In the special case  $\sigma = 1$ , (1.1) is known to be completely integrable. Global well posedness was studied in [18]. The author proved that for each initial data in  $H^1(\mathbb{R})$  with the mass bounded by  $4\pi$  the associated solution is global and uniformly bounded in  $H^1(\mathbb{R})$ . Later, [5] gave the same result by using variational argument. Moreover, the author showed that the solutions of (1.1) is global if its mass equals  $4\pi$  and its momentum is negative. Using integrability structure of (1.1), solutions of (1.1) is global in  $H^{2,2}(\mathbb{R})$  by [9] and in a subset of  $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$  by [14]. To our knowledge, the best result is given in [1] showing that (1.1) is global in  $H^{\frac{1}{2}}(\mathbb{R})$  and that the  $H^{\frac{1}{2}}(\mathbb{R})$  norm of the solutions remains globally bounded in time.

The equation (1.1) admits two parameters family of solitary waves solutions (or solitons)  $u_{\omega,c}$  for  $c \in (-2\sqrt{\omega}, 2\sqrt{\omega})$ . Stability and instability of such solutions were studied in many works (see e.g [12], [6], [11], [7], [4], [3]). It turns out that stability and instability of solitons depend on parameters  $\sigma, \omega, c$ . More precisely, we give instability for  $\sigma \geq 2$ , stability for  $0 < \sigma \leq 1$ . This is more complex in the case  $\sigma \in (1, 2)$ . In this case, there exists  $z_0 \in (-1, 1)$  such that solitons are stable if  $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$ , unstable if  $2z_0\sqrt{\omega} \leq c < 2\sqrt{\omega}$ . There are special solutions of (1.1) which behave in large time like sum of finite solitons [16], [17]. Stability of such solutions was studied in [10] when  $\sigma = 1$  and [15] when  $\sigma \in (1, 2)$ . More precisely, the authors proved that sum

of finite numbers of stable solitons is stable in some sense.

Scattering theory of (1.1) was first studied in [2]. The authors showed that when  $\sigma \geq 2$ , solutions are global and scatter for small initial data in  $H^s$  for  $\frac{1}{2} \leq s \leq 1$ . Moreover, when  $0 < \sigma < 2$ , there exist a class of solitons which converge to zero as  $c$  tends to some endpoint, which is against the small data scattering statement. Relating to scattering theory, in [13], the author proved existence of a modified wave operator under a small condition on asymptotic states given at infinity. In this paper, we prove existence of wave operator under an explicit smallness of the given asymptotic states.

The equation (1.1) admits three conservation quantities:

$$\begin{aligned} \text{(Energy)} \quad E(u) &= \frac{1}{2} \|u_x\|_{L^2}^2 - \frac{1}{2\sigma+2} \operatorname{Re} \int_{\mathbb{R}} i|u|^{2\sigma} \bar{u} u_x dx, \\ \text{(Mass)} \quad M(u) &= \|u\|_{L^2}^2, \\ \text{(Momentum)} \quad P(u) &= \operatorname{Re} \int_{\mathbb{R}} i u_x \bar{u} dx. \end{aligned}$$

Let  $(\omega, c) \in \mathbb{R}^+ \times \mathbb{R}$  such that

$$-2\sqrt{\omega} < c < 2\sqrt{\omega}. \quad (1.2)$$

Define, for  $\varphi \in H^1(\mathbb{R})$ :

$$\begin{aligned} S_{\omega,c}(\varphi) &= E(\varphi) + \frac{\omega}{2} M(\varphi) + \frac{c}{2} P(\varphi) \\ K_{\omega,c}(\varphi) &= \|\varphi_x\|_{L^2}^2 + \omega \|\varphi\|_{L^2}^2 + cP(\varphi) - \operatorname{Re} \int_{\mathbb{R}} i|\varphi|^{2\sigma} \bar{\varphi} \varphi_x dx, \\ \mu(\omega, c) &= \inf \{ S_{\omega,c}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\}, K_{\omega,c}(\varphi) = 0 \}, \\ \mathcal{K}_{\omega,c} &= \{ \varphi \in H^1(\mathbb{R}) : S_{\omega,c}(\varphi) \leq \mu(\omega, c), K_{\omega,c}(\varphi) \geq 0 \}, \\ \mathcal{K} &= \bigcup_{\substack{-2\sqrt{\omega} < c < 2\sqrt{\omega} \\ \omega > 0}} \mathcal{K}_{\omega,c}. \end{aligned}$$

