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Some examples of quasiperiodic tilings obtained with a simple grid method

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Abstract – A grid method using tiling by fundamental domain of simple bi-dimensional lattices is presented. It refers to a previous work done by Stampfli in 1986 using two grids by regular hexagons, one rotated by $\pi/2$ relatively to the other. This leads to a quasiperiodic structure with a twelve-fold symmetry made of regular triangles, squares and rhombuses. The tessellation of the plane by the overlap domains of two hexagons, each belonging to one of the two grids is considered. Vertices of the quasiperiodic tiling are the mid-point of the centers of the two overlapping hexagons. Edges of the quasiperiodic tiling are obtained by a Delaunay triangulation of the set of reference points. This method is extended to two other types of quasiperiodic tilings with other fundamental domains. A first example use two square grids leading to the octagonal Ammann-Beenker quasiperiodic tiling. The second example is also based on the hexagonal lattice, but with grids defined by the lattice fundamental rhombic cell.

Introduction. – Recent examples of quasiperiodic phases with dodecagonal symmetry are encountered in various fields of condensed matter like liquid crystals, [1,2] colloidal self-assembly [3,4] and bi-dimensional molecular materials [5,6]. Their structure can be described using tilings of the plane by regular tiles like squares, triangles and rhombuses. In some particular cases, the quasicrystalline order comes from the superposition of two identical and periodic layers rotated by a suitable angle, forming a Moiré pattern [7]. A straightforward experimental example is the case for dodecagonal graphene where two twisted graphene layers are stacked together with a rotation angle of $\pi/6$ [5]. In a quite different context, numerical simulations of liquid crystals with an appropriate interaction potential between spherical particles give rise to a stack of alternate hexagonal layers with dodecagonal symmetry [1]. In this context, the use of grid methods for building quasiperiodic tilings is highly relevant [8].

In a short paper published in 1986 Peter Stampfli [9] introduces the construction of a quasiperiodic tiling of the plane with regular triangles, squares and rhombuses having a global dodecagonal symmetry. He give a way to generate this tiling by a hierarchical decoration of tiles. But he introduces also a construction derived from the overlap of two similar grids. The two grids are two peri-

odic hexagonal tilings by identical regular hexagons. One grid is rotate by $\pi/2$ relatively to the other, so the two grids together have a dodecagonal symmetry. Two years later, inspired by the Stampfli proposition, Korepin [10] published a more developed and general paper, making a bridge between the grid method and the cut and project method.

The first purpose of this paper is to give a detailed construction showing how it is possible to go from the two tilings by regular hexagons to the quasiperiodic dodecagonal tiling by squares, regular triangles and rhombuses.

Then we show that this construction can be extended to other tilings and give two examples. The first example uses a square lattice to define a grid which is a tiling by squares and the second grid is obtained by a $\pi/4$ rotation. The resulting quasiperiodic tiling is the Ammann-Beenker tiling by squares and rhombuses with eight fold symmetry. The second example uses the hexagonal lattice like in the Stampfli case, but the two grids are a tiling by rhombuses instead of hexagons. A rhombus is made of two regular triangles, with resulting angles of $\pi/3$ and $2\pi/3$. The obtained quasiperiodic tiling contains squares, triangles and rhombuses like in the Stampfli example, but also three-fold star-like additional tiles. We conclude that the use of different grids, just by changing the fundamental domain

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associated to a given lattice, is a powerful way to get a large choice of quasiperiodic tilings.

The two hexagonal grids of the Stampfli example. – The first grid is generated using an hexagonal lattice defined by the two base vectors: $\mathbf{e}_1 = \{1, 0\}$, $\mathbf{e}_3 = \{1/2, \sqrt{3}/2\}$. At each nodes of this lattice an hexagonal motif is reproduced leading to a tiling of the plane by hexagons. The vertices of this hexagonal motif are $\{0, \frac{\sqrt{3}}{3}\}$, $\{-\frac{1}{2}, \frac{\sqrt{3}}{6}\}$, $\{-\frac{1}{2}, -\frac{\sqrt{3}}{6}\}$, $\{0, -\frac{\sqrt{3}}{3}\}$, $\{\frac{1}{2}, -\frac{\sqrt{3}}{6}\}$, $\{\frac{1}{2}, \frac{\sqrt{3}}{6}\}$. Then the second grid is obtained rotating the first one by a $\pi/2$ rotation around the origin, so constructed relatively to a lattice having base vectors $(\mathbf{e}_2, \mathbf{e}_4)$ orthogonal to $(\mathbf{e}_1, \mathbf{e}_3)$. The choice of these four vectors ($\mathbf{e}_1 = \{1, 0\}$, $\mathbf{e}_2 = \{\sqrt{3}/2, 1/2\}$, $\mathbf{e}_3 = \{1/2, \sqrt{3}/2\}$, $\mathbf{e}_4 = \{0, 1\}$) refers to the lift construction in a 4D hyper-space of dodecagonal tilings using a cut and project method (see the Supplementary Material and [12]). Here the grid method is made directly in the 2D plane without using the lift construction.

