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# Imaginary dialogues in secondary class mathematics lessons – understanding students' mathematical thinking processes

Petra Gössinger<sup>1</sup> and Stefan Götz<sup>2</sup>

<sup>1</sup>University College of Teacher Education, Austria; [petra.goessinger@ph-noe.ac.at](mailto:petra.goessinger@ph-noe.ac.at)

<sup>2</sup>University of Vienna, Austria

*The aim of this article is to discuss the use of imaginary dialogues in mathematics education in order to stimulate students' inner mathematical discourse of the scalar product and to make their thinking understandable for teachers. Imaginary dialogues are a form of mathematical writing that encourages students to continue a written and unfinished dialogue between two imaginary individuals discussing a mathematical problem. Analysing students' texts, teachers might be able to trace students' mathematical thinking processes. Results can be used by teachers to modify teaching instructions. In a research project, 10<sup>th</sup> grade students of Lower Austria are writing about their thoughts on the scalar product. A qualitative case study determines whether the concept of mental models is sufficient to represent students' thinking or whether there are other teaching-relevant aspects worth considering. First evaluation results are presented.*

*Keywords: Mathematical writing, mental models, imaginary dialogues, mathematical thinking processes, scalar product.*

## Introduction

Knowledge of students' thinking and learning processes has a substantial impact on teaching (Cai, 2005; Cai & Ding, 2017; Zuh et al., 2018). Clarkson and Presmeg (2008) note that understanding students' mathematical thinking is a key element for teachers' decision-making about further effective instructional steps. Wille (2009) states that writing imaginary dialogues is a way of initiating students' understanding processes and enabling a deep-level approach. Thus, teachers can figure out how students embed mathematical content in existing knowledge structures. They trace students' ways of thinking and their mathematical perceptions by evaluating written dialogues (Wille, 2017b). Therefore, teachers should not only become familiar with this specific teaching method, but also ought to know tools to analyse the content of the students' dialogues.

The concept of basic mental models becomes relevant when searching for an adaptable evaluation tool. It outlines the meaning people associate with specific mathematical content and the role these thoughts play in teaching and learning mathematics (Vom Hofe & Blum, 2016). The concept of mental models has a long tradition in the history of mathematics education in German-speaking areas (Vom Hofe et al., 2005). There is a lot of research related to this topic, regarding all school grades and various subject areas. Pre-service teachers learn about this concept at teacher training colleges and universities. The concept is also part of subject-specific didactic in-service trainings in further mathematics education. Therefore, it should be well-known by mathematics teachers. They should be able to set up and structure mathematics lessons based on the concept of mental models.

In the following, the authors attempt to analyse imaginary dialogues using mental models. We asked 10<sup>th</sup> grade students from Lower Austria to write self-created dialogues about the scalar

product. This topic of analytic geometry is part of the formal mathematics curriculum of the Austrian Ministry of Education in secondary school, level II. The pilot test and the main survey have already been carried out. First evaluation results will be presented and discussed within this paper. The results of the discussion shall enhance further evaluation processes.

## **Theoretical Framework**

This section has three main parts. First, theoretical facts and characteristics of imaginary dialogues are outlined. Then, the concept of mental models with the aim of clarifying the concept is illustrated and finally mental models described for the scalar product will be discussed.

### **The concept of imaginary dialogues**

An imaginary dialogue encourages students to compose written dialogues between two fictitious protagonists who discuss a mathematical task or question. This method encourages students to ask questions, it allows different perspectives and activates thinking processes. The dialogues reflect specific concepts or describe mathematical discoveries. In order to launch the writing process, a short initial dialogue is given. Students are then asked to continue this dialogue while elaborating the issue further (Wille, 2017a). Imaginary dialogues can be used in two different ways. Either students reflect mathematical facts they are already familiar with or independently evolve new mathematical ideas (Wille, 2017a; Wille, 2017b).