In [5], the authors gave some sufficient conditions for which solutions of (1.1) are global. We have the following result:

**Theorem 1.1** ([5]). *Let  $\sigma \geq 1$  and  $(\omega, c)$  such that (1.2). If the initial data  $u_0 \in \mathcal{K}_{\omega,c}$  then the associated solution  $u$  of (1.1) is global and  $H^1$ -norm is uniformly bounded in time. Specially, if  $u_0 \in \mathcal{K}$  then the associated solution of (1.1) is global.*

By using the idea in [17], we show that there exists a solution scattering to the given asymptotic state. This solution is well defined in large time. To show that the wave operator is well defined, we need to prove that this solution can extend to zero at least and this solution exists uniquely in some sense. Using Theorem 1.1, given some suitable conditions of asymptotic state, the associated solution scattering to this state exists globally in time.

Our first main result is the following.

**Theorem 1.2.** *Let  $\sigma > 2$  and  $u^+ \in H^2 \cap W^{2,1}$  be a given function. There exists a solution  $u$  of (1.1) scatters forward to  $u^+$ . The convergence is in polynomial rate, more precisely, for some  $T_0 \gg 1$ :*

$$\|u - e^{it\Delta} u^+\|_{H^1} \lesssim t^{1-\sigma}, \quad \forall t \geq T_0. \quad (1.3)$$

Furthermore, if  $v$  is a solution of (1.1) such that  $\|v_x\|_{L_x^\infty(\mathbb{R})}$  is uniformly bounded on  $[T_0, \infty)$  and  $v$  scatters to  $u^+$  in the sense of (1.3) then  $v \equiv u$ . If  $u^+$  satisfies

$$\|\partial_x u^+\|_{L^2}^2 + \omega \|u^+\|_{L^2}^2 < 2\mu(\omega, 0), \quad (1.4)$$

for some  $\omega > 0$  then the associated solution  $u$  of (1.1) is global and the wave map:  $u^+ \mapsto u_0 = u(0)$  is well defined.

For  $u \in H^2$ , we define Gauge transformations of  $u$  by

$$\begin{aligned} G_1(u) &= \exp\left(\frac{i}{2} \int_{-\infty}^x |u(t, y)|^{2\sigma} dy\right) u(t, x), \\ G_2(u) &= \exp\left(\frac{i}{2} \int_{-\infty}^x |u(t, y)|^{2\sigma} dy\right) \partial_x u(t, x). \end{aligned}$$

In the case where the initial data is small in  $H^2(\mathbb{R})$ , we show that the associated solution scatters up to Gauge transformations in sense of Theorem 1.3. Our second result is the following.

**Theorem 1.3.** *Let  $u_0 \in H^2$  be such that  $\|u_0\|_{H^2}$  is small enough. Then there exists a unique  $\varphi^\pm, \psi^\pm \in H^1$  such that*

$$\|G_1(u) - e^{it\Delta} \varphi^\pm\|_{H^1} + \|G_2(u) - e^{it\Delta} \psi^\pm\|_{H^1} \rightarrow 0,$$

as  $t \rightarrow \pm\infty$ .

Our strategy to prove Theorem 1.2 is to use Gauge transformations of  $u$  to obtain a system of two Schrödinger equations without derivative nonlinearities (3.2). In [17, Lemma 3.8], we study this system to prove existence of multi-solitons, where the profile  $H$  decays exponentially in time. In our case, the profile  $H$  only decays polynomially in time. Combining to the profile  $W$  decays polynomially, it suffices to show that there exists a unique solution of (3.2) which decays polynomial in time and hence give a desired solution of (1.1) scattering to the given asymptotic profile by polynomial rate.

Theorem 1.3 is proved by investigating system (2.2). We show that under small condition of initial data the associated solution of (2.2) scatters in both direction which gives the desired result.

This paper is organized as follows. In Section 2, we give some useful notations and preliminaries for the proof of our main results. The next section 3 proves the main results of this paper Theorem 1.2 and Theorem 1.3. In Appendix 4, we establish some useful tools used in the proof of the main results.