Figure 1 displays the two grids of hexagons and also how edges of hexagons cut each others. They can cut at $\pi/3$ when the two edges belong to the same hexagon; they can be orthogonal, so with an edge belonging to one grid and the other to the other grid; they also can cut with a $\pi/6$ angle also from different grids. Stampfli states that each edge overlap define a tile of the quasiperiodic tiling: regular triangle if they are at $\pi/3$ angle, square for $\pi/2$ angle and rhombus for $\pi/6$ angle. Nevertheless Stampfli does not give a complete demonstration of this fact and it is what is done in the next section.

Domains defined by the two grids. – A domain is the polygonal surface common to one hexagon of the first grid and another of the other grid if both are sufficiently close to intersect (Figure 2). Hexagons are characterized by the lattice vectors positioning their centers. Call (i, k) a first one define in the base $(\mathbf{e}_1, \mathbf{e}_3)$ and (j, l) another in the base $(\mathbf{e}_2, \mathbf{e}_4)$ defining the two hexagons intersecting to form the domain we are considering. We attribute to each domain a reference point, located at the vector $(i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4)/2$ which is the middle point between the two centers of the overlapping hexagons, respectively located at $i\mathbf{e}_1 + k\mathbf{e}_3$ and $j\mathbf{e}_2 + l\mathbf{e}_4$.

Notice that reference points can be seen as vectors in a $E_{//}$ space projected from a space with higher dimension as it is often used in quasicrystal construction [12]. In the Supplementary Material, the projection in perpendicular space E_{\perp} is provided [13].

The aim of this paper is to show that reference points of domains are vertices of the quasi-periodic tiling.

A polygonal domain is the overlap of two hexagons, one of the first grid (red (i, k)) and one of the other grid (blue (j, l)). We define, as reference point of this domain, the point which bisect the vector joining the two centres of intersecting hexagons.

Consider a domain close neighbour to the previous one,

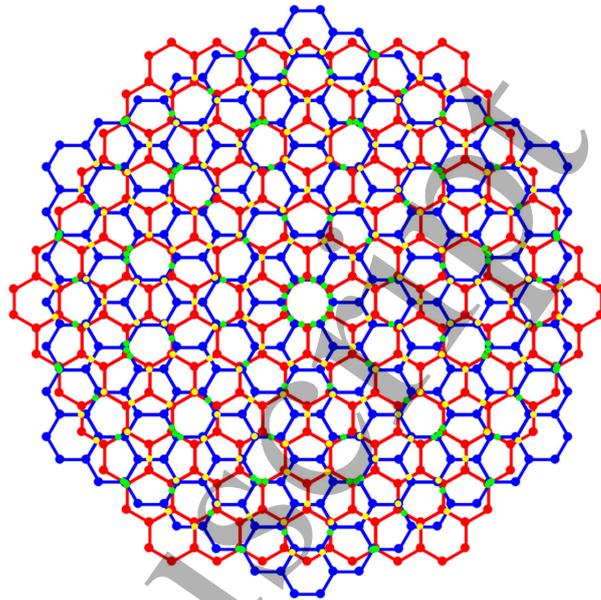


Fig. 1: The two hexagonal grids, one represented in blue, the other in red. Vertices between edges of hexagons are represented by points: red or blue for hexagons edges at hexagon vertices, yellow for two orthogonal edges, green for edges at $\pi/6$. Each edge intersections will be associated with tiles: triangles for red or blue points, squares for yellow points and rhombuses for green points.

so sharing one of its edges. Consequently this domain is necessarily the overlap of one of the previous hexagon (for instance the red one of the first grid (i, k)) and, in the other grid, a close neighbour with the hexagon (j, l) ; we can call it (j', l') . The vector joining (j, l) and (j', l') which are close neighbours in their own lattice, is a unit vector. So the length of the vector joining the two close reference points is the half of one unit vector defining lattices of grids (see Figure 3).