When teaching mathematics, writing texts is not used as a traditional method. Inspired by the work of Annika Wille, who introduced imaginary dialogues as a method to approach students' mathematical thinking processes, attention increased. During the last decade, she studied this extensively in different German classrooms. She found that the method offers students an opportunity to reflect on various mathematical issues – both guided and free-minded. Thus, their ability to build mathematical reasoning is promoted. In one of her papers, Wille gives hints on how to design an initial dialogue in order to unfold the potential of the method (Wille, 2017b). Ways to analyse the writings systematically have not been offered yet (Wille, 2017a). However, an overall concept that provides a foundation for analysing the dialogues, is still missing. It is suggested to have a closer look at the concept of mental models and to use this concept to examine the mathematical communication in the dialogues.

### **The concept of mental models**

Mental models on a mathematical concept are content-related interpretations that give meaning to specific terms (Vom Hofe et al., 2005). Greefrath et al. (2021) state that they are essential for understanding a mathematical phenomenon and capturing core elements. Researchers discuss normative and individual aspects of mental models. Normative aspects refer to what basic ideas students should ideally and generally already have created. They are related to subject specific aspects. They can be used by teachers to specify learning objectives for mathematics units and can guide them in designing and organizing their lessons (Greefrath et al., 2021; Vom Hofe & Blum, 2016). Individual mental representations are personal considerations, opinions and images in relation to mathematical terms or issues (Vohns, 2010). Teachers can trace students' individual

basic ideas by analysing students' dialogues. It is a way to listen to students' considerations attentively and allows them to gain insight into their progress across mathematical thinking.

### **Mental models on the scalar product**

The scalar or dot product is an algebraic operation that is calculated by multiplying the component of one vector with the corresponding entries of another vector. Dimensional spaces of both vectors must be equal. The result of this calculation is the sum of the products, turning to a single number. In Austrian secondary schools, the name "scalar product" is used in order to emphasize that the result is a scalar and not a vector.

According to the formal mathematics curriculum of the Austrian Ministry of Education, the scalar product is to be taught in higher stages of secondary school. Approaches can be both – algebraic and geometric. Algebraically, the scalar product is taught as the sum of the products of the corresponding components of two vectors given in cartesian form. It is used for defining energy and work relations in physics as well as for calculating scalar quantities in various everyday life problems. Geometrically, the scalar product is used for defining lengths, angles, and scalar projection. According to Frohn (2020), the concept of the scalar product is characterised with reference to four normative mental models:

- Product presentation: The scalar product as a generalized product that follows specific calculation rules;
- Orthogonality concept: The scalar product as an indicator of orthogonality;
- Angle concept: The scalar product as a dimension of the included angle;
- Projection presentation: The scalar product as a dimension of the length of the projection vector.

Teachers typically use all four mental models to approach the scalar concept in classroom. These approaches are also found in Austrian school textbooks.

Within this project, we use the mental models as a foundation for analysing and categorizing students' written texts. The research project intends a deepening of knowledge related to the concept of mental models on the scalar product.

## **Methods**

### **Research aims**

Based on the theoretical findings the aims of this study are: (a) to ask to what extent the four mentioned normative mental models on the scalar product mentioned by Frohn (2020) can be identified empirically, (b) to examine which individual mental concepts and cognitions students have actually developed in class, (c) how to reconcile these results with Frohn's findings and (d) how teachers can identify various students' concepts and drive the students towards mathematical goals.

### **Research questions**

- (a) How and to what extent can normative mental models on the scalar product be extracted from students' written dialogues empirically?

- (b) Do imaginary dialogues provide teachers insights into students' thinking about the scalar product?
- (c) Should there be offered a modification of the mental model concept on the scalar product?

### **Data collection**

The main survey of the research project took place in four 10<sup>th</sup> grade classes of secondary schools in Lower Austria in April 2022. Students should write imaginary dialogues on the topic scalar product. For this purpose, an initial dialogue was created (the dialogue was presented in German and translated by the authors):

Two students are talking to each other. We call them S1 and S2. Continue the dialogue.

S1: Hello, can you please explain something to me?

S2: Yes, of course. What do you want to know?

S1: We talked about the scalar product in class and got to know it as a tool for various calculations. But what exactly is the scalar product supposed to be?

S2: I can help you with that. It is best if I explain the meaning and the applications to you by giving examples. I can show you what the scalar product is and what it is used for.

S1: Thank you! However, count on me to be asking many questions.

...