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#### 2. PRELIMINARIES AND NOTATION

For convenience, we define  $L = i\partial_t + \Delta$  the Schrödinger operator.

Denote  $|(a, b)| = |a| + |b|$  and  $\|(a, b)\|_X = \|a\|_X + \|b\|_X$ , for any Banach space  $X$ .

A pair  $(q, r)$  is called a admissible in dimension one if  $r \geq 2$  and  $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ . We call  $\mathcal{A}$  is the set of all admissible pair.

Let  $I$  be a interval of  $\mathbb{R}$ . The Strichartz space  $S(I)$  is defined by

$$\|u\|_{S(I)} = \sup_{(q,r) \in \mathcal{A}} \|u\|_{L^q L^r}.$$

Let  $N(I)$  be dual space of  $S(I)$ . Define  $S^1, N^1$  by

$$\begin{aligned} \|u\|_{S^1(I)} &= \|(u, \partial_x u)\|_{S(I)}, \\ \|u\|_{N^1(I)} &= \|(u, \partial_x u)\|_{N(I)}. \end{aligned}$$

Let  $\varphi = G_1(u)$  and  $\psi = G_2(u)$ . Then,  $\psi = \partial_x \varphi - \frac{i}{2} |\varphi|^{2\sigma} \varphi$ . As in [17], we see that if  $u$  show (1.1) then  $(\varphi, \psi)$  show the following

$$\begin{cases} L\varphi &= P(\varphi, \psi) := i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \bar{\psi} - \sigma(\sigma-1) \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\psi^2 \bar{\varphi}^2), \\ L\psi &= Q(\varphi, \psi) := -i\sigma |\varphi|^{2(\sigma-1)} \psi^2 \bar{\varphi} - \sigma(\sigma-1) \psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\psi^2 \bar{\varphi}^2). \end{cases} \quad (2.1)$$

Let  $\eta = (\varphi, \psi)$  and  $F = (P, Q)$ . We may rewrite (2.1) as follows:

$$L\eta = F(\eta). \quad (2.2)$$

## 3. PROOF OF THE MAIN RESULTS

**3.1. Proof of Theorem 1.2.** The aim of this section is to prove Theorem 1.2. We use similar argument in [17].

Let  $R = e^{it\Delta}u_+$ . Then

$$LR + i|R|^{2\sigma}R_x = i|R|^{2\sigma}R_x := v(t, x).$$

Define  $h, k$  are Gauge transformations of  $R$ :

$$\begin{aligned} k &= G_1(R), \\ h &= G_2(R) = \partial_x h - \frac{i}{2}|h|^{2\sigma}h. \end{aligned}$$

By argument in [17], we have

$$\begin{aligned} Lh &= P(h, k) + m, \\ Lk &= Q(h, k) + n, \end{aligned}$$

where

$$\begin{aligned} m &= \exp\left(\frac{i}{2}\int_{-\infty}^x |R|^{2\sigma} dy\right) v - \sigma h \int_{-\infty}^x |R|^{2(\sigma-1)} \mathcal{I}m(\bar{R}v) dy \\ n &= \exp\left(\frac{i}{2}\int_{-\infty}^x |R|^{2\sigma} dy\right) (\partial_x v - \sigma \partial_x R \int_{-\infty}^x |R|^{2(\sigma-1)} \mathcal{I}m(\bar{R}v) dy). \end{aligned}$$

Let  $W = (h, k)$ ,  $H = -(m, n)$  and  $\tilde{\eta} = \eta - W = (\tilde{\varphi}, \tilde{\psi})$ . Then,

$$\partial_x \tilde{\psi} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^{2\sigma} - |h|^{2\sigma}h). \quad (3.1)$$

We see that if  $u$  show (1.1) then  $\tilde{\eta}$  show the following system

$$L\tilde{\eta} = F(\tilde{\eta} + W) - F(W) + H. \quad (3.2)$$

We divide the proof of Theorem 1.2 into three steps.

*Step 1. Existence of a solution of the system*

Consider (3.2). We show that (3.2) admits a solution which decays polynomially in large time.