The line joining the reference point of the first considered domain with this new one is now considered as an edge of the tiling. Its length is half of the module of base lattice vectors used to construct the grids (said $1/2$). It is orthogonal to the common edge of hexagon (j, l) and (j', l') . Extending this construction to all edges of the first domain we conclude that from the reference point of this domain, there are lines of length $1/2$ orthogonal to domain edges so possibly making $\pi/3$, $\pi/2$ or $\pi/6$ angles referring to angles possibly done by edges of domains (the red and blue points or the yellow and green points of Figure 1).

A reference point of a domain is connected by a segment to a reference point of a neighbouring domain, so that the two domains are sharing a common edge. The important property is that the segment orthogonal to the shared edge has a length half of that of base lattice vectors, so all such segment have the same length. The angles appearing between segments joining close domains are the angles characteristic of regular triangles, squares or rhom-

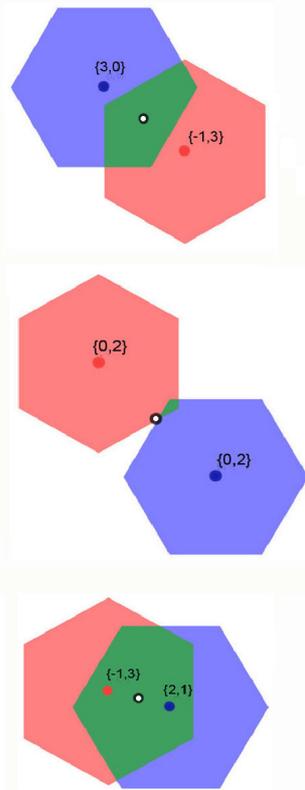


Fig. 2: Example of three different overlaps of two hexagons centered on nodes of the two lattices, one with coordinates (i, k) on the base $(\mathbf{e}_1, \mathbf{e}_3)$ (red point) and one with coordinate (j, l) on $(\mathbf{e}_2, \mathbf{e}_4)$ (blue points). Their common domain is shown in green, with its reference point (open circle) which is the middle point between the two centers of the overlapping hexagons. A reference point is a vertex of the quasiperiodic tiling.

buses. This is confirming that reference points of domains are vertices of a tiling by polygons whose edges have all the same modulus and angles are in the set $\pi/3, \pi/2$ or $\pi/6$. All tiles are necessarily regular triangles, squares or rhombuses forming a quasiperiodic tiling. All the overlap of hexagons, one of each grid, define polygonal domains entirely covering the $2D$ -plane (Figure 4). A domain has edges which are parts of edges of hexagons and have vertices which are the intersection points on hexagon edges, sometime from the same hexagon and otherwise from two hexagons from the two grids.

The quasiperiodic tiling is constructed from the set of all reference points of all domains using a Delaunay triangulation (see Figure 5). In practice, it is done using Mathematica code as detailed in the Supplementary Material [15]. In this triangulation there are edges with length different from $1/2$ like square diagonals. By keeping in the triangulation only edges of length $1/2$, on obtains the quasiperiodic tiling by regular triangles, squares and rhombuses.

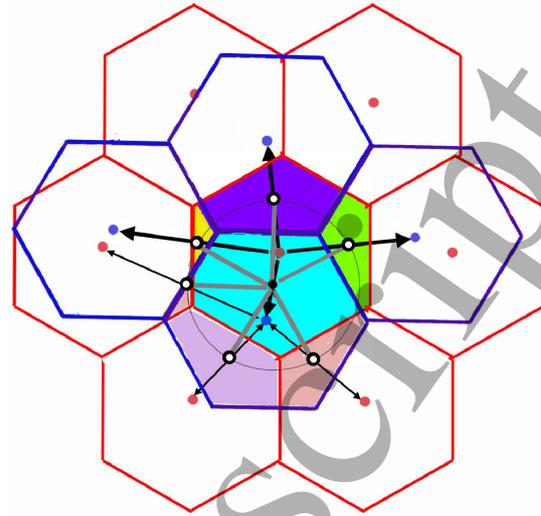


Fig. 3: Overlapping domains of two hexagons. The central red hexagon, centered on the central red point, intersect blue hexagons, centered on vertices of the other lattice (blue points). Reference points (open circles) of domains (vertices of the tiling) are mid points of vectors connecting the central red point to centres of blue hexagons. Edges of the tiling (in grey) join points marked by open circles, for instance the one between the blue and the green domains. The vector joining the centre of the right and bottom blue hexagons, which are close neighbours, is a vector of the blue lattice of unit modulus. So it appears that the length of the edge between the reference point of the blue domain and that of the green domain is half of this unit modulus. That is the same for all edges of the quasiperiodic tiling.