### **Development of the main categories and associated subcategories**

Sixty-one students' dialogues from four different classes are included in the research project. After the data collection has been completed, a coding scheme and coding guidelines have been developed as a central element of the content analytical procedure. Based on these, the text has been divided into segments (Mayring, 2015). All text passages in the dialogues are considered relevant for coding decisions, the developed system of categories is intended to represent the entire material. Inductive and deductive category formations are defined. The coding scheme includes definitions of categories according to the mental models suggested by Frohn (2020). Also, new-found categories are phrased, augmented step by step through text editing and incorporated into the coding guide. The aim of the analysis is to be sensitive and to cover all aspects of students' individual cognitions.

To extract normative mental models on the scalar product from the written dialogues, the following three mental models presented by Frohn (2020) represent main categories: "product presentation", "orthogonality concept" and "angle concept". As none of the dialogue contents matches "projection presentation", this category has been discarded. First, all text components matching the main categories are extracted from the material. After screening 10% of the material, it becomes obvious that students' ideas about the scalar product are complex and the main categories do not represent the content of the dialogues well. Mental models mentioned in the literature do not reflect the students' ideas distinctively. Regarding the research question (c) it seems necessary to divide each of the main categories into subcategories.

The main category "product presentation" represents students' references to the scalar product as a specific algebraic operation. It indicates how students understand the scalar product as a specific operation, an aspect that is particularly emphasized in the classroom. Students give individual examples to calculate a scalar product by using cartesian components of two vectors. They describe their statements verbally and explain their calculations. Very few specify their ideas by giving the

formula written in mathematical symbols. Students also reflect on possible applications. Based on the listed aspects of the text material, the subcategories “concept presentation”, “result presentation”, “algorithm presentation” and “application presentation” are generated.

Similar subcategories can also be derived for the main category “orthogonality concept”. Contents in the dialogues referring to the main category “angle concept” can be assigned to the subcategories “concept presentation” and “algorithm presentation”.

After analysing the dialogues by means of the category system, several text passages remain uncoded. It has been observed that a large number of students underpin their dialogues with detailed further information or refer to previous knowledge. For example, they provide information about solution strategies or sources of error when calculating the scalar product. To capture these various statements and cognitions, a fourth main category (which does not refer to Frohn) “Advanced information on vector calculation” was developed. The main category “Advanced information on vector calculation” is multifaceted. In order to carry out data analysis, the diversity of the students’ presented topics has to be reduced to core contents. Five subcategories are defined: “prior knowledge”, “transfer”, “sources of error”, “solution strategies” and “interpretation”.

It is important for students to repeat existing knowledge such as adding or subtracting two vectors by adding or subtracting the corresponding components. They try to recall this previous knowledge and to combine what they have already learned with the new aspects of the scalar product. These observations are represented in the category “prior knowledge”. The category “transfer” refers to students’ questions about transferring the multiplication of two two-dimensional vectors to three-dimensional or multi-dimensional spaces. The consideration of sources of error and possible solution strategies poses an additional factor in the dialogues and justifies the definition of two further categories. Students discuss the meaning of the scalar product as well as possible applications. Questions such as “What’s the purpose of the scalar product?” appear in the dialogues but remain unanswered. This and similar statements made by students are summed up in the “interpretation” category. So far, these categories have not been evaluated in detail.

**Table 1: Overview of main categories and associated subcategories**

Main categories	Subcategories
product presentation	concept presentation
	result presentation
	algorithm presentation
	application presentation
orthogonality concept	concept presentation
	result presentation
	algorithm presentation
	application presentation
angle concept	concept presentation
	algorithm presentation
advanced information on vector calculation	prior knowledge
	transfer
	sources of error
	solution strategies
	interpretation