**Proposition 3.1.** *Let  $H = H(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$ ,  $W = W(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$  be given functions which satisfies for some  $C_1, C_2 > 0$ ,  $T_0 > 0$ :*

$$t^{\frac{1}{2}}\|W(t)\|_{L^\infty \times L^\infty} + \|W(t)\|_{L^2 \times L^2} + t^\sigma \|H\|_{L^2 \times L^2} \leq C_1, \quad \forall t \geq T_0, \quad (3.3)$$

$$t^{\frac{1}{2}}\|\partial_x W(t)\|_{L^\infty \times L^\infty} + \|W(t)\|_{L^2 \times L^2} + t^\sigma \|\partial_x H\|_{L^2 \times L^2} \leq C_2, \quad \forall t \geq T_0. \quad (3.4)$$

*There exists a unique solution  $\tilde{\eta}$  to (3.2) such that*

$$t^{\sigma-1}(\|\tilde{\eta}\|_{S[t, \infty) \times S[t, \infty)} + \|\partial_x \tilde{\eta}\|_{S[t, \infty) \times S[t, \infty)}) \leq 1,$$

*for all  $t \geq T_0$ .*

*Proof.* Fix  $T_0 > 1$  large enough. Define

$$B = \{\tilde{\eta} : t^{\sigma-1}\|(\tilde{\eta}, \partial_x \tilde{\eta})\|_{S[t, \infty) \times S[t, \infty)} \leq C, \quad \forall t \geq T_0\},$$

for some  $C > 1$  large enough and

$$\Phi(\tilde{\eta})(t) = i \int_t^\infty e^{i(t-s)\Delta} (F(\tilde{\eta} + W) - F(W) + H)(s) ds.$$

If  $\tilde{\eta}$  tends to zero in  $H^1$ -norm then the system (3.2) is rewritten as follows

$$\tilde{\eta} = \Phi(\tilde{\eta}).$$

Thus, it suffices to prove that  $\Phi$  is a contraction mapping on  $B$  for  $T_0$  large enough. We divide the proof in two steps.

*Step 1. Proof  $\Phi$  maps  $B$  into  $B$*

Let  $t \geq T_0$ ,  $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2) \in B$ ,  $W = (w_1, w_2)$  and  $H = (h_1, h_2)$ . By Strichartz, we have

$$\|\Phi\tilde{\eta}\|_{S[t, \infty) \times S[t, \infty)} \lesssim \|F(\tilde{\eta} + W) - F(W)\|_{N[t, \infty) \times N[t, \infty)} \quad (3.5)$$

$$+ \|H\|_{L^1 L^2([t, \infty)) \times L^1 L^2([t, \infty))}. \quad (3.6)$$

For (3.6), we have

$$\begin{aligned} \|H\|_{L^1 L^2([t, \infty)) \times L^1 L^2([t, \infty))} &= \|(h_1, h_2)\|_{L^1 L^2([t, \infty))} \\ &\lesssim \int_t^\infty s^{-\sigma} ds = t^{1-\sigma}. \end{aligned}$$

Using Lemma 4.2, we have

$$\begin{aligned} &\|F(\tilde{\eta} + W) - F(W)\|_{N[t, \infty) \times N[t, \infty)} \\ &\lesssim \|F(\tilde{\eta} + W) - F(W)\|_{L^1 L^2[t, \infty) \times L^1 L^2[t, \infty)} \\ &\lesssim \|\tilde{\eta}\|_{L^q L^r} \|\tilde{\eta}\|_{X_1}^{2\sigma} + \|\tilde{\eta}\|_{L^1 L^2} \|W\|_{L^\infty L^\infty}^{2\sigma} \\ &\quad + \|\tilde{\eta}\|_{L^q L^r} \|\tilde{\eta}\|_{X_2}^{2\sigma} + \|\tilde{\eta}\|_{L^1 L^2} \|W\|_{L^\infty L^2}^2 \|W\|_{L^\infty L^\infty}^{2\sigma-2} \\ &\lesssim t^{(1-\sigma)(2\sigma+1)} + t^{2-\sigma} t^{1-\sigma} \lesssim t^{1-\sigma}. \end{aligned}$$

This implies that

$$\|\Phi \tilde{\eta}\|_{S[t, \infty) \times S[t, \infty)} \leq C t^{1-\sigma},$$

for some constant  $C > 0$ .