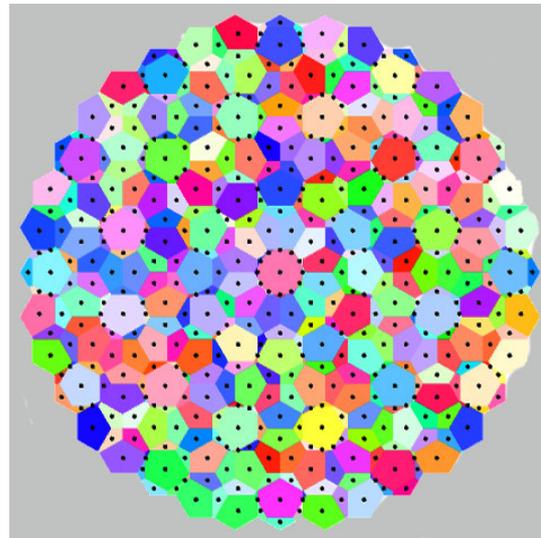


Fig. 4: Tessellation of the plane by overlap domains. All domains have a different shape and are plotted with random colors. To each domain corresponds a single reference point as explained in Figures 2 and 3. The set of all reference points is pictured with black dots. The location of a reference point is related to the shape of its overlap domain. The reference points are the vertices of the quasiperiodic tiling (see Figure 5).

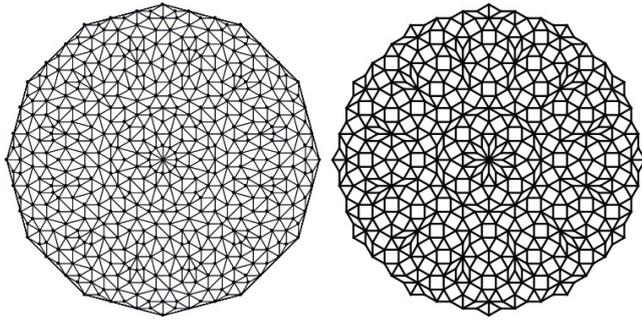


Fig. 5: Dodecagonal Stampfli's tiling. Left: black points are reference points of domains resulting from the overlap of hexagons. The figure is the Delaunay triangulation of this set of points. Right: Tiles of the quasiperiodic tiling are regular triangles, squares and rhombuses. This is derived from the Delaunay triangulation by suppressing edges whose length is not $1/2$ (diagonal of squares and rhombuses and lines at the border).

Another example with square grids. — In place of the two hexagonal grids we consider instead two grids of squares, one with squares centered on the vertices of a square lattice with unit vector of unit length, the other grid being simply rotated around the origin by a $\pi/4$ angle. Then we construct all the polygonal domains corresponding to the overlap of squares of the two grids, with one square on one grid and the other square on the other grid. The reference point of such an overlap domain is the mid-point of the two square centers.

This set of points can be triangulated to form a Delaunay set. In this set some edges have a $1/2$ length (in term of base vector length); selecting these edges we get a tiling by squares and rhombuses which is quasiperiodic. In fact we recognize the well known Ammann-Beenker [16,17] tiling obtained using inflation rules or by a different grid method (see Figure 6).

An example with hexagonal lattices but with less symmetric tiles for the grids. — In order to get a grid, the tile which is reproduced by the lattice has to be a fundamental domain of the lattice. In the Stampfli example, a fundamental domain of the hexagonal lattice is a regular hexagon (a Voronoi cell of lattice nodes). But we can also choose another fundamental domain with the same hexagonal lattice. To illustrate this idea, we consider for the fundamental domain a rhombus made by two glued regular triangles of unit size. Note that it corresponds to the standard unit cell of the hexagonal lattice. The grid method is applied exactly in the same way as for the two previous examples. The resulting quasiperiodic tiling is shown in Figure 7. All edges have the same length, which is equal to $1/2$. Tiles are regular triangles, squares, rhombuses, like for the Stampfli tiling, but in addition, a new type of tile is present. The shape of this tile is a three-arm star made of three half rhombuses radiating around a small triangle. Interestingly, this three-arm star is not