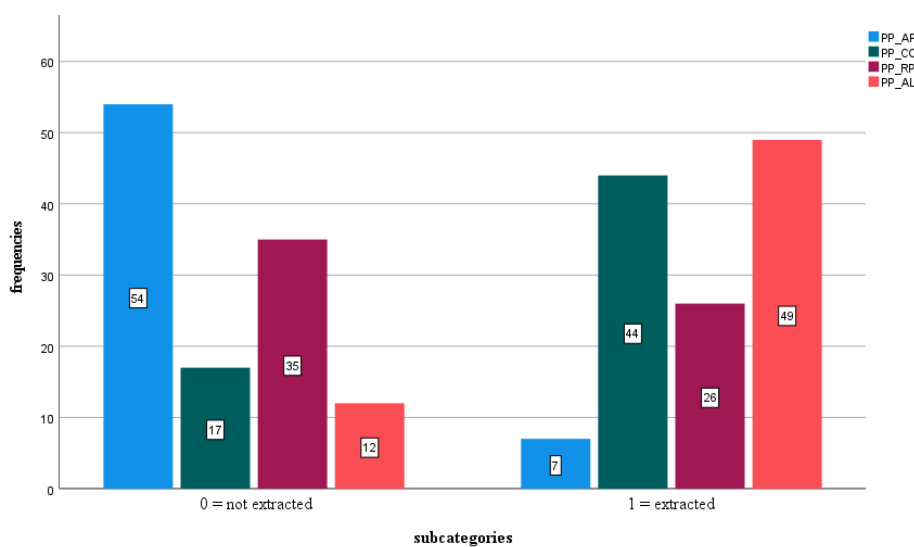
Table 1 gives an overview of the current coding guideline. All extracted categories to be considered relevant for the coding scheme are listed.

## Outcomes

After categories and content-analytical rules have been set, the material has been worked through and coded a second time. Afterwards a sample of 15 dialogues has been analysed using an intra-coder agreement test in order to prove reliability. Only small differences in the reproduction of the categories have been detected. A measurement for retest-reliability via inter-code agreement has not been established yet.

The focus of the analysis is set on the main category “product presentation” and the defined subcategories. The objective is to provide an insight into these categories. The dialogues show 126 references to the main category “product presentation” of the scalar product. This analysis has not revealed any content-related details; therefore, text passages have been assigned to subcategories.

The results of the associated quantitative analysis are presented in Figure 1. The term “extracted” in Figure 1 refers to the occurrence of subcategories in the dialogues (1 = extracted means that subcategory X has been found Y times in the material; 0 = not extracted means that subcategory X has not been found Y times in the material).



**Figure 1: Frequencies – extracted subcategories**

Students refer to two subcategories most frequently: the “algorithm presentation” (PP\_AL) and the “concept presentation” (PP\_CO). 49 students (PP\_AL) remember how to calculate the scalar product of two vectors given the cartesian components of the two vectors. The majority provide individually designed examples. They choose column vectors, multiply the components, and add the results. Doing so they use the formula to calculate a scalar product, but only few write down the formula in mathematical symbols.

The category “concept presentation” (PP\_CO) is represented by 44 students’ elaborations. Students mainly discuss a geometric approach to the concept of the scalar product but only very few use the scalar product in geometrical applications afterwards. Almost a fifth of the students mention

uncertainties relating the term “scalar product”. Uncertainties mainly concern the delimitation of the scalar product to the multiplication of a vector with a scalar. Students are also uncertain with the number of vectors that need to be multiplied to obtain the scalar product (“What if I multiply three vectors?”). Nobody states important properties of the scalar product.

Only 26 out of 61 students discuss the result of the multiplicative linking of two vectors. These statements are summarized in the category “result presentation” (PP\_RP). About a quarter of the students describe the result of the scalar product as a scalar value rather than a vector (“as the name suggests it is a scalar but no vector...”).

Within the category “application presentation” (PP\_AP), seven students discuss ways in which the scalar product can be used either in daily life (“What are real-life applications?”) or in physics (“to calculate work done by a force”). Five students give an example by choosing a quantity vector, which acts on a cost vector and returns a total cost.

## Conclusion

In this project, 61 students are writing about their thoughts on the scalar product. The written imaginary dialogues are analysed by using content-analytical procedures. Text analysis procedures are based on the concept of mental models. According to defined categories, written material is divided into segments. The main category “product presentation” has been evaluated in this paper. To classify the content more sensitively, the category has been divided into four subcategories to draw an extensive picture of the content and to present adequately teachers’ insights into students’ thinking about scalar products.

Based on this, we conclude that if teachers want to analyse imaginary dialogues systematically, the concept of mental models will provide an initial basic structure. To capture all contents presented by the students, the concept has yet to be modified.

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