Moreover,

$$\begin{aligned} \|\partial_x \Phi \tilde{\eta}\|_{S[t, \infty) \times S[t, \infty)} &\lesssim \|\partial_x (F(\tilde{\eta} + W) - F(W))\|_{N[t, \infty) \times N[t, \infty)} \\ &\quad + \|\partial_x H\|_{L^1 L^2[t, \infty) \times L^1 L^2[t, \infty)}. \end{aligned} \quad (3.7)$$

For (3.8), we have

$$\begin{aligned} \|\partial_x H\|_{L^1 L^2([t, \infty)) \times L^1 L^2([t, \infty))} &= \|(h_1, h_2)\|_{L^1 L^2([t, \infty))} \\ &\lesssim \int_t^\infty s^{-\sigma} ds = t^{1-\sigma}. \end{aligned}$$

Furthermore, using Lemma 4.2, we have

$$\begin{aligned} &\|\partial_x (F(\tilde{\eta} + W) - F(W))\|_{N[t, \infty) \times N[t, \infty)} \\ &\lesssim \|\partial_x \tilde{\eta}\|_{L^1 L^2} (\|\tilde{\eta}, W\|_{L^\infty L^\infty}^{2\sigma} + \|\tilde{\eta}\|_{L^\infty L^{2\sigma}}^{2\sigma}) + \|\tilde{\eta}\|_{L^1 L^2} \|\partial_x (\tilde{\eta}, W)\|_{L^\infty L^\infty} \|\tilde{\eta}, W\|_{L^\infty L^\infty}^{2\sigma-1} \\ &\quad + \|\partial_x W\|_{L^\infty L^2} \|\tilde{\eta}\|_{L^1 L^2} \|\tilde{\eta}, W\|_{L^\infty L^2} \|\tilde{\eta}, W\|_{L^\infty L^\infty}^{2\sigma-2} + \|\tilde{\eta}\|_{L^1 L^2} \|\tilde{\eta}, W\|_{L^\infty L^\infty}^{2\sigma} \\ &\lesssim t^{1-\sigma}. \end{aligned}$$

Thus,  $\Phi$  map  $B$  into  $B$  if we chose  $C$  large enough.

*Step 2. Proof  $\Phi$  is a contraction map on  $B$*  By using (3.3), (3.4) and a similar argument as in the step 1, we may prove that, for any  $\tilde{\eta}, \tilde{\kappa} \in B$ , we have

$$\|\Phi \tilde{\eta} - \Phi \tilde{\kappa}\|_X \leq \frac{1}{2} \|\tilde{\eta} - \tilde{\kappa}\|_X,$$

where

$$\|\tilde{\eta}\|_X = \sup_{t \geq T_0} t^{\sigma-1} \|(\tilde{\eta}, \partial_x \tilde{\eta})\|_{S[t, \infty) \times S[t, \infty)}.$$

Thus,  $\Phi$  is a contraction map on  $B$ . This implies the desired result.  $\square$

*Remark 3.2.* By Strichartz, we have  $\|R\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \|u^+\|_{L^1}$ . The assumption on  $u^+$  ensures that all conditions in Proposition 3.1 are satisfied.

*Step 2. Existence of a desired solution*

We prove that the solution  $\tilde{\eta}$  found in Step 1 satisfying the relation 3.1. Let  $\tilde{v} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^{2\sigma}(\tilde{\varphi} + h) - |h|^{2\sigma}h)$ .

Using argument in [17], we have for  $N \gg t$ :

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N K_1 + K_2 + K_3 ds\right), \quad (3.9)$$

where

$$\begin{aligned} K_1 + K_2 + K_3 &= \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi} + k)^2\|_{L^1} + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\ &\quad + \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x\varphi\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\varphi^{2\sigma-1}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\ &\quad + \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}. \end{aligned}$$

Hence, by using  $\sigma > 2$ , we see that

$$(K_1 + K_2 + K_3)(s) \lesssim s^{1-\sigma} \in L^1,$$

Thus, (3.9) converges to zero as  $N \rightarrow \infty$  and  $\tilde{\psi} = \tilde{v}$ .

Define

$$u = G_1^{-1}(\tilde{\varphi} + h).$$

We have

$$\|u - R\|_{H^1} \lesssim \|\tilde{\varphi}\|_{H^1} \lesssim t^{1-\sigma} \rightarrow 0.$$

This implies that  $u$  scatters forward to  $u^+$ . It is clear that  $\|\partial_x u\|_{L_t^\infty([T_0, \infty), L_x^\infty(\mathbb{R}))} = \|\psi\|_{L_t^\infty([T_0, \infty), L_x^\infty(\mathbb{R}))}$  is bounded.