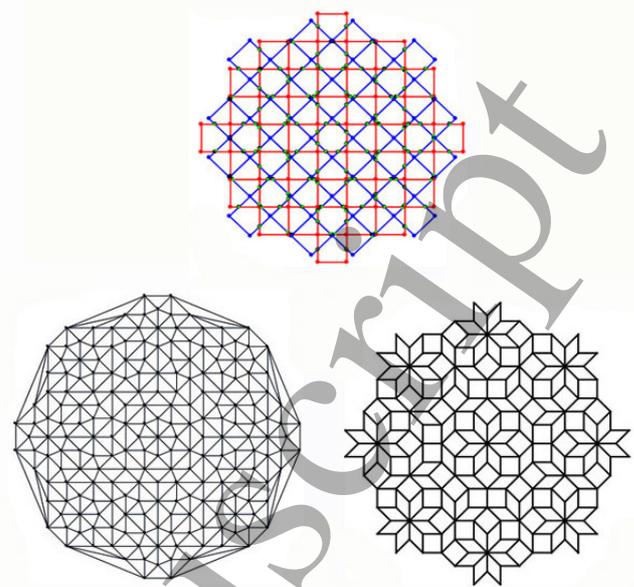


Fig. 6: Octagonal tiling. Top: The two square grids with intersections of edges. In the quasiperiodic tiling, vertices of grids lead to the two orientations of squares when intersections between blue and red edges lead to rhombuses. Bottom left: black points are reference points of domains resulting from the overlap of squares. The figure is the Delaunay triangulation of this set of points. Bottom right: Tiles of the quasi-periodic tiling are squares and rhombuses. This is derived from the Delaunay triangulation by suppressing edges whose length is not $1/2$ (diagonal of squares and rhombuses and lines at the border). This is the Ammann-Beenker tiling.

convex. In the quasiperiodic tiling, it is always surrounded by three rhombuses, forming a shield shape, a shape which is often encountered in experimental dodecagonal tilings as defects [3]. Note that the same three-arm star tile has been recently obtained in a dodecagonal tiling using a near coincidence rule between pair of points on two triangular lattices [18].

Conclusion. — There are different methods which generate quasiperiodic tilings. The present one based on grids of periodic tessellations of the plane is interesting as it makes a bridge between methods derived from projections of high dimension lattices and methods related to Moiré patterns.

The advantage of this grid method is that it is done directly in the plane. It contains only two simple steps. First the overlapping domains and the set of their reference points is build. Then the quasiperiodic tiling is obtained from the Delaunay triangulation of this set of points, after removing extra edges. Both steps are implemented using Wolfram Mathematica language. The code is provided in the Supplementary Material and is available online at the Wolfram Cloud.

Using this grid method, the type of tiles in the quasiperiodic tiling depends on the intersections between edges of

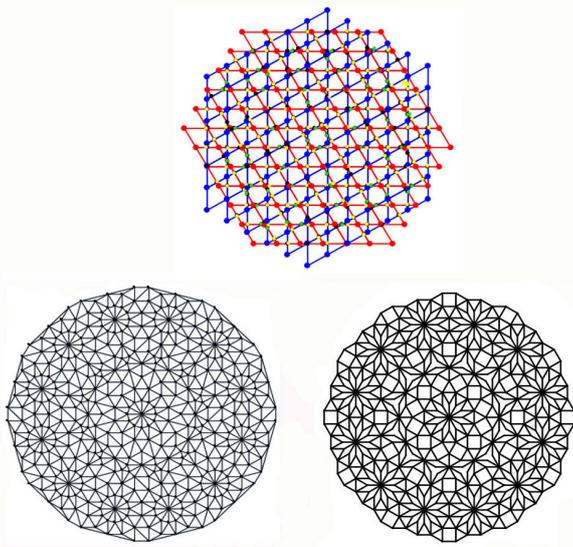


Fig. 7: Another dodecagonal tiling. Top: The two grids are made of rhombuses formed by gluing two regular triangles. Intersections of edges form $\pi/3$, $\pi/2$ or $\pi/6$ angles (same color code as in Figure 1). Bottom left: Delaunay triangulation of the set of reference points. Bottom right: The dodecagonal quasiperiodic tiling obtained after keeping only edges of length $1/2$. In addition to square, triangle and rhombus tiles, new tiles in the shape of three-arms stars are present in this example of dodecagonal tiling.

the tiles of the grid which have to be fundamental domain of the considered lattices. For instance, using hexagonal fundamental domain, intersections are with $\pi/6$, $\pi/2$ angles between grid tiles of the two grids, and with $\pi/3$ angles on grid vertices. The ratio between the different types of tiles is the ratio between the different types of intersection. An open question is that fundamental domain can have very complex shapes, eventually fractal [19], nevertheless the number of types of intersection remains probably small.

The rule that a reference point is the mid-point of the centers of two fundamental domains if they overlap is a very simple and general rule and has the advantage that it can be applied locally even on a small portion of two superimposed grids. Generalisation to other kinds of grids is straightforward and should give rise to new quasiperiodic tilings, that might be relevant to many condensed matter systems.

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