Now, assume that  $u^+$  satisfies the condition (1.4). Since  $u^+ \in H^2 \cap W^{2,1}$ , we have  $\|R\|_{L^\infty} \rightarrow 0$  as  $t \rightarrow \infty$  and hence

$$\mathcal{R}e \int_{\mathbb{R}} i \partial_x R \bar{R} dx \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow \infty} S_{\omega,0}(R) &= \frac{1}{2} \|\partial_x u^+\|_{L^2}^2 + \frac{\omega}{2} \|u^+\|_{L^2}^2 < \mu(\omega, 0), \\ \lim_{t \rightarrow \infty} K_{\omega,0}(R) &= \|\partial_x u^+\|_{L^2}^2 + \omega \|u^+\|_{L^2}^2 > \omega \|u^+\|_{L^2}^2 > 0. \end{aligned}$$

Since  $\|u(t) - R(t)\|_{H^1} \rightarrow 0$  as  $t \rightarrow \infty$ , we have  $K_{\omega,0}(u(t)) \geq \omega \|u^+\|_{L^2}^2 > 0$  and  $S_{\omega,0}(u(t)) < \mu(\omega, 0)$  for  $t$  large enough. By Theorem 1.1,  $u$  exists globally in time. It remains to show that the existence of  $u$  is unique in some sense.

*Step 3. Uniqueness of the desired solution*

Assume that  $u, v$  are two solutions of (1.1) on  $[T_0, \infty)$  such that  $\|(u_x, v_x)\|_{L_t^\infty([T_0, \infty), L_x^\infty(\mathbb{R}))} \lesssim 1$  and for some  $T_0 \gg 1$ :

$$\sup_{t \geq T_0} t^{\sigma-1} (\|u(t) - R\|_{H^1} + \|v(t) - R\|_{H^1}) \lesssim 1.$$

We have

$$i(u - v)_t + \Delta(u - v) + i(|u|^{2\sigma}u_x - |v|^{2\sigma}v_x) = 0.$$

Multiplying two sides of the above equation by  $\overline{(u - v)}$ , taking imaginary part and integrating by space, we have

$$0 = \frac{1}{2} \partial_t \|u - v\|_{L^2}^2 + \mathcal{R}e \int_{\mathbb{R}} (|u|^{2\sigma}u - |v|^{2\sigma}v) \overline{(u - v)} dx.$$

This implies that

$$\begin{aligned} \frac{1}{2} \partial_t \|u - v\|_{L^2}^2 &= -\mathcal{R}e \int_{\mathbb{R}} (|u|^{2\sigma}u - |v|^{2\sigma}v) \overline{(u - v)} dx \\ &= -\mathcal{R}e \int_{\mathbb{R}} (|u|^{2\sigma} - |v|^{2\sigma}) u_x \overline{u - v} + |v|^{2\sigma} \overline{u - v} (u_x - v_x) dx \\ &= -\mathcal{R}e \int_{\mathbb{R}} (|u|^{2\sigma} - |v|^{2\sigma}) u_x \overline{u - v} dx - \frac{1}{2} \mathcal{R}e \int_{\mathbb{R}} |v|^{2\sigma} \partial_x (|u - v|^2) dx \\ &= -\mathcal{R}e \int_{\mathbb{R}} (|u|^{2\sigma} - |v|^{2\sigma}) u_x \overline{u - v} dx + \frac{1}{2} \mathcal{R}e \int_{\mathbb{R}} \partial_x (|v|^{2\sigma}) |u - v|^2 dx. \end{aligned}$$

Hence,

$$\begin{aligned} |\partial_t \|u - v\|_{L^2}^2| &\lesssim \int_{\mathbb{R}} |u - v|^2 (|u|^{2\sigma-1} + |v|^{2\sigma-1})(|u_x| + |v_x|) dx \\ &\lesssim \|u - v\|_{L^2}^2 \|(u, v)\|_{L_x^\infty}^{2\sigma-1} \|(u_x, v_x)\|_{L_x^\infty} \\ &\lesssim \|u - v\|_{L^2}^2 t^{\frac{1}{2}-\sigma} \end{aligned}$$

Define  $M(t) = \|u(t) - v(t)\|_{L^2}^2$ . Assume  $M(T_0) > 0$ . We have

$$\left| \int_{T_0}^b \frac{\partial_t M}{M} ds \right| \lesssim \int_{T_0}^b \left| \frac{\partial_t M}{M} \right| ds \lesssim \int_{T_0}^b s^{\frac{1}{2}-\sigma} ds$$

This implies that

$$|\log(M(b)) - \log(M(T_0))| \lesssim T_0^{\frac{3}{2}-\sigma} - b^{\frac{3}{2}-\sigma}.$$

Since  $\sigma > 2$  and  $M(b)$  tends to zero as  $b$  tends to infinity, the above estimate gives us a contradiction by letting  $b$  tends to infinity. Thus,  $M(T_0) = 0$ . By uniqueness of  $H^1$  solutions of (1.1), we have  $u \equiv v$ . This implies the uniqueness of existence of  $u$  and hence the wave map  $u^\pm \mapsto u_0 = u(0)$  is well defined. This completes the proof.

**3.2. Proof of Theorem 1.3.** For this, we prove that the solution  $\eta$  to (2.2) scatters provided  $\eta(0)$  small in  $H^1 \times H^1$ . Let us consider (2.2). By Strichartz, we need only to prove that  $F(\eta) \in N^1(\mathbb{R}) \times N^1(\mathbb{R})$ . By Strichartz, we have

$$\|\eta\|_{S^1(\mathbb{R}) \times S^1(\mathbb{R})} \lesssim \|\eta(0)\|_{H^1 \times H^1} + \|F(\eta)\|_{N^1(\mathbb{R}) \times N^1(\mathbb{R})}.$$

Using Lemma 4.3, we have

$$\begin{aligned} \|F(\eta)\|_{N(\mathbb{R}) \times N(\mathbb{R})} &\lesssim \|\eta\|_{L^q L^r} (\|\eta\|_{X^1}^{2\sigma} + \|\eta\|_{X^2}^{2\sigma}) \\ &\lesssim \|\eta\|_{L^q L^r} (\|(\eta, \partial_x \eta)\|_{L^m L^h}^{2\sigma} + \|(\eta, \partial_x \eta)\|_{L^{2\sigma} L^k}^{2\sigma}), \end{aligned}$$

where  $(q, r)$ ,  $(m, n)$  are as in Lemma 4.3 and  $(m, h)$ ,  $(2\sigma, k)$  are admissible pairs.

Using Lemma 4.3 again, we have

$$\begin{aligned} \|\partial_x F(\eta)\|_{N(\mathbb{R}) \times N(\mathbb{R})} &\lesssim \|\partial_x \eta\|_{L^q L^r} \|\eta\|_{X^1 \cap X^2}^{2\sigma} + \|\eta\|_{L^q L^r} \|\eta\|_{X^1}^{2\sigma} \\ &\lesssim \|\partial_x \eta\|_{L^q L^r} \|(\eta, \partial_x \eta)\|_{L^m L^h \cap L^{2\sigma} L^k}^{2\sigma} + \|\eta\|_{L^q L^r} \|(\eta, \partial_x \eta)\|_{L^m L^h}^{2\sigma}. \end{aligned}$$

Define  $Z = L^q L^r \cap L^m L^h \cap L^{2\sigma} L^k$ . As the above, we have

$$\|(\eta, \partial_x \eta)\|_Z \leq C \|\eta(0)\|_{H^1 \times H^1} + C \|(\eta, \partial_x \eta)\|_Z^{2\sigma+1},$$

for some constant  $C > 0$ . Thus, for  $\|\eta(0)\|_{H^1 \times H^1}$  small enough (since  $\|u_0\|_{H^2}$  is small), by continuous argument, we have  $\eta, \partial_x \eta \in Z(\mathbb{R})$  and  $F(\eta) \in N^1(\mathbb{R}) \times N^1(\mathbb{R})$ . By classical argument, there exists  $\varphi^\pm, \psi^\pm$  such that

$$\lim_{t \rightarrow \pm\infty} \|\eta - e^{it\Delta}(\varphi^\pm, \psi^\pm)\|_{H^1 \times H^1} \rightarrow 0.$$

Hence, the proof is completed.

#### 4. APPENDIX

**Lemma 4.1** ([17]). *Fix  $\alpha, \beta \in \mathbb{R}$  with  $\alpha + \beta > 0$ . We have*

$$|(w + z)^\alpha (\bar{w} + \bar{z})^\beta - w^\alpha \bar{w}^\beta| \lesssim |z|^{\alpha+\beta} + |z||w|^{\alpha+\beta-1}. \quad (4.1)$$

As a consequence of Lemma 4.1, we introduce (without proof) the following estimates for nonlinearities:



**Lemma 4.2.** For  $\eta_1 = (\varphi_1, \psi_1), \eta_2 = (\varphi_2, \psi_2) \in H^1 \times H^1$ , we have

$$\begin{aligned} |F(\eta_1) - F(\eta_2)| &\lesssim |\eta_1 - \eta_2|(|\eta_1|^{2\sigma} + |\eta_2|^\sigma + \int_{-\infty}^x |\eta_1|^{2\sigma} dy) \\ &\quad + |\eta_2| \int_{-\infty}^x |\eta_1 - \eta_2|(|\eta_1|^{2\sigma-1} + |\eta_2|^{2\sigma-1}) dy, \\ |\partial_x(F(\eta_1) - F(\eta_2))| &\lesssim |\partial_x(\eta_1 - \eta_2)|(|\eta_1|^{2\sigma} + |\eta_2|^{2\sigma} + \int_{-\infty}^x |\eta_1|^{2\sigma}) + |\eta_1 - \eta_2| |\partial_x(\eta_1, \eta_2)|(|\eta_1|^{2\sigma-1} + |\eta_2|^{2\sigma-1}) \\ &\quad + |\partial_x \eta_2| \int_{-\infty}^x |\eta_1 - \eta_2|(|\eta_1|^{2\sigma-1} + |\eta_2|^{2\sigma-1}) + |\eta_1 - \eta_2|(|\eta_1|^{2\sigma} + |\eta_2|^{2\sigma}). \end{aligned}$$

For  $(q, r)$  is a admissible pair,  $(m, n)$  satisfies  $\frac{2}{m} + \frac{1}{n} = \frac{1}{\sigma}$ . Define

$$\|u\|_{X_1} = \|u\|_{L_t^m L_x^n},$$

and

$$\|u\|_{X_2} = \|u\|_{L_{t,x}^{2\sigma}}.$$

We see that

$$\|u\|_{X_1 \cap X_2(I)} \lesssim \|\langle \nabla \rangle u\|_{S(I)}.$$

We note that

$$\left| \int_{-\infty}^x f(y) dy \right| \leq \|f\|_{L^1}.$$

Thus, as a consequence of the above lemma, we have the following result.

**Lemma 4.3.** We have

$$\begin{aligned} \|F(\eta_1) - F(\eta_2)\|_{N^0} &\lesssim \|\eta_1 - \eta_2\|_{L^q L^r} (\|(\eta_1, \eta_2)\|_{X_1}^{2\sigma} + \|\eta_1\|_{X_2}^{2\sigma}), \\ &\quad + \|\eta_2\|_{L^q L^r} \|\eta_1 - \eta_2\|_{X_2} \|(\eta_1, \eta_2)\|_{X_2}^{2\sigma-1} \\ \|\partial_x(F(\eta_1) - F(\eta_2))\|_{N^0} &\lesssim \|\partial_x(\eta_1 - \eta_2)\|_{L^q L^r} (\|(\eta_1, \eta_2)\|_{X_1}^{2\sigma} + \|\eta_1\|_{X_2}^{2\sigma}) + \|\eta_1 - \eta_2\|_{X_1} \|\partial_x(\eta_1, \eta_2)\|_{L^q L^r} \|(\eta_1, \eta_2)\|_{X_1}^{2\sigma-1} \\ &\quad + \|\eta_1 - \eta_2\|_{X_2} \|\partial_x \eta_2\|_{L^q L^r} \|(\eta_1, \eta_2)\|_{X_2}^{2\sigma-1} + \|\eta_1 - \eta_2\|_{L^q L^r} \|(\eta_1, \eta_2)\|_{X_1}^{2\sigma}. \end{aligned}$$

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